SURVEYING

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PREFACE

In the Preface to the First Edition the hope was expressed that the book would prove useful to students of Civil Engineering at Universities and Technical Colleges ; to students studying privately for the examinations of the Institution of Civil Engineers and other bodies ; and to many with no examinations in view. The reader was advised to work through the various examples given at the end of each chapter, particularly as, in many cases, these were chosen to emphasize points not elaborated in the text. Some of these examples are, by permission, taken from the examination papers of the Universities of Birmingham, London and Wales, and of the Institution of Civil Engineers.

The necessity for practical work was stressed, as it is only by the actual handling of the various instruments in the field that an efficient working knowledge of Surveying can be obtained.

Since the original edition was published in 1920, various additions and modifications have been introduced, and appendices were added from time to time to deal with special topics and to note modern developments. In this, the Fourth Edition, the type has been reset and considerable changes have been made.

Firstly, the notes from the appendices have been elaborated where necessary and incorporated in the text, and in particular the appendix on Errors in Surveying, kindly revised by Dr. F. Garwood, now forms the first portion of Chapter XIX—a new chapter which deals also with the Adjustment of Errors, a subject which is receiving an increasing amount of attention in the more advanced branches of the subject.

Other sections which have been partially rewritten and revised include the notes on the adjustments of the level and theodolite ; on the effects of various errors to which theodolite observations are liable ; on the comparison of the methods of balancing traverse surveys ; and on the three-point problem in hydrographic and plane-table surveying. The few notes previously given on Aerial Survey have been replaced by a fuller account, though it is realized that such a specialized branch of Surveying cannot be adequately dealt with in a general book such as this. A brief description is given of the New National Grid and Reference system which is being applied to Ordnance maps and plans. Throughout the book, references have been added and data revised.

In the Preface to the First Edition, the author expressed his thanks to the Astronomer-Royal, the Director-General of the Ordnance Survey, and the Surveyor-General of Canada, for kindly supplying information ; to Professor F. C. Lea, O.B.E., D.Sc., and Professor S. M. Dixon, M.A., M.Sc., for permission to use results obtained by the students of the University of Birmingham in River Gauging and Photographic Survey-

ing respectively, and to Dr. G. A. Shakespear, M.A., D.Sc., who kindly read the chapter on Optics and Magnetism.

The author's thanks are also given to the following firms who kindly supplied blocks of their instruments : Messrs. Barr & Stroud, Messrs. C. F. Casella & Co., Messrs. Cooke, Troughton & Simms, Messrs. J. Halden & Co., Messrs. Harling, Messrs. J. J. Hicks, Messrs. W. F. Stanley & Co., Messrs. J. H. Steward, Messrs. A. G. Thornton, and Messrs. E. R. Watts & Son. In addition, Fig. 8.18 is reproduced from *The Modern Rangefinder*, by Cheshire, and Fig. 13.2 from Clark's *Geodesy*, by the kind permission of Messrs. Harrison & Sons and the Clarendon Press respectively.

The author thanks correspondents from various countries, and reviewers, for helpful suggestions which have been adopted where such would not have unduly extended the scope of the book. He will be grateful to receive notification of any errors he may inadvertently have overlooked.

W. N. T.

CARDIFF
October 1947

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PLATE

CHAPTER I

INTRODUCTION, CHAIN SURVEYING, PLANS AND MAPS, CALCULATION OF AREAS, ACCURACY OF CHAINING

INTRODUCTION

Surveying is the art of determining the relative positions of distinctive features on portions of the earth's surface.

Generally, this term includes **Levelling**—*i.e.* the determination of the relative heights of different positions and objects—though not infrequently Surveying and Levelling are considered as distinct operations.

The "plotting" of plans and sections, the calculation of areas and volumes, the setting out of proposed "works", and the determination of latitude and longitude, etc., are also included among a Surveyor's duties.

The object of a survey may be the preparation of a plan or map, and in this case a representation of the horizontal projection of the area is usually required—not a representation of the actual area—as it is obvious that the relative positions of *all* points on the different slopes of a hill, for instance, cannot be correctly represented on a single plane surface. Thus, on a plan, horizontal distances are shown, though a more or less correct representation of vertical distances may be made by the use of contour lines, hachures, or some other of the methods described later.

Vertical distances, however, are more correctly shown by means of sections.

A plan is not invariably required, for, if areas or volumes only are wanted, they may often be calculated direct from the field book.

Classification.—Surveys may be classified under headings which define the use or purpose of the resulting maps:

Thus **Topographical** Surveys are made to determine the natural features of a country, such as rivers and streams, lakes, woods, hills, etc.; and such artificial features as roads, railways, canals, towns, and villages.

Cadastral Surveys are usually plotted to a larger scale than topographical surveys, and determine additional detail, such as the boundaries of fields, houses, and other property.

Geodetic Surveys are conducted with a very high degree of accuracy to furnish data concerning the size and shape of the earth, or to locate the positions of widely distant points which shall afterwards act as control points for less accurate surveys.

Engineering Surveys are undertaken for the determination of quantities or to afford sufficient data for the designing of engineering works, such as roads and reservoirs, or those connected with sewage disposal or water supply.

Railway Surveys in connection with proposed railway schemes are of several classes—Rough Exploratory or **Reconnaissance** Surveys to choose possible routes; **Preliminary** Surveys to obtain sufficient data to enable the best route to be chosen and laid down definitely on the plan; **Location** Surveys to set out the adopted line on the ground and to obtain all the necessary data for quantities, etc.

Geographical and **Exploratory** Surveys, **Military** Surveys, **Geological** and **Magnetic** Surveys are other types, each with a purpose of its own.

An alternative classification may be based upon the instruments or methods employed, the chief types being:

(*a*) Chain Surveys,
(*b*) Theodolite Surveys,
(*c*) Traverse Surveys—closed or unclosed,
(*d*) Triangulation Surveys,
(*e*) Tacheometric Surveys,
(*f*) Plane Table Surveys,
(*g*) Marine or Hydrographic Surveys,
(*h*) Photographic Surveys, and
(*i*) Aerial Surveys.

Chain Surveying

The simplest of these is the "Chain Survey", but this is only suitable for moderately small areas. The chief appliances used are the chain, tape, arrows, ranging rods, offset staff, and occasionally a cross staff, optical square, box-sextant, or prismatic compass.

The **Chain** is generally divided into 100 links, sometimes into 50—but there are several varieties and lengths in ordinary use.

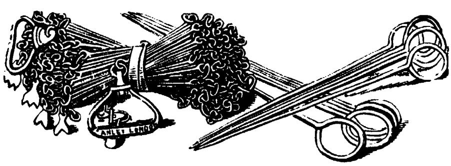


Fig. 1.1.—Chain and Arrows

The links are composed of lengths of iron or steel wire, and—except at the centre of the chain, and at the 25th link from each end, where swivel joints (S, Fig. 1.2) are provided—these are connected at their extremities by three small oval rings, preferably welded. At every 10th link from each end of the chain a brass tag or teller is fastened to the small central connecting ring. The teller which has only one point indicates ten links from either end of the chain—*i.e.* the 10th or the 90th link measuring in the same direction; that with two points marks the 20th or the 80th link; three points indicate the 30th or the 70th link; four points the 40th or the 60th link; and a circular tag the centre of the chain.

The brass tellers are sometimes designed to be inserted in the

length of the chain, but though less liable to catch in hedges, etc., they are perhaps hardly as distinctive as the usual type.

The ends of the chain are furnished with brass handles attached by

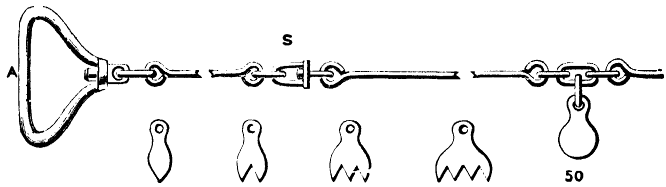


FIG. 1.2.—Brass Tellers

means of swivel joints, and the length of 100 links is measured from the outside of one handle to the outside of the other.

The **Gunter's** chain—so named after its inventor—is generally used by the Land Surveyor. It is 66 ft. in length, each link measuring 7·92″, and is very convenient when it is required to calculate areas in acres and decimals of an acre, since 10 sq. chains = 1 acre: also when linear dimensions are required in miles and furlongs, since 10 chains = 1 furlong and 80 chains = 1 mile. When the term " chain " or " link " is used in a general sense, without reference to any particular unit of measurement, the Gunter's chain is inferred.

The **Engineer's** chain is 100 ft. long, each link being 1 ft. in length. It is heavier than the Gunter's chain, but being longer does not need to be laid down so frequently in the measurement of a definite distance; for this reason there is less liability to error from the inaccurate marking of the ends of the chain. Again, as the levelling staff is usually graduated in feet and decimal parts of a foot, this chain is more convenient than the 66 ft. chain when used in connection with levelling or tacheometric operations. In municipal work, too, the 100 ft. chain is employed, as, in this case, dimensions are required to be expressed in yards, feet, and inches.

Metre chains are also in use, the commonest lengths being 10, 20, and 25 metres. They are subdivided into one-fifth parts of a metre and tallied at every two metres from each end.

A **Steel Band** chain (Fig. 1.3) $\frac{3}{4}''$ or $\frac{5}{8}''$ wide, or even less, wound on a steel cross, is frequently used for more accurate work, as, unlike the ordinary chain which stretches by continual use, it is practically unalterable in length; but since much more care has to be taken over its manipulation and preservation, and since the readings are not nearly so distinct, a chain has advantages for ordinary work.



FIG. 1.3.—Steel Band

Tapes of steel or linen are used for subsidiary measurements. They are generally 33, 50, 66, or 100 ft. in length, and are graduated in feet and inches on one side, and in links on the reverse.

The **Offset Staff** is a round, wooden rod, 10 links long, painted in two or three colours (red, black, and white) to show 1 link lengths, as in Fig. 1.4. It is fitted with an iron shoe and sometimes also with a brass

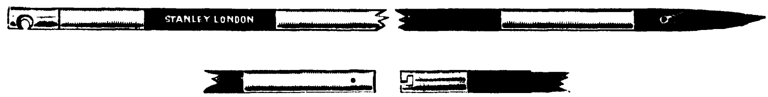


FIG. 1.4.—Offset Staff

hook for dragging the chain through a hedge or other awkward place. Occasionally it is jointed at the centre. It is used for measuring short distances—or offsets—from the chain line to objects near.

Ranging Rods are similar rods, 6, 8, or 10 ft. long, painted in two or three colours, and shod with iron shoes. They are used to mark station or other points in a survey, so that straight lines may be ranged out over the ground and chained if necessary. If the painted divisions are in feet or link lengths, the rods may be used as offset staves.

Whites are small pieces of cleft stick, cut from the hedges, and carrying slips of paper to enable them to be easily distinguished. They are used to assist in ranging out lines, and render unnecessary a large number of ranging rods.

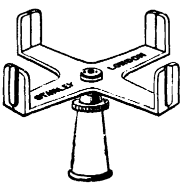


FIG. 1.5

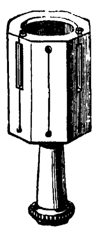


FIG. 1.6

The Cross-Staff

Arrows are iron or steel pins (Fig. 1.1) 15″ to 24″ in length, sold in sets of 10, and used to mark the end of each chain length when measuring a long line.

The **Cross Staff** is a small instrument, $2\frac{1}{2}''$ to 4″ across, and is used to set out right angles in the field, particularly in the case of offsets which are too long to allow of their direction being judged by the eye alone. There are several forms in common use, such as those by Stanley shown in Figs. 1.5 and 1.6, which fit on to a short iron-shod staff. The four vertical arms in Fig. 1.5 and the eight faces in Fig. 1.6 are provided with vertical slits or hair lines. The rod is held vertically over a point on the chain line, and the instrument turned until the line of sight through one pair of opposite slits is along the chain line. Then the line of sight through the two other slits (in Fig. 1.5) is at right angles to this, and a ranging rod may be directed into a convenient position for marking it. With the octagonal type, angles of 45° or 90° may be set out.

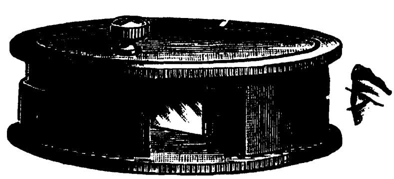


FIG. 1.7.—Optical Square

The point on the chain line where falls the perpendicular from any definite object may easily be found by a few trials.

The **Optical Square** is used for the same purpose as a cross staff. It

usually consists of a small cylindrical box about 2″ in diameter, in which are fitted two mirrors at right angles to the plane of the instrument.

Fig. 1.7 gives a general view of the instrument as made by Harling. Prisms are used in place of mirrors in some optical squares, and one such instrument made by Stanley is shown in Fig. 1.8. In a similar way prisms are fairly widely used in the optical systems of other surveying instruments.

The optical square shown in Fig. 1.8 is of the double type. It contains two prisms : rays from the left are deflected through 90° to the eye by two successive reflections in one prism, and rays from the right are similarly deflected through 90° to the eye in the other prism.

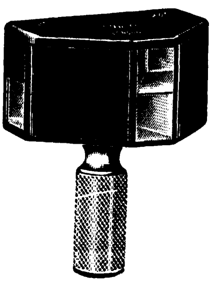


Fig. 1.8.—Double Prism Optical Square

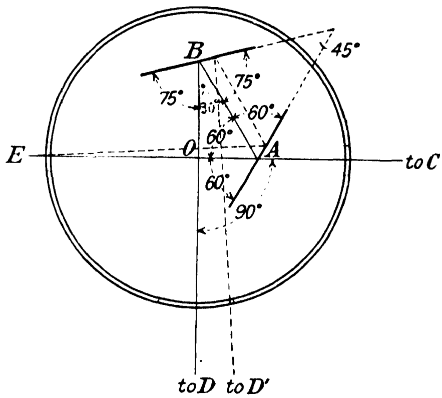


Fig. 1.9.—Principle of Optical Square

The included angle between the two reflecting surfaces of each prism is hence 135°.

While sighting directly over the instrument at O to a selected object C, such as a ranging rod, in one direction, objects D_1 and D_2 on the perpendiculars to OC, either to the left or to the right, can be seen in the prisms ; C, D_1, and D_2 coinciding.

The double prism can also be used as a line-ranging instrument, *i.e.* the observer can range himself into line between two objects, say D_1 and D_2, one of which is to his left and the other to his right, merely by moving along OC until the images D_1 and D_2, one in each prism, coincide.

Fig. 1.9 is a diagrammatic view of a mirror instrument. E represents the position of the eye, and the object at C can be observed through small apertures in the sides of the box, and through the lower unsilvered part of the mirror A. Usually EC would lie along the chain line and C would be the pole, towards which the chain line was being ranged. An object D approximately at right angles to EC would be seen in the top silvered portion of the mirror A, after reflection from the other mirror B. When the angle DOC is exactly 90°, the image of D would be immediately above that of C (Fig. 1.10 *a*), the rays being

as shown in full lines in Fig. 1.9. When, however, D′OC is not quite a right angle, D′ would appear to one side of C (Fig. 1.10 *b*), the rays being as shown in dotted lines in Fig. 1.9.

If D is a fixed point and it is required to find where the perpendicular from this cuts the chain line, the instrument would be held to the eye while the Surveyor walked along the chain line towards C. When the two images appeared in the same vertical line, the point immediately below the instrument would be the foot of the perpendicular required.

The optical square is a special case of the box sextant, and its theory will be dealt with later. Like that instrument, too, it must be kept horizontal when observations are being made so that its use is restricted to moderately level ground.

To test if the instrument is in correct adjustment, range out a straight line (Fig. 1.11) upon fairly level ground, and, by standing at any point B on this and looking towards A, direct a staff man at D to the right or left until the image of the pole he is holding coincides with the pole A seen by direct vision in the lower part of the glass. Now hold

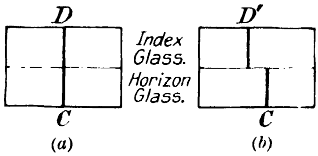


FIG. 1.10—Optical Square.

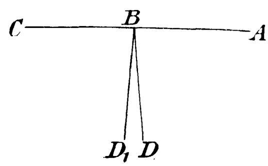


FIG. 1.11

the instrument upside down, look along the line BC towards C, and set out a point D_1 at right angles to BC. If the mirrors in the instrument are in their true positions, B, D, and D_1 should be in line, *i.e.* the two perpendiculars BD and BD_1 should coincide.

If D and D_1 do not coincide, but the angles ABD and CBD_1 overlap, then the angle between the mirrors is too large, *i.e.* is greater than 45°. If D and D_1 do not coincide, and the angles do not overlap, as in Fig. 1.11, the angle between the mirrors is too small. In some instruments the mirrors are permanently fixed by the maker, and so cannot be easily adjusted : in other forms a special key is provided by means of which one mirror may be rotated relatively to the other as explained on p. 76 for the box sextant.

The Adjustment of the Chain.—To **unfold** the chain, the two handles are grasped in the left hand and a few pairs of links unfolded. The remainder of the chain is then taken in the right hand and thrown forward. The 50 tally is found and held by an assistant, and the two lengths separated, after which one handle is taken ahead and the chain stretched into line ready for use.

To **fold** up the chain after use, the two halves are brought to lie alongside each other so that the two handles come together. The folding is commenced at the 50 tally. The first two pairs of links from this point are folded across the left hand, then by grasping the end of the fourth pair with the right hand the next two pairs are folded and

laid diagonally across the first set, and by a continuation of this process the chain is neatly folded into a wheatsheaf bundle as in Fig. 1.1.

Before the chain is used on a survey, any bent links should be straightened, and the length of the chain compared with that of a standard chain, or with some other permanent standard distance previously marked out, *e.g.* on a curb or on the plinth of a building.

If the chain is found to be too long after all the bent links have been straightened, it may be adjusted—

(*a*) By closing up any of the joints of the small connecting rings that may have opened—especially if these are not brazed.

(*b*) By hammering back to shape, and so shortening, a number of the small rings that may have become elongated with the continual strain.

(*c*) By replacing some of the worn rings by smaller new rings.

(*d*) By removing altogether, if necessary, one or more whole rings.

The adjustment should be made as evenly as possible along the whole length of the chain, so that fractional parts as well as the whole chain shall be restored as nearly as is practicable to the correct length.

If the chain is found to be too short, it is probably due to the fact that some of the longer links are bent, and consequently the length may be adjusted by straightening these. If the length is still found to be below the standard, *e.g.* if the chain has been overcorrected when too long, it may be lengthened by flattening some of the small connecting rings, or by replacing a few of these by larger sizes, or if necessary by inserting one or more new rings.

The Principle of a Chain Survey lies in the provision of a skeleton framework of straight lines, that can be plotted to scale, when the lengths of these lines are determined.

To enable this to be done the framework must consist essentially of triangles, as a triangle is the only simple plane figure enclosed by straight lines which can be plotted from the lengths of its sides alone. Thus, if the four sides of a figure ABCD (Fig. 1.12) are known, the figure can only be plotted—unless the magnitude of one of the angles is known—by dividing it into two triangles by a diagonal AC or BD, whose length has previously been determined, or by a measured tie line such as *bd*, which is equivalent in effect to BD, because it fixes the points B and D. But it is advisable to have some check upon the accuracy of the framework, and this is furnished by obtaining the lengths of a few more lines than are absolutely necessary for the plotting alone, and ascertaining that they agree with the scaled lengths on the plan. Thus, in Fig. 1.12, if the lengths of AB, BC, CD, DA, and BD are known, the quadrilateral can be plotted, but mistakes in any of the lines would

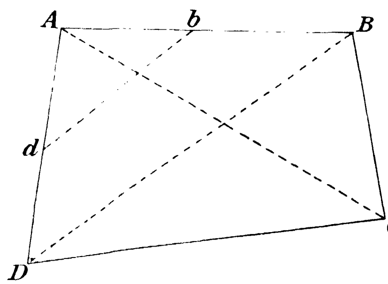


Fig. 1.12

be undetected unless large enough to alter very materially the shape of the figure, or to cause two sides of one of the triangles to appear less than the third side, when the work could not be plotted. A check line such as AC would therefore be advisable, and its length scaled from the plan should agree with its length measured on the ground.

The exact arrangement of chain lines to be adopted for any particular survey can only be decided upon after full consideration of the peculiar conditions of the case. The best arrangement is that which enables the whole of the required data to be obtained with the least expenditure of time and labour and with the necessary degree of accuracy. (*Vide* Fig. 1.30 and Example 1, p. 46.)

Certain broad rules may be observed, but otherwise the scheme depends upon the judgment and skill of the Surveyor.

(1) The ground should first of all be inspected, as a likely scheme on paper will often be found faulty or impossible on the ground, *e.g.* on account of thick hedges, trees, and other obstacles.

(2) The framework should contain at least one long chain line which will prevent distortion or twisting of the plan when plotting, and which will serve as a base line upon which the whole scheme may be built.

(3) A sufficient number of lines should be measured to enable the framework to be plotted, *i.e.* the system should virtually be composed of triangles as explained above. These triangles should be as well proportioned, *i.e.* as nearly equilateral, as the conditions permit.

(4) A few extra lines should be skilfully chosen and measured to act as checks and tie lines.

(5) As many as possible of the chain lines, including the check lines and ties, should run sufficiently near to the hedges to enable detail to be surveyed with a minimum expenditure of time and trouble, *i.e.* the offsets should be as short as possible.

(6) The lines should be measured in such order as to prevent as much as possible any useless walking between stations.

Method of Procedure.—The general scheme having been decided upon, the station points, *i.e.* the angular points of the framework, are fixed on the ground with wooden pegs, and a key plan showing the various lines, numbered or lettered, is sketched in the field book.

Chaining commences from one of these points and is carried throughout all the lines of the frame as continuously as possible.

To chain a line at least two persons are required, but frequently three are employed—the Surveyor himself, who does the reading and booking, the " leader ", and the " follower ".

The direction of the first line having been marked out by means of ranging rods (or " whites " if necessary), the leader taking with him the ten arrows, drags the chain ahead.

The follower holds the rear end of the chain at the station point, and by movements of his arm directs the arrow or ranging rod held by the leader for the purpose, into true alignment. The leader then pulls the chain taut, and inserts an arrow in the ground to mark the end. After ascertaining that none of the connecting rings is knotted,

measurements are taken from the chain line to any near objects that are to be surveyed, and these measurements are termed "offsets".

As already mentioned, it is a general principle that these subsidiary measurements should be as small as possible without undue elaboration of the skeleton framework, the limit allowable depending upon the scale to which they are to be plotted.

An object may be fixed with reference to the chain line by (*a*) a single offset at right angles to the chain line; (*b*) a single oblique offset at any definite angle, other than 90°, to the chain line; (*c*) two linear measurements from different points on the chain, forming a small triangle; or (*d*) two angular measurements from two points on the chain.

(*a*) and (*c*) are generally employed in a chain survey, the latter for the more important points, but (*b*) and (*d*) may be resorted to in exceptional cases.

When the Surveyor has taken any offsets that he requires, either with the offset staff or with a tape, the leader again pulls on the chain, leaving an arrow to mark the position of the end of the first length. The follower holds the rear end of the chain against this, and directs the leader into alignment as before. After the chain has been pulled taut, and the further end marked by the second arrow, the follower picks up the first and carries it with him. The number of arrows in the hand of the follower at any time should thus indicate the number of complete chain lengths measured. After 10 chains have been laid down, the follower hands over the 10 arrows to the leader and the same procedure is carried out for the next 10 lengths.

Field Book.—The field book is an oblong book hinged at the narrow edge and the chain is represented in it either by one or by two red or blue lines ruled down the centre of the length of each page.

The booking is commenced at the bottom of the end page and carried continuously through the book, so that when the operator in the field is inserting any dimensions the chain line on the ground and that in the book run in the same direction.

Methods of booking vary considerably. Specimen field notes for a single-line book are given on pp. 20-22, but many surveyors prefer the double-line book, in which the main chain line figures are separated from the offsets, and are hence less liable to become confused with them. Also, a rule is sometimes observed that all offsets are booked on the inner or chain side of the points to which they refer.

Station points are sometimes lettered or numbered, and a small oval or rectangle is drawn in the field book to enclose the chainage figures, or the points are distinguished by a small circle at the side. The lines meeting at the station point are also indicated and lettered or numbered as in Fig. 1.13.

At the commencement of a line in the book are sometimes written the direction in which the chaining is carried out and also the numbers or letters of the station points connected, *e.g.* "Line 7, go north from J to K"; or the entry may be simply "Line 7" or "commencement

of Line 7 ". In order to facilitate reference, no line should be commenced on the same page that contains the finish of another line.

A fence, hedge, or other object cutting the chain line obliquely is

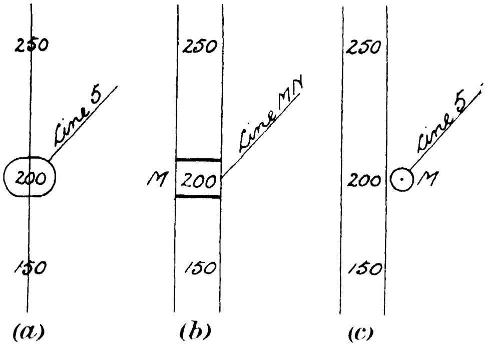


FIG. 1.13

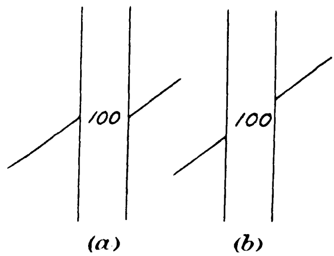


FIG. 1.14

shown on the double line field book as in Fig. 1.14 (*a*) ; not as in Fig. 1.14 (*b*).

When a field is bounded by a hedge and ditch, the true boundary is generally at the edge of the ditch (Fig. 1.15 (*a*)), but as this is nearly always very ill defined, offsets are taken from the chain line to the centre or root of hedge, and usually this is the line which is delineated on the plan. The position of the ditch is indicated by the small T, and the distance of the true boundary from the plotted line is expressed as, say, " 4 ft. R.H. ", where R.H. signifies Root of Hedge, *e.g.*

T
4 ft. R.H.

In the calculation of areas, an allowance must then be made for the fact that the continuous line on the plan is not the true boundary.

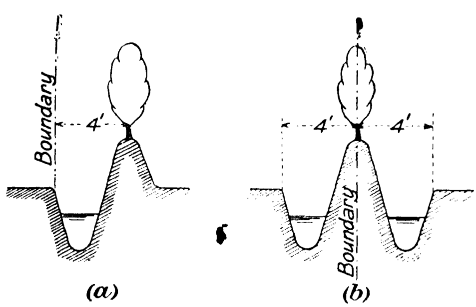


FIG. 1.15.—Boundaries

Frequently, although the *offsets* are taken to the root of the hedge, the correct allowance is made while plotting, and the true boundary or edge of ditch is represented by the continuous line on the plan, while the T sign indicates to which property the ditch belongs—*i.e.* to which side of the boundary the *hedge* lies. In this case the areas may be ascertained directly from the plan without any further allowance.

Occasionally again, the continuous line represents the " root of hedge ", while the T shows to which property the ditch belongs—*i.e.* the T is on the *opposite* side of the hedge to the ditch. This method,

however, is not as common as the first two methods. Each of the first two methods has its adherents, and it is very often impossible to tell from the plan alone which notation has been adopted.

The distance from the " root of hedge " to the true boundary varies in different parts of the country, and has such values as $4\frac{1}{2}$ links, 6 links, or 4 feet.

When there is a ditch on each side of the hedge, the boundary would usually be the centre of the hedge (Fig. 1.15 (*b*)).

The general principle of a chain survey having been noted, a few special problems in connection with the field work will now be explained.

To set out a right angle on the ground the following are the chief methods adopted :

(i) In the case of small offsets such as those taken with the offset staff, the angle is merely judged by the eye.

(ii) For larger offsets taken with the tape, if the zero be held at the object, and the tape swung round this as a centre, the foot of the perpendicular from the object on to the chain line may be determined by noting that position which gives the minimum reading on the tape. This point may be judged approximately by the eye, so that the arc through which the tape is to be moved is small.

(iii) Angular instruments may be employed, such as the cross staff, optical square, and box sextant, which have already been referred to, or more precise instruments such as the theodolite.

(iv) The chain only may be employed, or the chain and tape together ; these methods mostly depend upon Euclid I. 47 or its converse I. 48, and it may easily be seen that triangles having their sides in any of the following ratios contain a right angle opposite the longest side :

3	:	4	:	5		
5	:	12	:	13		
1	:	1	:	$\sqrt{2}$ or 1·414		
1	:	$\sqrt{3}$ or 1·732	:	2		(*i.e.* half an equilateral triangle).

As an example of the use of the first and most common of these ratios, assume the perpendicular is required to be set out at the end B of any particular chain length AB (Fig. 1.16). While the chain is stretched on the ground, mark the position of the 70 tally at C by means of an arrow, *i.e.* CB = 30 ft. Slip the handle of the chain over an arrow at B and skewer or hold the 10 tally at C. If the 60 tally be now found and the chain pulled taut to D as in Fig. 16 the angle at B will be 90°, because CB = 30 ft., BD = 40 ft., and DC = 50 ft. To check this result a similar procedure may be adopted on the far side of B from A and another right angle set out exactly as when using the optical square (Fig. 1.11). If the two operations do not yield the same result a mean may be adopted provided that the discrepancy is not large.

Another method is to measure 25 ft. (Fig. 1.17) along the chain line to each side of the point B, slip the handles over arrows at these two points (A and C) and pull the chain taut from the 50 tally at D, when BD will be the perpendicular required. The angles at A and C are each 60°.

To chain across obstacles such as ponds or rivers which do not obstruct the view, or to obtain their widths :

(1) Small distances can be measured by stretching the chain or

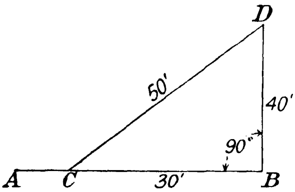


Fig. 1.16

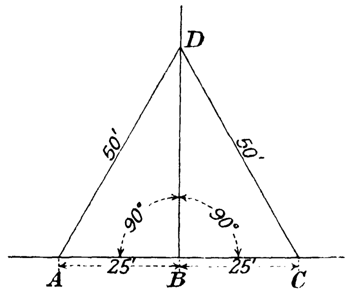


Fig. 1.17

Setting out the Right Angle with the Chain

tape tightly across, but the sag—especially in the case of the chain—is likely to cause considerable error if the distance is at all large.

(2) Let AB (Fig. 1.18) be the chain line. At any point C near the edge of the pond set out an angle of 90° as explained above, and measure a length CE sufficiently long to clear the obstacle. At E set out another right angle and measure to F, where a third right angle is set out. Measure FD = EC; then as a check upon the work C, D, and B should be in line, and EF is equal to the required length CD.

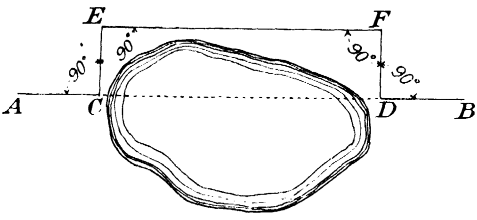


Fig. 1.18

(3) As before, let AB (Fig. 1.19) represent the chain line. At the point C, near the edge of the pond, set out an angle of 60°, either with the box sextant or with the chain and tape, and measure the length CE. The position of E must be so judged that ED shall lie at a convenient distance from the far edge of the pond. At E set out another angle of 60° and measure the length ED equal to CE. As a check upon the accuracy of the work D should be in line between C and B. The distance CD will be equal to either of the lengths CE or ED.

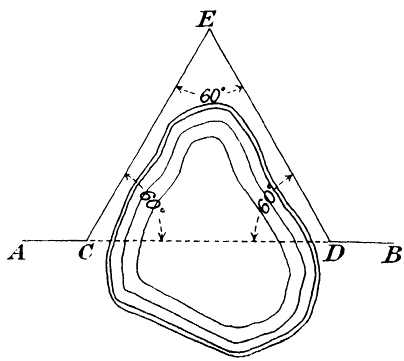


Fig. 1.19

(4) Fix rods at the points C and D (Fig. 1.20) on the chain line near the opposite edges of the pond, and also at any point E which is in

such a position that CE and DE are clear of the banks. Measure the length CE and prolong this line to F, making EF some multiple of CE, *i.e.* EF $= n$. CE say. Measure DE and prolong to G, making EG $= n$. DE. Then measure FG, which should be equal to n . CD because the two triangles CDE, FGE are similar.

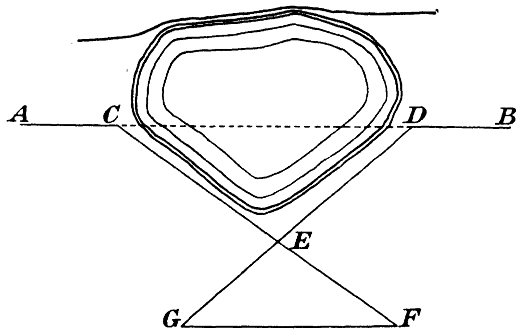


Fig. 1.20

(5) In the case of a river, let poles be fixed on the chain line at C and D, points on its opposite banks. Measure any length AC, and at A and C set out angles of 90°—or *any* other equal angles if more convenient. Fix any point E in the line from C and another point F in the line from A so that F, E and D shall be in line. Then because the triangles AFD, CED are similar, if the lengths AC, CE, and AF are measured, CD can be deduced :

$$i.e.\ \text{CD} = \frac{\text{CE} \cdot \text{AC}}{(\text{AF} - \text{CE})}.$$

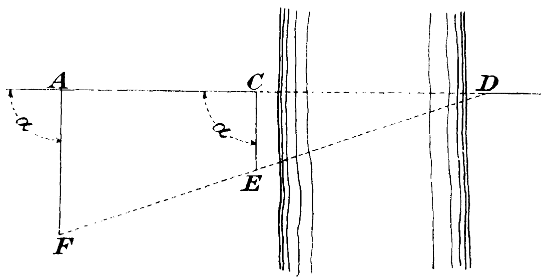


Fig. 1.21

(6) Let poles be fixed on the chain line (Fig. 1.22) at C and D, points on the banks of the river. At C set out CE at right angles to AD (preferably with an optical square) and measure any convenient length CE. At E set out EA at right angles to ED and prolong this line to A, its intersection with the main chain line AD. Measure AC. Then because the triangles ACE, ECD are similar,

$$\text{AC} : \text{CE} :: \text{CE} : \text{CD},$$

$$\text{from which CD} = \frac{\text{CE}^2}{\text{AC}}.$$

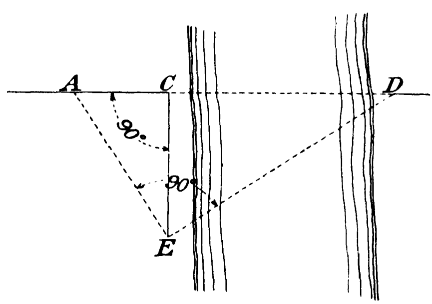


Fig. 1.22

(7) The points C and D (Fig. 1.23) being on the chain line near the banks of the river as before, set out a definite angle at C and measure any length CE : prolong CE to F, making EF some multiple of CE, *i.e.* n . CE (say), where n may be unity if desired. At F set out an angle EFG equal to the angle DCE, and prolong the line FG to G so

that G is seen to be in line with E and D. Measure the length FG. Then as the triangles CDE and FGE are similar, CD the length required is equal to FG ÷ n or $\frac{CE}{EF}$. FG.

It may be noticed that it is not necessary that the angles at C and F shall be a number of even degrees, though angles of 60° and 90° are very convenient. The angle a is definitely fixed if a small triangle is formed with sides of a feet along AC, b feet along CH, and c feet from H to AC as in Fig. 1.23. This fixes the direction of HC and hence of CF. A similar triangle is then set out at F as shown, to fix the direction FG.

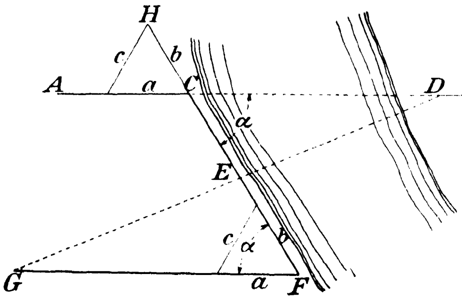


FIG. 1.23

Numerous modifications of the above methods, involving the use of the chain and tape only, may be employed. Inaccessible distance may also be determined by triangulation with a theodolite or other instrument; by tacheometry; by telemetry, or by means of a plane table, but these methods do not come within the scope of the present chapter.

When an obstacle such as a house obstructs the view, it is often necessary to fix the direction in which the line is to be produced after passing the object, as well as to ascertain the intercepted distance.

The simplest method of doing this is to choose points A and B (Fig. 1.24) 100 ft. or more apart on the chain line, and to set out right angles at these points. Along the perpendiculars measure equal lengths AA_1 and BB_1 sufficiently great to clear the object. Prolong the line A_1B_1 and set off perpendiculars at C_1 and D_1—similar points on the far side of the object. Measure along these perpendiculars distances C_1C and D_1D, each equal to A_1A and B_1B. Then the length B_1C_1 is equal to the required length BC, and CD gives the direction in which the main line is to be produced.

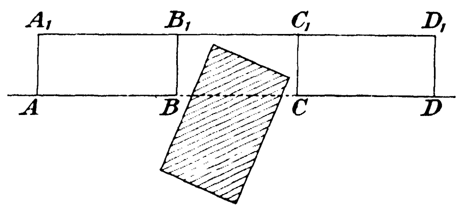


FIG. 1.24

To check the direction of the line CD, and to ensure that it is a true prolongation of AB, the method may be duplicated by running a similar set of chain lines round the opposite side of the obstacle, and taking a mean of the two results if the discrepancy is but slight.

Although many of the above methods appear very satisfactory theoretically, in practice errors are very liable to occur. In measuring

the width of a river, for instance, by the chain and tape methods, an error of 1 per cent or more may easily result unless great care is taken. It is more preferable to employ some instrumental method, as angles set out with the chain are not very reliable, especially if the chain has been adjusted. It is obvious that if some of the small connecting rings have been removed, the total length of the chain and also a few of the fractional parts may be correct, but most points will still be more or less inaccurate. If angles were set out with a chain in which the total inaccuracy was evenly divided along its whole length, no error would be introduced on this account, but since the errors are *unevenly* distributed, angles set out with an adjusted chain may be appreciably incorrect. Also the length of line defining the angle is necessarily very short, so that points at a distance located from the prolongation of this line are subject to two serious sources of error, (1) due to the initial angle being inaccurate, and (2) due to the difficulty of accurately producing the short length for any considerable distance.

To range a Line over a Fold in the Ground.—Let A and B (Fig. 1.25) be two station points, each hidden from view of the other by a fold in the ground, and let it be required to chain from A to B. The line can generally be ranged by two operators, who take up intermediate positions C and D, so that A can be seen from the more distant point D, and B can be seen from C. These first points C and D are as nearly in line with A and B as can be roughly judged, but are shown in the figure an exaggerated distance from AB for the sake of clearness. The person at C holds up his ranging rod and directs the ranging rod held by the operator at D to a new position D_1 in line with himself and B. Then D_1 directs C to C_1 in the line D_1A. Then C_1, repeating the process, directs D_1 to D_{11} in the line C_1B, etc. In this way A, C, D, and B are very quickly ranged into one line, and AB can be chained in the usual manner.

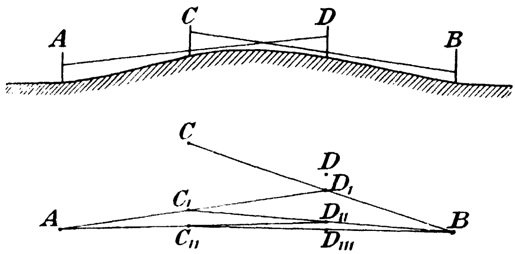


FIG. 1.25.—Ranging a Straight Line

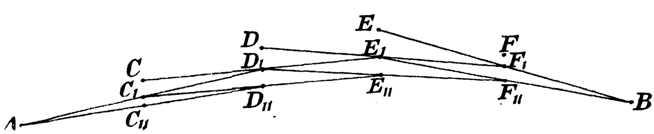


FIG. 1.26.—Ranging a Straight Line

If it is found impracticable to fix two points as above so that D can be seen from A and B from C, a similar method may be carried out with three or more rods C, D, E, F, . . . Fig. 1.26 indicates the commencement of one such solution, the notation being as in Fig. 1.25.

Inclined Measurements.—The horizontal projection of a line which

lies along sloping ground is found by one of two methods. Either (1) the slope length (l) is measured with the chain in the usual way and the inclination β of the ground taken with a clinometer, Abney level, or other instrument, when the horizontal distance is easily calculated from tables as $l \cos \beta$, or (2) the horizontal distance is measured directly in the field by the process of "stepping". In this method a portion of the chain, say 25 to 50 links, the length depending upon the steepness of the hill, is held horizontally with one extremity on the ground, while the point vertically below the other extremity is found by means of a plumb-bob and marked. The next "step" is commenced from this point and the process continued until the whole horizontal distance is measured (Fig. 1.27). Instead of using a plumb-bob, sometimes a stone or an ordinary arrow is dropped to mark the point, but this is *not* to be recommended. A special form of drop arrow is shown in Fig. 1.28. The difference between the hypotenuse and base is given on some instruments, *e.g.* on the back of the vertical circle of a theodolite, and small tables are also published. The deduction to be made from the slope length is negligible unless the angle of slope approaches 3° or more—for 3° the amount to be deducted is 0·137 link per chain.

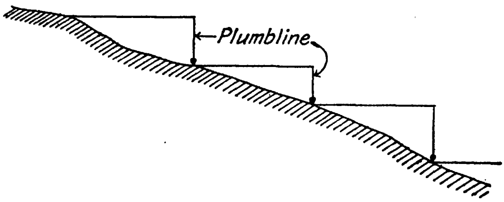


Fig. 1.27.—Stepping

If the ground is uniformly sloping, it is probably more accurate and expeditious to measure the angle, and chain along the slope as in the first method, the horizontal distance being deduced afterwards, or the difference may be allowed for at the end of each chain length as the work proceeds; but very often the slope is not uniform and stepping is probably then to be preferred.

Plotting the Survey.—The skeleton framework of main chain lines is plotted by selecting and setting off on the paper one of the longest lines as a base, then plotting the various triangles, etc., from this by means of beam or other compasses.



Fig. 1.28 Drop Arrow

The choice of a suitable base can best be determined by an inspection of the small key plan; and if this is not already given in the field book, it should be compiled from the notes before commencing the proper plan.

The accuracy of the framework is verified by means of the tie or check lines taken for this purpose in the field, and if any large error is detected, its whereabouts on the survey can probably be located approximately, when that portion of the work must be rechained unless sufficient data have been booked to obviate this.

(The error "allowable" in ordinary chain surveys is mentioned later.)

It is advisable to ink in the framework lines in a faint red colour, or at least so to mark the station points and a small piece at the extremity of each chain line that they can be quickly ruled in again if required.

The offsets are readily plotted in pencil by means of an offset scale, which is a short rule 2″ or 3″ long, sliding along the ordinary scale.

The plan is next inked in, coloured, and the title, etc., printed, a scale is drawn, and also a "North Point": the magnetic north is often indicated as well as the true north.

If convenient the plan should be oriented, *i.e.* the side margins should lie due N. and S., the north being at the top of the plan.

Conventional Signs.—Some of the conventional signs used for delineating various objects upon plans are shown in Fig. 1.29, p. 18. Others are described in the keys on the various Ordnance plans and maps.

Plans and Maps

Scales.—The scale of a plan or map is expressed (i) as x units of length on the ground to 1 different unit of length on the paper, *e.g.* x feet to 1 inch, (ii) as y units of length on the paper to 1 different unit of length on the ground, *e.g.* 6 inches to the mile, and (iii) as a representative fraction (R.F.) which is the ratio between a length of 1 unit on the paper to z similar units on the ground, *e.g.* 1/500, signifying 1 ft. on the paper represents 500 ft. on the ground, or 1 cm. on the paper represents 500 cms. of the ground, etc.

The choice of a suitable scale depends upon the amount of detail required to be shown on the plan, and this again influences the choice of survey technique to be employed in the field. Other factors, such as convenience in size for handling, need to be considered also. The following are some of the more usual scales employed by British surveyors and engineers, the choice being very wide:

Chain Scales.—1, 2, 3, 4, 5, 6, 8, or 10 chains to 1 inch.

Foot Scales.—10, 20, 30, 40, 50, 80, 100, 200, 400; 33, 44, 88 ft. to 1 inch. Details may be drawn to $\frac{1}{8}''$, $\frac{1}{4}''$, $\frac{3}{8}''$, $\frac{1}{2}''$, $\frac{3}{4}''$, $1''$ or $1\frac{1}{2}''$ to 1 foot.

Some scales, usually made of boxwood, though other materials are used, are divided along one edge into 10, 20, 30, or more divisions to one inch, and can be used to plot a number either of chains or of feet to 1 inch. If these divisions are taken to represent chains and links, the opposite edge may be marked "feet equal" and be so graduated that the divisions represent feet *to the same scale.* Thus, if a survey be made with a Gunter's chain, and plotted in chains and links with one edge of the rule, distances in feet may be scaled directly from the plan, if required, with the opposite edge.

Ordnance Scales, etc.—The dividing line between plans and maps is not very marked. The Ordnance Survey take it that plans include all drawn to a scale of 6 inches to 1 mile, or larger scale (*i.e.* with R.F. larger than 1/10,560).

The Ordnance large-scale plan includes the well-known town plans

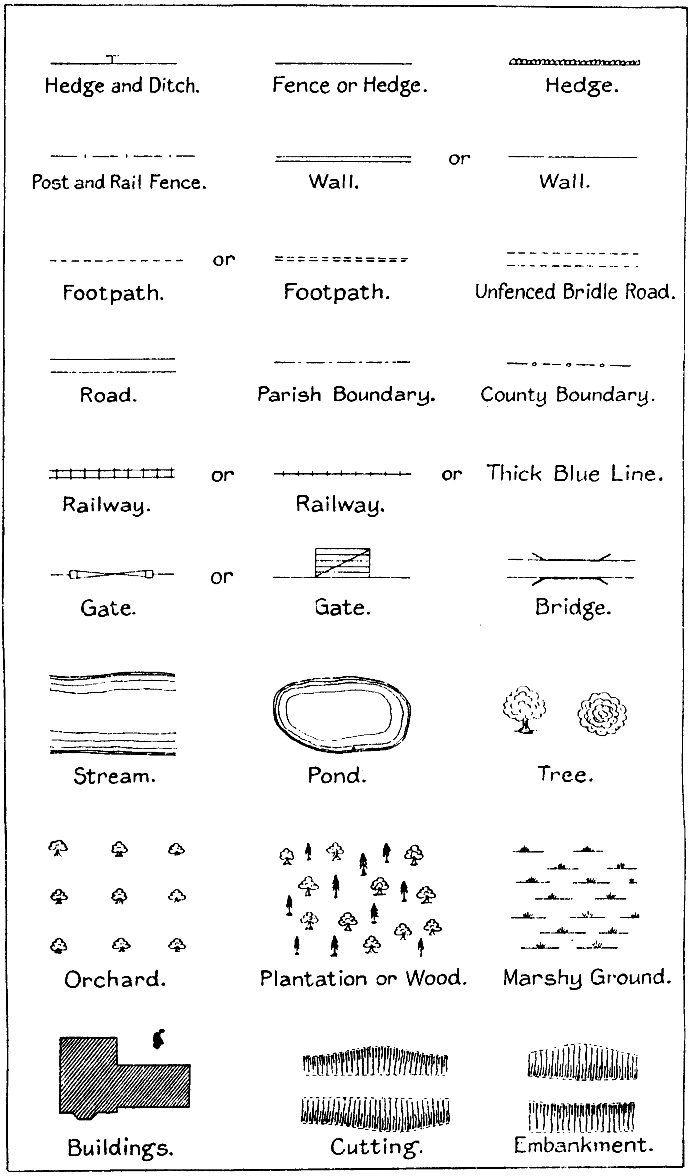


FIG. 1.29.—Conventional Signs

Example.—The following is an example of a simple chain survey, and Fig. 1.30 shows the plan plotted from the field notes on pages 20, 21, and 22. The student is recommended to plot this himself to a scale of 1 or 2 chains to 1 inch.

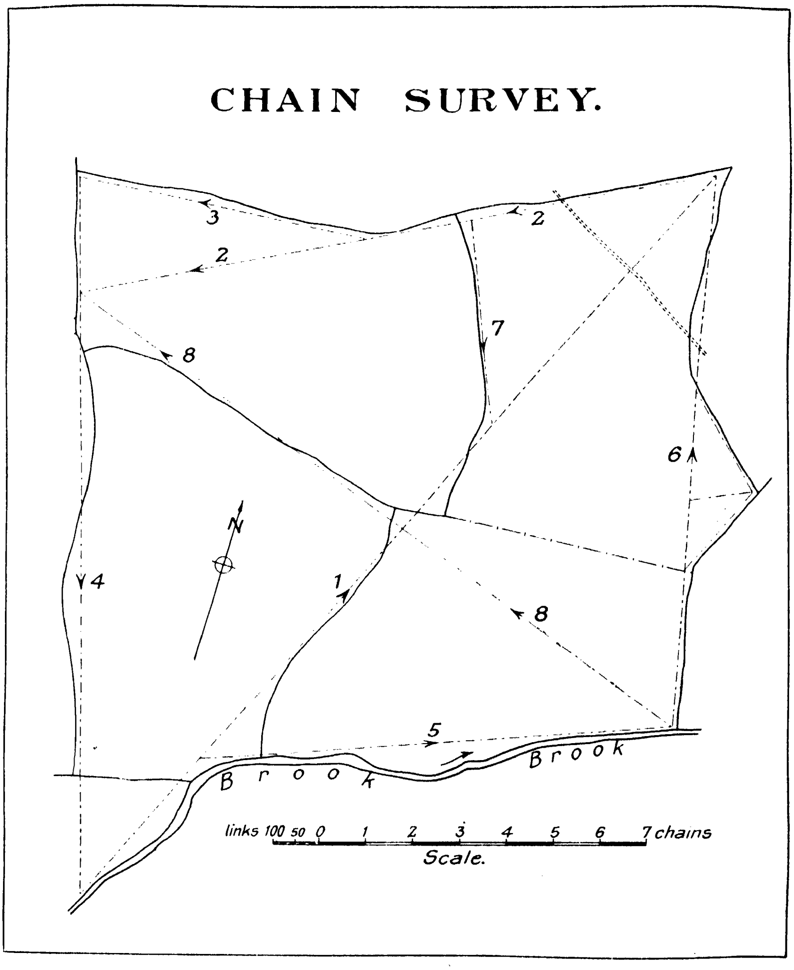
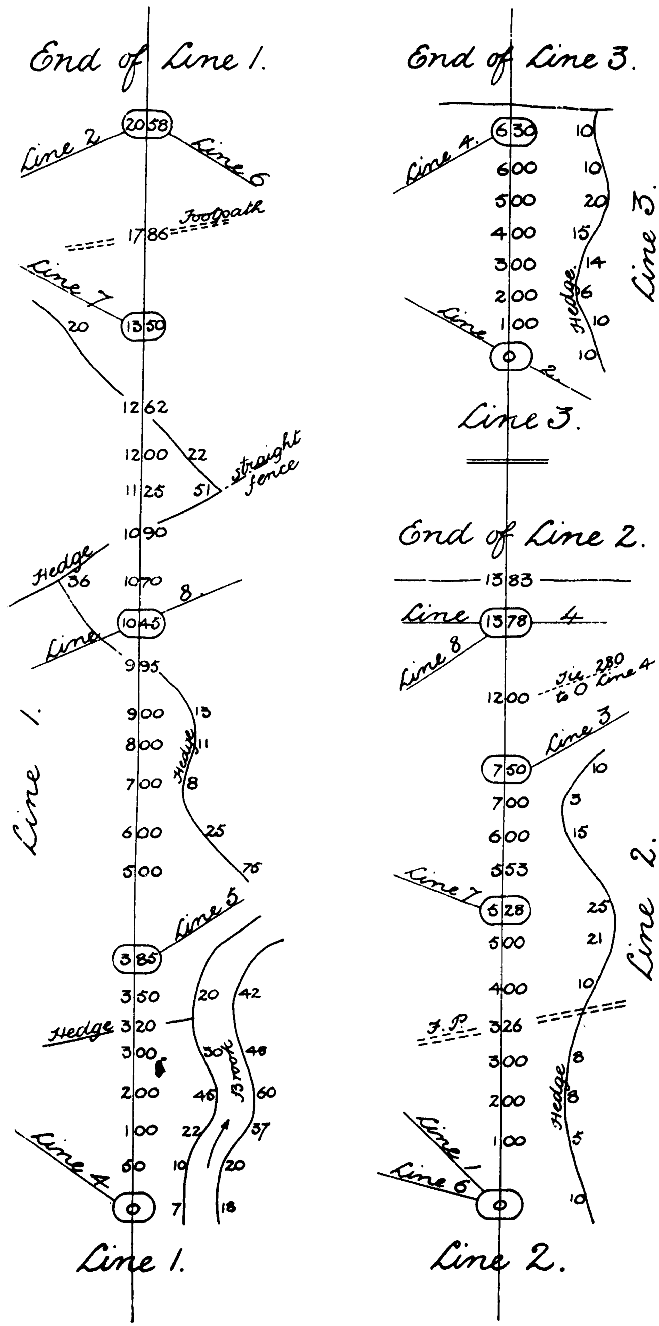
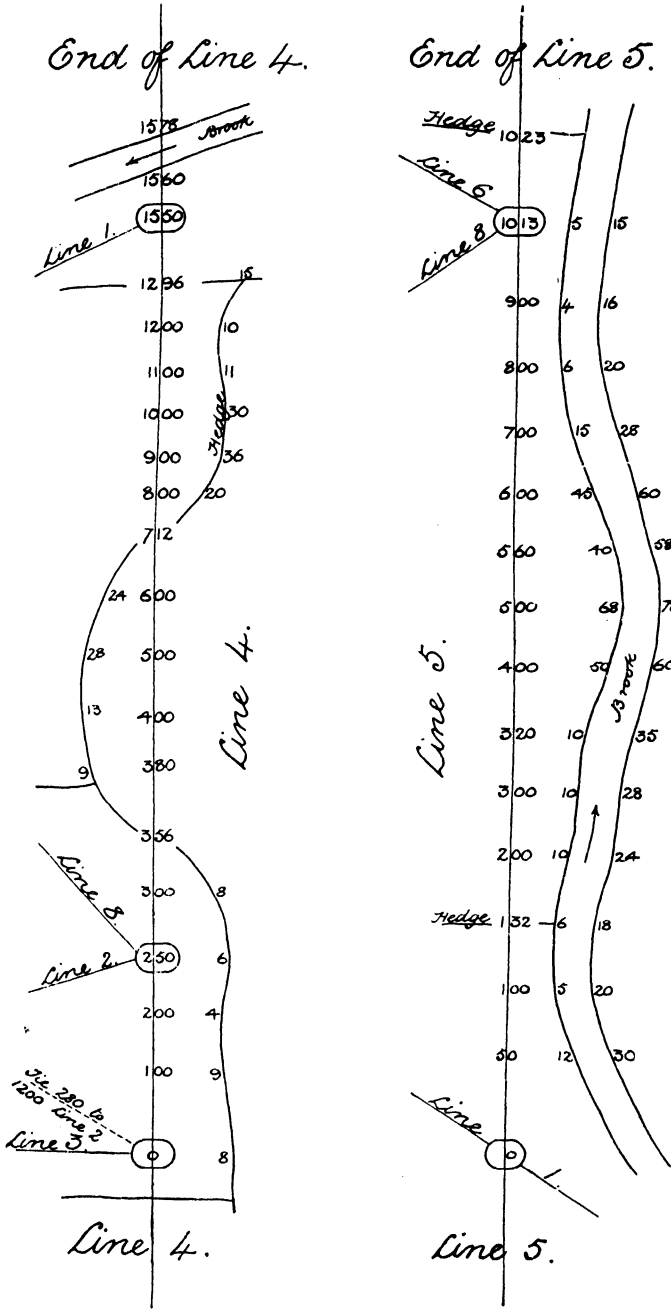
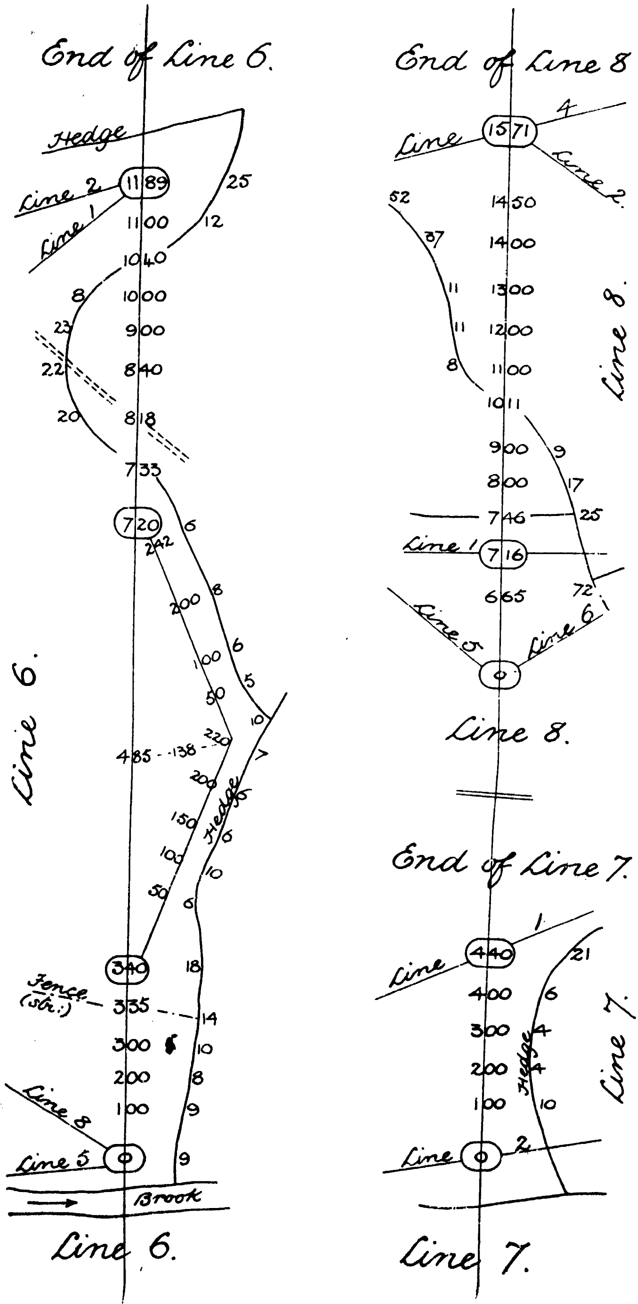


FIG. 1.30


End of Line 1.
2058
Line 2
Line 6
Footpath
1786
Line 7
20
1350
1262
1200 22
straight fence
1125 51
1090
Hedge
36
1070
8.
1045
Line
995
900 13
800 11
Hedge
700 8
600 25
500 76
Line 1.
Line 5
385
350 20 42
Hedge 320
300 30 48
Brook
200 46 60
100 22 37
50 10 20
Line 4
0 7 18
Line 1.
End of Line 3.
Line 4.
630 10
600 10
500 20
400 15
300 14
200 6
100 10
Hedge
Line
0 10
Line 5.
Line 3.
End of Line 2.
1383
Line 1378 4
Line 8
1200 Tie 280 to 0 Line 4
Line 3
750 10
700 3
600 15
553
Line 7
528 25
500 21
400 10
F.P. 326
300 8
200 8
100 5
Hedge
Line 1
Line 6
0 10
Line 2.
Line 2.


End of Line 4.
1578
Brook
1560
1550
Line 1.
1296
15
1200
10
1100
11
1000
30
Hedge
900
36
800
20
712
24
600
28
500
13
400
380
9
Line 4.
356
Line 8.
300
8
Line 2.
250
6
200
4
100
9
Tie 280 to 1200 Line 2
Line 3.
0
8
Line 4.
End of Line 5.
Hedge
1023
Line 6
Line 8
1013
5
15
900
4
16
800
6
20
700
15
28
600
45
60
560
40
58
500
68
78
400
50
60
Brook
Line 5.
320
10
35
300
10
28
200
10
24
Hedge
132
6
18
100
5
20
50
12
30
Line
0
1.
Line 5.


End of Line 6.
Hedge
Line 2
Line 1
11 89
25
11 00
12
10 40
8
10 00
23
9 00
22
8 40
20
8 18
7 33
7 20
6
242
8
200
6
100
5
50
10
220
485 - 138
200
Hedge
150
6
100
10
50
6
3 40
18
Fence (str.)
3 35
14
3 00
10
2 00
8
1 00
9
Line 8
Line 5
0
9
Brook
Line 6.
Line 6.
End of Line 8
4
Line
15 71
Line 2.
52
14 50
37
14 00
11
13 00
Line 8.
11
12 00
8
11 00
10 11
9 00
9
8 00
17
7 46
25
Line 1
7 16
6 65
72
Line 5
Line 6
0
Line 8.
End of Line 7.
1
Line
4 40
21
4 00
6
3 00
4
Hedge
2 00
4
1 00
10
Line 7.
Line
0
2
Line 7.

to a scale of 10·56 ft. to 1 mile, or 41·66 ft. to 1 inch, *i.e.* 1/500. There are also a few at 10 ft. to 1 mile, or 1/528 ; and a few at 5 ft. to 1 mile, or 1/1056, are also published.

The 25-inch plans, actually 25·344 inches to 1 mile, or 1/2500, are the parish plans. They show frequent spot levels along the roads, but no contours, and 1 square inch approximately represents 1 acre. Each field or " parcel " has a serial number, and the area, given in acres, is that enclosed by the material boundaries which are not necessarily the true boundaries of the property (see p. 10).

The 6-inch, or county plans, show contour lines and ordnance level marks.

Of the maps, mention may be made of the 10 miles to the inch, and the $\frac{1}{4}''$, $\frac{1}{2}''$, and 1″ to the mile series, the R.F.'s being 1/633,600 ; 1/253,440 ; 1/126,720 ; 1/63,360 respectively.

The Ordnance Survey in 1913 published two sheets covering Great Britain to a scale of one-in-a-million, 1/M (1/1,000,000 or 15·782 miles to 1 inch), and this scale has been adopted for an *International* 1/M series of maps. A number of sheets, each covering 6° of longitude × 4° of latitude, have been prepared by various Governments.

Foreign Scales.—Various parts of the world have been mapped to scales smaller than 1/M, *e.g.* 1/2M, 1/3M, 1/4M, 1/5M, 1/6M, 1/10M, 1/15M, 1/16M. Larger scale maps of foreign countries include 1/10,000 ; 1/20,000 ; 1/25,000 ; 1/50,000 ; 1/80,000 ; 1/100,000 ; 1/200,000 ; 1/250,000 ; 1/300,000 ; 1/500,000 ; and British Military maps are consequently sometimes also reproduced to one or other of these scales.

Map References.—It may be convenient here to mention briefly methods adopted for locating positions upon maps. Readers will be familiar with the system employed in many guide-books, etc., of dividing the map up into rectangles, two opposite edges of the map being numbered and the other two lettered. Any rectangle can then be identified by its reference number and letter, and the point referred to sought within it. As a refinement, a point within the rectangle can be defined by coordinates taking the left lower corner as origin and imagining the sides of the rectangle divided into 10 or 100 parts as desired.

Other systems, varying in detail, have been developed for application to maps covering a number of sheets—or even to a series of maps of different scales covering the same area on the ground. The lines forming the rectangles (usually squares) comprise a " grid ". The vertical lines of the grid are directed to " Grid North ", and usually that at the centre of the system would be arranged to coincide with the meridian at that place, so that along this line Grid North would coincide with True North.

Away from the centre, the grid lines would usually diverge from the meridians, the amount depending upon the " projection " of the map (see p. 406). The deviation on any particular sheet, either at the centre or at the edges, might be marked on the sheet—or, if the meridians were shown, the difference between Grid North and True North could be measured directly from the map. If the relationship between

True North and Magnetic North were known (see p. 60), the relationship between Grid North and the Magnetic North could readily be determined.

Some systems that have been used on British Maps are described in the War Office publication *Manual of Map Reading, Photo Reading and Field Sketching*. Other information on these (and on many other points of interest concerning maps) may be found in a small book, *A Key to Maps*, by Brigadier H. S. L. Winterbotham, sometime Director-General of the Ordnance Survey.

On pre-war 1-inch and smaller scale maps the grid was in yard-units: *e.g.* the sides of the squares on the 1-inch map were each 5000 yds., on the ½-inch map 10,000 yds., and on the 3-inch map 1000 yds. The origin of the system of co-ordinates was arbitrarily chosen 1,000,000 yards south and 1,000,000 yards west of a point used as an origin in connection with the survey itself (2° W., 49° N.).

For post-war maps of this country a *National Projection*, which is a slightly modified Transverse Mercator projection, has been adopted by the Ordnance Survey. It is based on the new Primary Triangulation, and the origin chosen for it is 2° W. longitude 49° N. latitude.

A *National Grid*[1] has also been adopted for all the large-scale plans and all the small-scale maps (with certain exceptions), and this will provide one reference system for all the maps of the whole of Britain. The origin of co-ordinates is a point arbitrarily chosen a little to the south and to the west of Land's End, 400 Km. west and 100 Km. north of the survey origin referred to above. This ensures that the co-ordinates of all points on the maps will be positive quantities, *i.e.* E. and N. of the origin, and no co-ordinate will exceed 1000 Km. for points on the mainland.

The adoption of the international metre in lieu of the yard previously employed, has several advantages, since various units differing by factors of 10 can be readily used for different purposes and for maps and plans of different scales. The spacings of the grid lines vary according to the scale of the map—*e.g.* on the 1-inch, 1/25,000, and 6-inch maps the spacings represent 1 Km. intervals: on the 1/2500 and 1/1250 scale plans the intervals are 100 metres: on the 10-mile and ¼-inch maps, 10 Km. These grid lines are appropriately numbered in the margins of the maps, the same system being used throughout.

The grid itself, to include the whole of Great Britain, consists of 70 squares—each 100 Km. × 100 Km.—extending 1000 Km. North and 700 Km. East from the origin. Each square can be denoted by the co-ordinates in hundreds of Kms. to its South-West corner. Thus the square within which London is situated is square 51, and Cape Wrath is in square 29.

The *Normal National Grid Reference* gives the co-ordinates of points within such a square to the nearest 100 metre or 0·1 Km. Each co-ordinate thus consists of three figures, and the whole is a six-figure reference. For example, a point in square 51 whose Normal National Grid Reference was 046287 would lie 4,600 metres to the East and 28,700 metres to the North of the S.W. corner of square 51. These

[1] See Ordnance Survey Booklet No. 1/45, *A Brief Description of the National Grid and Reference System.*

same six figures would denote a point similarly situated in any of the 70 squares, so that if there were any doubt as to the square referred to, the *Full National Grid Reference* should be used, and this is written as 51/046287. Note that the same number of figures must be used for each co-ordinate and that the Eastings always precede the Northings. The figures just quoted should *not* be written as 51/46287, as this would be quite ambiguous.

The location of points to the nearest 1 Km. would be given by a Four-Figure Reference or *Normal Kilometre Reference—e.g.* 0428 or *Full Kilometre Reference* 51/0428.

The 1/2500-scale plans are now published in sheets each 1 Km. square, and bounded by grid lines. The S.W. corner reference is thus given by a four-figure reference number, *e.g.* 51/0428, and this suffices to define the sheet. For the 1/1250 scale the same area occupies four sheets distinguished by the letters N.W., N.E., S.W., S.E.—so that the point 51/046287 would lie in sheet 51/0428 N.E., and the sheet is numbered in this way. To define on the large-scale plans the position of a point in the 1-Km. square to the nearest 10 metres, a further four figures would be required, or to the nearest 1 metre, a further six figures—*e.g.* 51/0428/6271 or 51/0428/625714. These respectively give the *Full Ten-Metre Reference* and the *Full One-Metre Reference.* Usually, however, the plan number, *i.e.* 51/0428, can be omitted without ambiguity, and only the last four or six figures need be used to give the *Ten-Metre Reference* or the *One-Metre Reference* respectively.

To recapitulate, it may be recalled that the co-ordinates of the last point referred to would, in other systems, be written E.504625 m., N.128714 m.

Copying, Reducing, and Enlarging.—If a copy of a plan is required, one of the following methods may be employed, to obviate the necessity of plotting the whole afresh from the field notes.

(i) Place over the plan a piece of tracing paper or tracing linen, and trace on to this in ink all the details required. When using linen the drawing is usually done upon the dull, and the colouring upon the glazed side. The colour and the ink will be found to flow more evenly if a little ox-gall is added, or if the surface of the cloth is rubbed over with powdered chalk before commencing operations.

(ii) A copy may be made upon ordinary drawing paper by using a sheet of transfer paper. This is simply a sheet of thin paper such as tracing paper, one side of which has been rubbed with black lead or with a soft pencil, and it is pinned down with the black side in contact with the sheet on which the plan is to be copied. Above it is pinned either the original plan or a tracing, and the outline of this is traced over with a hard pencil. An imprint will thus be made on the lower sheet, which can be inked in and a finished drawing prepared.

(iii) An uncoloured tracing having been prepared, any number of copies can readily be printed upon sensitized paper, linen, or mounted paper, giving an almost black or a blue line upon a white ground. A white line upon a blue ground is not used for survey plans, but chiefly for mechanical drawings.

The plan is printed on the prepared surface, like an ordinary

photograph, by placing it behind the tracing in a frame and exposing to sunlight or to an electric arc for a definite time. Generally the print is "developed" in a clean water bath, dried, and coloured in the usual way.

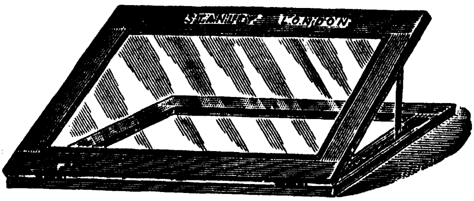


Fig. 1.31.—Glass Tracing Table

(iv) A copy may be made by pricking through a number of points from the original drawing, inking in, lettering, and colouring.

(v) By placing the sheet of drawing paper over the original plan laid on a plate-glass tracing board or table (Fig. 1.31) beneath which is a reflecting mirror, a copy can be made by tracing without such injury to the plan as would be caused by pricking.

(vi) Lithography is employed when a large number of copies is required. The plan is transferred from a tracing in lithographic ink on to a stone, from which the copies are printed off. There is a certain amount of shrinkage due to the drying, so care should be taken that the scale is drawn upon the tracing before being transferred.

In all the above methods the copy is to the same scale as the original; by the following methods copies may be obtained to the same scale, or to either a reduced or an enlarged scale.

(vii) There are several modern methods of reproducing drawings involving photographic techniques, and carried out by specialist firms. They permit of the drawings being reduced, or enlarged, or copied to the same scale, and transparencies need not be prepared. Photostats or other prints may be in the form of negatives with white lines on a dark background, and for certain purposes are attractive, but for most survey purposes, dark lines on a white background are preferred.

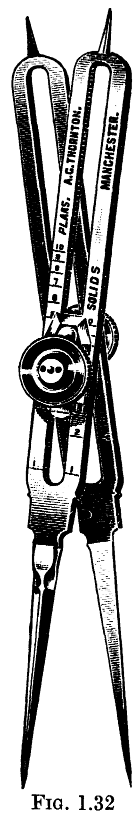


Fig. 1.32 Proportional Compasses

(viii) By the use of proportional compasses (Fig. 1.32) a plan may be copied to almost any scale. Distances are measured from the plan by means of the points at one end of the instrument, and are plotted on the copy with the other pair of points. The ratio of these distances is fixed by the slide and is indicated by the scale marked on the face of the instrument.

(ix) Draw lightly in pencil on the original plan, or on a sheet of tracing paper covering it, a network of squares as in Fig. 1.33. If the survey is to be copied to a scale n times the original scale (n being $>$ = or $<$ unity), draw on the new sheet a similar network of squares having their linear dimensions $\frac{1}{n}$ times

those of the first set. *E.g.* if the original plan is plotted to 2 chains to 1 inch scale, and the copy is required to 3 chains to 1 inch scale, $n = \frac{3}{2}$ and the linear dimensions of the squares will be reduced in the ratio of $\frac{2}{3}$.

Note where the lines on the plan cut the sides of the squares and

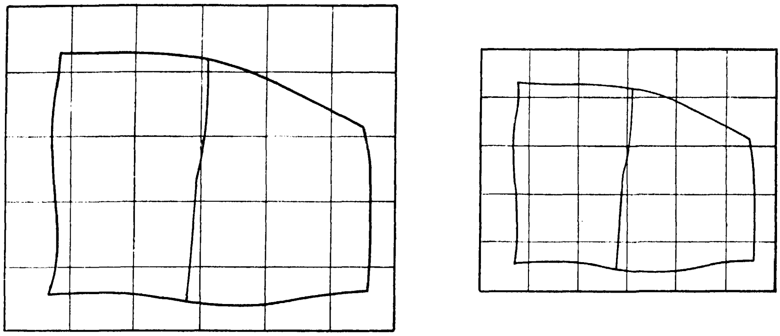


FIG. 1.33.—Method of reducing or enlarging Drawings

mark the corresponding points on the copy, sketching in other detail by the eye.

(x) The pentagraph or pantagraph is chiefly used for making reduced copies of drawings. As shown by Stanley in Fig. 1.34, it consists of a light brass framework of tubular construction, mounted on castors, and freely hinged at the joints; it is provided with a heavy block B as a fulcrum about which the instrument moves, and with two tracing points. The outer of these, C, is moved over the lines of the original drawing while a pencil at the other point D marks out the copy. This pencil may be raised off the paper by means of the light cord passing to the outer tracing point. The position of the fulcrum and that of the middle tracing point may be moved along scales marked on the framework to give the requisite reduction,

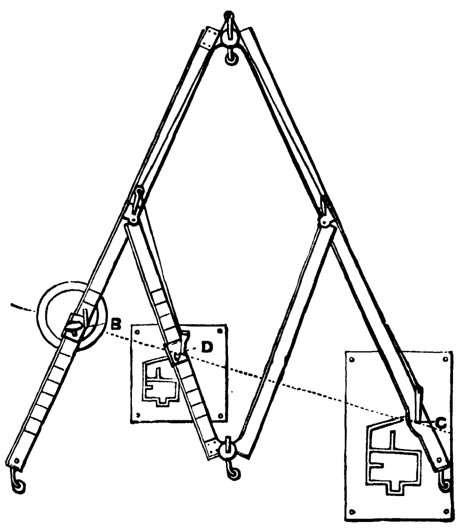


FIG. 1.34.—Pantagraph

and may be interchanged. Let F (Fig. 1.35, *a*) denote the fulcrum, P_1 the pencil, and P_2 the tracer. Then these three points must be in one straight line when the instrument is set.

The frame *abcd*, having the lengths of its sides fixed, always remains

a parallelogram, *i.e.* dP_1 is parallel to aP_2; dF and aF are in one line and the angle P_1dF is equal to the angle P_2aF, so that if $P_1d : dF :: P_2a : aF$, then triangles P_1dF and P_2aF are similar and P_2F and P_1F lie in the same straight line. Hence when the length Fd is fixed and P_1 set

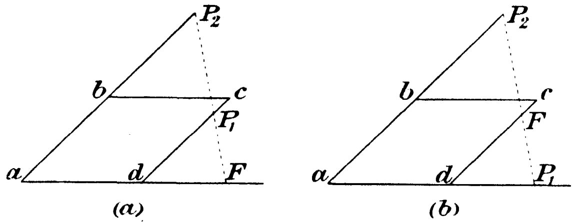


FIG. 1.35.—Pantagraph

in its proper relative position on dc, the ratio of FP_1 to FP_2 is a constant, being equal to $\frac{Fd}{Fa}$.

Any movement of P_2 through a distance x produces a corresponding movement of P_1 though $\frac{Fd}{Fa} \cdot x$, and consequently the linear reduction in size of the copy is $\frac{Fd}{Fa}$.

With this setting of the instrument the greatest value of this fraction is $\frac{1}{2}$, which means the reduction must be *at least* $\frac{1}{2}$ the linear dimensions.

If F and P_1 are interchanged (Fig. 1.35, *b*), $\frac{FP_1}{FP_2} = \frac{P_1d}{da}$ which is constant as before, and this fixes the ratio which the size of the copy bears to the original. It may have any value less than or be equal to

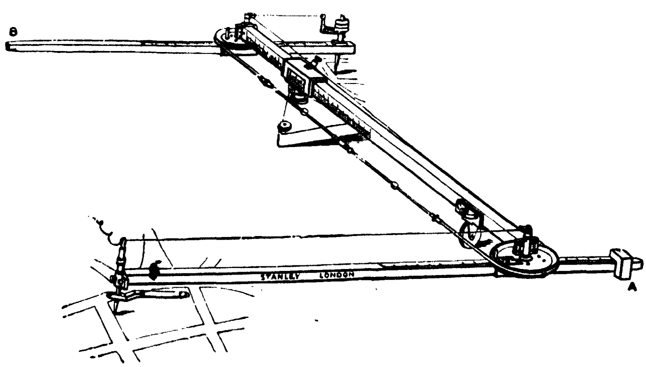


FIG. 1.36.—The Eidograph

unity. To make an enlarged copy, P_1 and P_2 would be interchanged in either of the above arrangements.

(xi) The eidograph, though more expensive, is a more reliable instrument than the pantagraph, and it can be used to reduce or

enlarge drawings in *any* proportion. The pantagraph is more or less limited to the specific proportions shown by the markings, unless the required positions are found by trial.

By reference to Fig. 1.36 the eidograph will be seen to consist of three bars of tubular construction, each graduated as shown. At the ends of the centre bar are pulley wheels, on the underside of which are boxes through which the outer bars may slide until adjusted and clamped. The fulcrum may also be moved along the centre bar. Connecting the two pulleys is an adjustable steel band which ensures that the two cross-bars always remain parallel to each other.

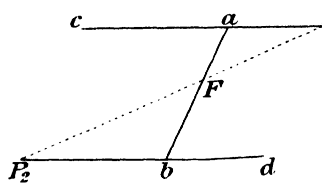


FIG. 1.37.—The Eidograph

To set the instrument, the bars are moved through the slides until P_1c, P_2d, and ab are divided in the same ratio (Fig. 1.37), *i.e.*

$$\frac{P_1a}{ac} = \frac{db}{bP_2} = \frac{aF}{Fb},$$

and as P_1c and dP_2 are equal,

$$\frac{P_1a}{aF} = \frac{P_2b}{bF};$$

also as P_1c and P_2d always remain parallel, the angles P_1aF and P_2bF are equal, and consequently the triangles P_1aF, P_2bF similar. P_2, F, and P_1 are thus in line as in the pantagraph, $\frac{FP_1}{FP_2}$ is constant for all movements of the instrument, and, as before, this gives the value of the reduction. If each of the bars is graduated outwards in 100 divisions from the centre, and the scale is required to be increased from n ft. to 1 inch to m ft. to 1 inch, *i.e.* the linear dimensions on the plan are to be reduced in the ratio of n to m, then if x be the number of divisions each point P_2, F, and P_1 is to be moved from the centre of the bar,

$$\frac{n}{m} = \frac{FP_1}{FP_2} = \frac{aF}{bF} = \frac{100 - x}{100 + x},$$

and

$$x = \frac{100\,(m - n)}{m + n},$$

e.g. to reduce from a scale of 1 chain to 1 inch to 80 ft. to 1 inch

$$x = \frac{100\,(80 - 66)}{80 + 66} = \frac{1400}{146} = 9{\cdot}59 \text{ divisions on each bar.}$$

Calculation of Areas

Areas.—The chief methods of determining areas will now be described. In some cases, it will be noted, it is necessary to have previously plotted the plan to scale, while in other cases the calculations can be made direct from the field notes.

1. If the surface whose area is required is bounded by straight lines, the enclosed area may be found by dividing the figure into a

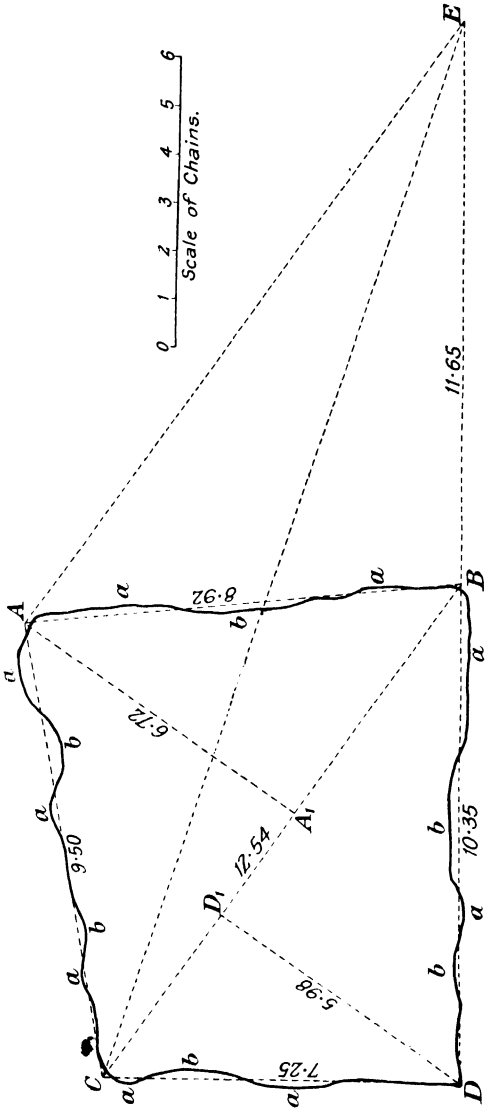


FIG. 1.38.—Calculation of Areas

number of triangles and finding the area of each of these in turn by the well-known formula

$$\text{Area} = \tfrac{1}{2}bh, \quad . \quad . \quad . \quad (1)$$

where b is the length of any side, and h is the perpendicular distance of the opposite vertex from this side. Generally the triangles can be taken in pairs so that one base serves for two triangles and the calcula-

tions are simplified. If the sides of the figure are not straight, by using the edge of a transparent celluloid set square or a black thread stretched across the paper "give and take" lines may be drawn to average these, and the uneven boundaries replaced by straight lines. An example is given in Fig. 1.38, where the dotted lines are drawn to include as nearly as possible the same area as the original boundaries, *i.e.* the sum of the small pieces marked *a* should be equal to the sum of those marked *b*.

The length BC and the perpendicular distances AA_1 and DD_1 are scaled from the plan as 12·54, 6·72, and 5·98 chains respectively. The area is then $\frac{1}{2}BC \,.\, AA_1 + \frac{1}{2}BC \,.\, DD_1 = \frac{1}{2}BC \,.\, (AA_1 + DD_1) = \frac{1}{2} \times 12{\cdot}54$ $(6{\cdot}72 + 5{\cdot}98) = 79{\cdot}63$ sq. chains or 7·963 acres about.

2. Another formula[1] which is sometimes useful for determining the area of a triangle when the sides and angles are known is

$$\text{Area} = \tfrac{1}{2}bc \sin A, \qquad . \quad . \quad . \quad (2)$$

where *b* and *c* are any two sides and A is the angle included between them.

Applying this to Fig. 1.38, if $\angle A = 85°\text{-}46'$, $\angle D = 89°\text{-}03'$, AB = 8·92 chains, BD = 10·35 chains, DC = 7·25 chains, CA = 9·50 chains, then the area of the triangle $ABC = \frac{1}{2} \times 9{\cdot}50 \times 8{\cdot}92 \times \sin 85°\text{-}46'$ sq. chains.

$$\begin{aligned}
\log \tfrac{1}{2} &= \bar{1}{\cdot}69897 \\
\log 9{\cdot}50 &= \;\;{\cdot}97772 \\
\log 8{\cdot}92 &= \;\;{\cdot}95036 \\
\log \sin 85°\text{-}46' &= \bar{1}{\cdot}99881 \\
\hline
\log \text{area} &= 1{\cdot}62586
\end{aligned}$$

area = 42·253 sq. chains or 4·225 acres.

Similarly the area of the triangle BDC = 37·514 sq. chains or 3·752 acres, and total area = 7·977 acres about.

3. Another formula[1] which may be used when the three sides of each triangle are known is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}, \qquad . \quad . \quad (3)$$

where *a*, *b*, and *c* are the lengths of the sides and $s = \dfrac{a+b+c}{2}$.

Applying this to Fig. 1.38, in the triangle ABC $a = 12{\cdot}54$, $b = 9{\cdot}50$, $c = 8{\cdot}92$,

$$\begin{aligned}
\text{and } s &= 15{\cdot}48 & \log s &= 1{\cdot}18977 \\
\therefore s-a &= 2{\cdot}94 & \log s-a &= {\cdot}46835 \\
s-b &= 5{\cdot}98 & \log s-b &= {\cdot}77670 \\
s-c &= 6{\cdot}56 & \log s-c &= {\cdot}81690 \\
& & &\;2)\overline{3{\cdot}25172} \\
& & \log \text{area} &= 1{\cdot}62586
\end{aligned}$$

and area of triangle ABC = 42·253 sq. chains.
Similarly area of triangle BCD = 37·514 sq. chains,
and total area as before = 79·767 sq. chains,
or 7·977 acres.

[1] Any text-book on Elementary Trigonometry.

4. A plane figure bounded by a number of straight lines as ABCDE may be reduced geometrically to a single triangle as shown in Fig. 1.39.

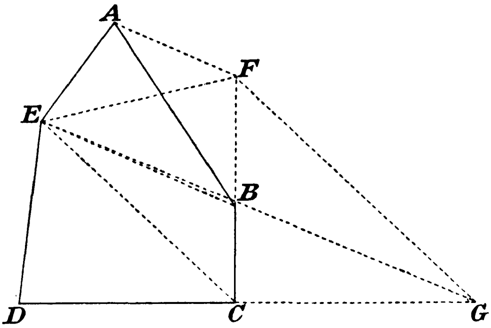


FIG. 1.39

Join BE, and draw a parallel line AF through A. Produce CB to cut this in F and join FE. Then the triangle EFB is equal in area to the triangle EAB because they are on the same base EB and between the same parallels, so that the whole figure EFCD is equal to the original figure ABCDE, and the sides are reduced by one in number. Similarly join EC, draw a parallel line through F, and produce DC to cut this in G. Join EG.

As before, the figure GED is equal to the figure EFCD, which is equal to ABCDE. Proceeding in this way, any such figure may be reduced to a single triangle whose area can be deduced as already explained.

By the application of this method to Fig. 1.38, ABDC is reduced to the equivalent triangle ECD, from which by scaling ED is equal to 22·00, and the perpendicular distance from C = 7·25, and the area by method (1) = 7·975 acres.

5. In the case of a large survey the area enclosed by the survey framework lines may be calculated by subdivision into triangles, but the long narrow strips of ground between these straight framework lines and the irregular boundaries may best be calculated by means of ordinates from the survey line.

In a general case in which neither boundary is straight a base line

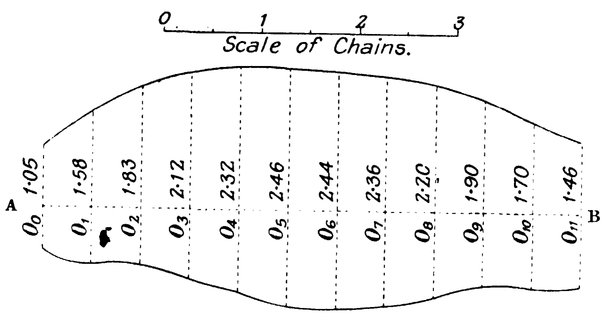


FIG. 1.40.—Calculation of Areas

is taken through the area as in Fig. 1.40 and divided into a number (n) of equal parts. The ordinates at each of the division points are scaled from the plan or measured direct on the ground, as, for instance,

$$O_0\ O_1\ O_2\ O_3\ .\ .\ .\ O_n.$$

The area may then be calculated approximately by taking an average of these ordinates and multiplying by the length AB ($=l$ say),

$$i.e.\ \text{Area} = \frac{O_0 + O_1 + O_2 + O_3 + \ldots O_n}{n+1} \times l. \quad . \qquad . \quad (4)$$

The area of the plot shown in Fig. 1.40 by this formula is found to be $\frac{23\cdot42}{12} \times 5\cdot5 = 10\cdot73$ sq. chains or 1·073 acres.

6. A more accurate formula is that given by the "Trapezoidal Rule", which assumes that the boundaries between the extremities of the ordinates are straight lines.

$$\therefore \text{ the area of the first division} = \left(\frac{O_0 + O_1}{2}\right)\frac{l}{n},$$

$$\text{the area of the second division} = \left(\frac{O_1 + O_2}{2}\right)\frac{l}{n},$$

. . . . . .

$$\text{the area of the last division} = \left(\frac{O_{n-1} + O_n}{2}\right)\frac{l}{n}.$$

By summing these it is seen that the total area

$$= \frac{O_0 + 2O_1 + 2O_2 + \ldots 2O_{n-1} + O_n}{2} \cdot \frac{l}{n}, \quad . \quad (5)$$

and the area of Fig. 1.40, by the use of this formula, is $\frac{44\cdot33}{2} \times \frac{5\cdot5}{11}$ $= 11\cdot08$ sq. chains or 1·108 acres.

7. Simpson's Rule assumes that the boundaries being curved may be considered as portions of parabolic arcs, and consequently the rule is sometimes known as the "Parabolic Rule".

Thus let aa_1, bb_1, cc_1 (Fig. 1.41) be any three consecutive co-ordinates from the straight line to the curved boundary, and at equal distances d apart. Join a_1c_1 cutting bb_1 in b_2.

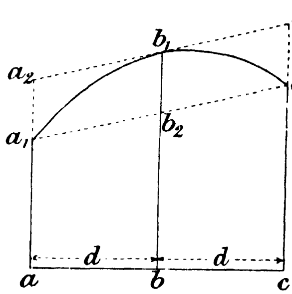


FIG. 1.41.—Simpson's Rule

Draw $a_2b_1c_2$ through b_1 parallel to a_1c_1.

Then the area required $aa_1b_1c_1cba$ may be considered as composed of the trapezoid $aa_1b_2c_1ca$, together with the segment $a_1b_1c_1b_2a_1$ under the curve.

But if the arc $a_1b_1c_1$ is assumed to be a parabolic arc having its axis parallel to the ordinates, then the area $a_1b_1c_1b_2a_1$ is equal to two-thirds of the circumscribing parallelogram $a_1a_2b_1c_2c_1a_1$,

$\therefore$ area of figure between ordinates a and c $= (aa_1b_2c_1c) + \frac{2}{3}(a_1a_2b_1c_2c_1)$

$$= \frac{aa_1 + cc_1}{2} \cdot 2d + \frac{2}{3}\left(bb_1 - \frac{aa_1 + cc_1}{2}\right) 2d$$

$$= \frac{d}{3}\left\{aa_1 + 4bb_1 + cc_1\right\}.$$

This proof holds for any three consecutive co-ordinates on either side of the base line, so that the area included between

$$O_0 \text{ and } O_2 \text{ is } \frac{d}{3}(O_0 + 4O_1 + O_2),$$

$$\text{between } O_2 \text{ and } O_4 \text{ the area is } \frac{d}{3}(O_2 + 4O_3 + O_4),$$

. . . .

$$\text{between } O_{n-2} \text{ and } O_n \text{ the area is } \frac{d}{3}(O_{n-2} + 4O_{n-1} + O_n).$$

Adding these, the total area is

$$\frac{d}{3}(O_0 + 4O_1 + 2O_2 + 4O_3 + \ldots 2O_{n-2} + 4O_{n-1} + O_n). \qquad (6)$$

It must, however, be borne in mind that this total is made up by considering the partial areas in pairs, so that to use this formula there must be an *even* number of divisions and hence an *odd* number of ordinates. If an odd number of divisions has been taken, the area of one of these must be calculated separately and added to the result of Simpson's Rule applied to the remainder.

Simpson's Rule, therefore, states that if the line be divided into any even number of divisions, the area is equal to the sum of the first and last ordinates + twice the sum of the remaining odd ordinates + four times the sum of all the even ordinates, multiplied by one-third of the common distance apart.

Applying the rule to the example in Fig. 1.40, we get

Sum of first and 11th ordinate	= 2·75
4 × sum of even ordinates = 4 × 10·42	= 41·68
2 × sum of odd ordinates = 2 × 8·79	= 17·58
Total	= 62·01

The area from O_0 to O_{10} is therefore $62{\cdot}01 \times \frac{{\cdot}5}{3} = 10{\cdot}33$ sq. chains.

The area of the extra division O_{10} to $O_{11} = \frac{1{\cdot}70 + 1{\cdot}46}{2} \times {\cdot}5 = {\cdot}79$ sq. chain.

∴ Total area = 10·33 + ·79 = 11·12 sq. chains or 1·112 acres.

8. The method of mid-ordinates, like the Trapezoidal Rule, assumes the boundary lines to be straight. Ordinates are taken at the mid points of each division, and the area is given by the formula

$$\frac{O_1' + O_2' + O_3' + \ldots O_n'}{n} \times l \quad . \quad . \quad (7)$$

or

$$(O_1' + O_2' + O_3' + \ldots O_n')d. \quad . \quad . \quad (8)$$

The area of the example in Fig. 1.40 by the application of this method, is found to be $\frac{22{\cdot}39}{11} \times 5{\cdot}5 = 11{\cdot}19$ sq. chains or 1·119 acres.

9. A piece of tracing paper may be ruled in squares, each having an area representing a definite number of square links, chains, or feet. By placing this over the plan the number of whole squares falling within the boundaries may be counted and the areas of those portions of the squares cut by the boundaries and included within them can be estimated. Celluloid sheets ruled in acre squares for different scales are also manufactured.

10. The tracing paper, instead of being ruled into squares, may have a series of parallel lines drawn upon it at a distance representing 1 chain or other convenient measure apart. When this is placed over the plan, the mean length of each strip is scaled between "give and take" lines at its ends (Fig. 1.42) by means of the ordinary plotting scale, so that if the distance apart of the lines is 1 chain and the total length of all the strips is 152·40 chains, the total area is 152·4 sq. chains or 15·24 acres.

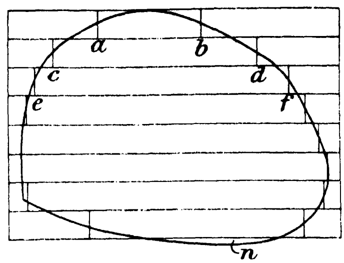


FIG. 1.42.—Computation of Areas

The length of each strip need not be written down separately, as the addition can be done mechanically on the scale. For instance, if the zero of the scale is fixed against *a* (Fig. 1.42), and the reading at *b* is 2·34 chains, this reading may be placed against *c* and the reading at *d* gives the sum of the first two strips: the reading at *d* is then placed in line with *e*, and the process repeated until the whole length of the strips is ascertained. As an alternative, the lengths may be marked off continuously upon the edge of a long strip of paper. In Fig. 1.42 the small portion *n* is not included in this calculation, and its area must therefore be determined separately and added to the result.

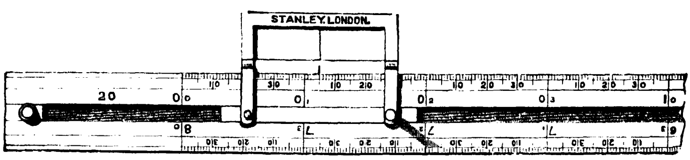


FIG. 1.43.—Computing Scale

Generally, however, by turning the tracing paper relatively to the plan a position can be found at which the boundary is exactly included between two of the parallel lines: no odd pieces such as *n* then remain to be separately computed.

11. The computing scale (Fig. 1.43) is a contrivance for still further simplifying this process. Parallel lines 1 chain apart are drawn, and the length of the strips measured off continuously as before, but instead of reading the scale at the end of each strip and then fixing this same reading at the commencement of the next strip, the position on the rule is marked by means of the line on the sliding cursor, and is

mechanically transferred to the next strip. There is no need to draw the give and take lines on the paper in this case, as the cursor line serves the purpose. It will be seen that this method is very much more expeditious, and more accurate results are to be expected than when

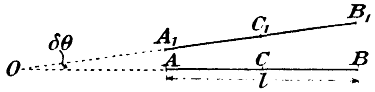


FIG. 1.44

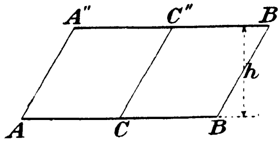


FIG. 1.45

Theory of Planimeter

a plain scale is used. The instrument is graduated, so that for a particular scale the main divisions give the area directly in acres, and, as the main divisions are subdivided into four parts, each of these represents 1 rood. The edge of the scale may be further subdivided, as in Fig. 1.43, to read to poles or perches; or the cursor may be provided with a small independent scale for this purpose.

12. The planimeter is an extremely handy instrument for measuring areas from plans. The principle [1] may be understood by considering the motion on a plane surface of a rod AB of any length l. Suppose this is given a very small displacement to a new position A_1B_1 (Fig. 1.44); then if BA and B_1A_1 are produced they will meet at some point O (though not necessarily at a finite distance away). The area swept over by AB, *i.e.* AA_1B_1B, may be considered as the difference between the two triangles BB_1O and AA_1O, *i.e.* $\frac{1}{2}OB.OB_1 \sin\delta\theta - \frac{1}{2}OA.OA_1 \sin\delta\theta$, where $\delta\theta$ is the small angle at O, *i.e.*

$$\tfrac{1}{2}\sin\delta\theta\,(OB.OB_1 - OA.OA_1), \qquad (9)$$

but, as the motion is very small, $\sin\delta\theta = \delta\theta$, $OB_1 = OB$ and $OA_1 = OA$,

$$\begin{aligned}\therefore \text{ area swept over} &= \tfrac{1}{2}\delta\theta\,(OB^2 - OA^2)\\ &= \tfrac{1}{2}\delta\theta\,(OB + OA)(OB - OA)\\ &= l\,.\,\frac{OB + OA}{2}\,.\,\delta\theta.\end{aligned}$$

But $\dfrac{OB + OA}{2}\,.\,\delta\theta$ is the small movement of the centre of the rod at right angles to $AB = \delta c$ say,

$$\therefore \text{ area swept over } = l\,.\,\delta c,$$

and the total area swept over during any finite movement of AB is

$$\int l\,.\,dc \text{ or } l\int dc. \qquad (10)$$

As an example of this, suppose the line AB moves directly to a parallel position A″B″, as shown in Fig. 1.45.

Then as $\int dc$ is the sum of all the small movements of C *at right angles*

[1] Lamb's Infinitesimal Calculus.

to AB, $\int dc = h$, the perpendicular distance between the lines, *i.e.* the area of the parallelogram ABB″A″, is lh and does not depend upon the *actual* displacement of C to C″.

When the line AB returns to its original position, as shown in Fig. 1.46, the area swept over in the outward motion is $BB_1B_2A_2A_1A$, which may be considered as positive, and in the return motion $B_2B_3BAA_3A_2$, which may be taken as negative. $l\int dc$ represents the difference between

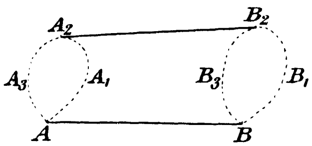


FIG. 1.46

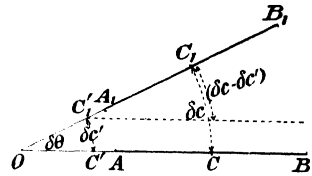


FIG. 1.47

Theory of Planimeter

these, *i.e.* the area marked out by B ($BB_1B_2B_3$) minus that marked out by A ($AA_1A_2A_3$), or *vice versa.*

Again, if during the motion of AB neither end describes a complete circle about the other before returning to its original position, it may be shown that $\int dc$ for the centre point C is equal to $\int dc'$ for any other point C′ on AB.

Thus during the small motion in Fig. 1.47 of AB, let δc and $\delta c'$ be the small displacements of C and C′ to C_1 and C_1' respectively. Then from the fig. it is seen that

$$\delta c = \delta c' + C'C \, . \, \delta\theta,$$

$$\therefore \int dc = \int dc' + C'C\int d\theta, \qquad (11)$$

and if C′C returns to its original position, and a complete circle is not described,

$$\int d\theta = 0,$$

and

$$\int dc = \int dc'. \qquad (12)$$

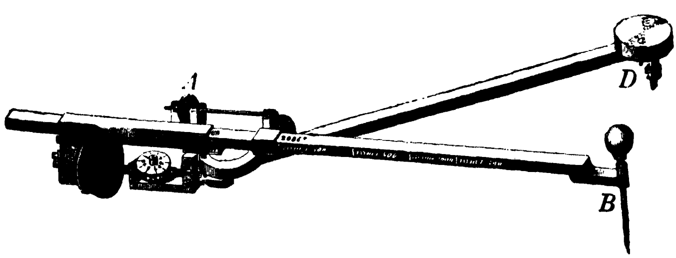


FIG. 1.48.—Planimeter

The planimeter (Fig. 1.48) consists of two bars DA and AB hinged at A, and when the instrument is in use the point D, which is provided with a needle-point and a weight, is fixed to the paper at some con-

venient position outside the area to be measured. At B is the tracer, and the outline of the field or other surface is carefully traced over with this, the position of B at completion being exactly at the starting-point. The motion of A in the meantime, if D is outside the area traced by B, is along the arc of a circle of radius DA, and the area of the figure corresponding to $AA_1A_2A_3$ (Fig. 1.46) is zero. Therefore $l\int dc'$ is in this case the area traced by B alone, corresponding to $BB_1B_2B_3$ (Fig. 1.46), and this is the area to be determined.

The value of the integral $\int dc'$ is measured directly by a small roller, the axis of which is parallel to AB at any point on its length. It is rotated by friction against the paper, and only movements perpendicular to its axis (*i.e.* to AB) are recorded. Its edge is graduated into ten main divisions, each subdivided into ten parts, and it is provided with a vernier reading to $\frac{1}{100}$th of a main division. There is also a horizontal dial graduated into ten divisions, each being equal to one complete revolution of the roller with which it is geared. The instrument reads directly, therefore, to 4 significant figures, and areas may be expressed in square inches, square centimetres, or other units by adjusting the length of AB (*i.e.* l) to graduated marks on the slide near A.

If the diameter of the roller is d units, its circumference is πd, and the area traced over for one revolution of the wheel is $l \,.\, \pi \,.\, d$ units, where l is the distance from the tracing point to the pivot between the two arms. The value of this would generally be some power of 10, *e.g.* for inch units $l\pi d = 10$, so that one revolution of the roller corresponds to 10 sq. inches area, and the vernier reads to ·01 of a square inch. If the reading of the wheel after the outline of the plan has been traced over in a clockwise direction by B is r, and the scale of the plan is p chains to 1 inch, the area is p^2r sq. chains. When the area is too large to be measured with one setting of the instrument, the plan may be divided into an arbitrary number of parts, and the areas of these determined separately.

A large area can sometimes be measured without subdivision by placing the point D *inside* the boundaries, when the instrument will have a much bigger range. In this case, however, a constant, which is marked on the bar AB for different values of l, must be taken into account, for, as explained above,

$$l\int dc = \text{area traced by B} - \text{area traced by A},$$
$$\text{and } \int dc = \int dc' + CC'\int d\theta.$$

Here, when D is within the diagram, the area traced out by A is no longer zero, but is a circle, of area $\pi \,.\, DA^2$, while $\int d\theta$ is 2π radians.

Hence $l\int dc = l(\int dc' + CC' \,.\, 2\pi) = \text{area traced by B} - \pi \,.\, DA^2$,

i.e. the area traced out by $B = l\int dc' + (l \,.\, CC' \,.\, 2\pi + \pi \,.\, DA^2)$

$$= l\int dc' + \text{constant.} \qquad (13)$$

That is to say, if the outline is traced in a clockwise direction by the point B, and the reading on the wheel and disc taken as before, the area is obtained by adding this result to the constant engraved on the

instrument, the sign of $lfdc'$ being taken into account, as it may sometimes be negative.

The area of the plot shown in Fig. 1.40, by the average of several readings, was found to be 11·13 sq. inches, which, as the scale is 1 chain to 1 inch, represents 1·113 acres.

It will be seen that this agrees almost exactly with the result obtained by Simpson's Rule (7), which is generally considered to be the most accurate of the ordinate rules. The results of Methods 5, 6, and 8 should also be compared.

13. In the case of a traverse survey (see Chapter V) the area enclosed by the survey lines may be deduced directly from the tabular co-ordinates, thus eliminating errors due to scaling, while the smaller areas between the straight survey lines and the uneven boundaries may be ascertained either directly from the field notes or from the plan by one of the preceding methods. If " Northings " and " Eastings "

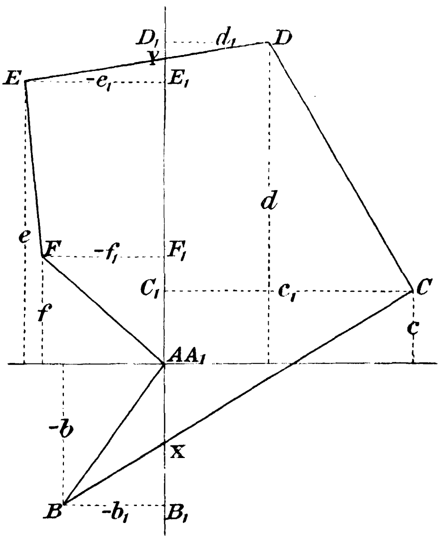


FIG. 1.49.—Calculation of Area from Co-ordinates

are positive, and the " Southings " and " Westings " negative, the area enclosed by the traverse lines is expressed by the formula

$$\text{Area} = \Sigma \frac{d_2 + d_1}{2} (l_2 - l_1) \text{ or } \tfrac{1}{2} \Sigma (d_2 + d_1)(l_2 - l_1), \quad . \quad (14)$$

where l_1 and l_2 are the latitudes and d_1 and d_2 the departures at the beginning and end of each line.

The lines should be taken in order, proceeding round the figure, either in a clockwise or a counter-clockwise direction as given in the traverse table ; it is immaterial in which direction if the *sign* of the result is ignored.

A general case is considered below. Let ABCDEF (Fig. 1.49) be

the station-points of the traverse, $A_1B_1C_1$ . . . their projections on the meridian, and a, a_1, b, b_1, c, c_1, etc., their latitudes and departures. The area is deduced as shown by the following table, and is given by the summation of column 2, in which is shown the application of the formula to each line in turn ; it explains the method of dealing with positive and negative co-ordinates. In column 1 the geometrical figure corresponding to each of these terms is given for purposes of comparison, though not necessary in making the computation.

ABB_1A_1	$\left[\frac{(-b_1)+o_1}{2}\right]\left[(-b)-o\right]$
$\left.\begin{matrix}-BB_1X\\ XCC_1\end{matrix}\right\}$	$\left[\frac{+c_1-b_1}{2}\right]\left[c-(-b)\right]$
CDD_1C_1	$\left[\frac{d_1+c_1}{2}\right]\left[d-c\right]$
$\left.\begin{matrix}-DD_1Y\\ YEE_1\end{matrix}\right\}$	$\left[\frac{-e_1+d_1}{2}\right]\left[e-d\right]$
EFF_1E_1	$\left[\frac{-f_1-e_1}{2}\right]\left[f-e\right]$
FAA_1F_1	$\left[\frac{o_1-f_1}{2}\right]\left[o-f\right]$

$$\text{Area}=ABCDEF \qquad =\Sigma\left[\frac{d_2+d_1}{2}\right](l_2-l_1).$$

An example illustrating this method is given in Chapter V, p. 151.

Accuracy of Chaining

Errors in Chaining.[1]—The chief sources of error in the chaining or taping of distances are :

(1) *Mistakes*, such as :

- (*a*) The omission of one or more chain lengths due to the miscounting or the loss of arrows, etc.
- (*b*) Confusion of the tallies, such as 40 and 60, 30 and 70, *e.g.* booking 42·7 instead of 62·7 or *vice versa.*
- (*c*) The miscounting of the odd links between tallies, *e.g.* booking 45·7 instead of 44·7 or 46·7.
- (*d*) The mistaking of a 9 for a 6 upon a tape, or, when badly worn or dirty, the mistaking of other figures.
- (*e*) Confusion when the direct chainage is interrupted by the intervention of obstacles such as pools, trees, buildings, etc.

[1] See Chapter XIX.

(*f*) The mistaking of the positions at which the end of the chain should be held—especially if marks are relied upon instead of arrows being used.
(*g*) Incorrect booking, *e.g.* 1354 instead of 1345.
(*h*) Incorrect booking when one person reads and calls out the dimensions, *e.g.* 7 and 10 sound very much alike when called over some distance.

(2) *Cumulative Errors.*—Positive errors causing the observed length to be in excess of the true distance, such as those due to

(*a*) The length of the chain or tape being less than the standard, on account of

(i) The bending of the wire links.
(ii) " Knots " in the connecting rings.
(iii) The removal of too many rings during adjustment.
(iv) The clogging of the rings with clay, etc.
(v) Low temperatures.
(vi) Shrinkage of a tape after becoming wet.

(*b*) The omission of a correction for sloping ground.
(*c*) Incorrect alignment.
(*d*) The sag in chains or tapes through not being pulled taut, especially over hollows or on uneven ground.
(*e*) The bellying out of a tape due to wind.

Negative errors, causing the observed length to fall short of the true distance, such as those due to

(*a*) The length of the chain or tape being greater than the standard on account of

(i) The wear or flattening of the connecting rings; the opening of the ring joints if not brazed; or the stretching of tapes.
(ii) High temperatures.

(3) *Compensating Errors*, such as those due to

(*a*) The incorrect markings of the ends of the various chain lengths owing to

(i) The leader holding the marking arrow in an inclined position, such as \\···· or /·····
(ii) The leader or the follower inserting a ranging rod or arrow through the chain handle and thus causing at either end a $+^{ve}$ error equal to the thickness of the chain handle plus half the thickness of the rod.

(iii) The follower not holding the rear end of the chain exactly at the mark left by the leader, but allowing it to slip forward as the chain is tightened—the error being $-^{ve}$.

(Errors (ii) and (iii), if considered separately, may be classed as cumulative.)

(*b*) Fractional parts of the chain being incorrect, if the total length of the chain has been adjusted by the insertion or removal of a few connecting rings.

(*c*) Incorrect plumbing during stepping operations, particularly if some cruder method is adopted in lieu of the proper plumb-bob method.

(*d*) Incorrect determination of inaccessible distances owing to the errors introduced into the angles on account of

(i) The fractional parts of the chain being incorrect, or

(ii) Discrepancy between tape and chain measurements when these are used together.

Accuracy of Linear Measurements.[1]—As mentioned above, the final error in a linear measurement, exclusive of actual " mistakes " which follow no known law, is composed of two portions :

(*a*) Cumulative errors which are proportional to L, and

(*b*) Compensating errors which are proportional to $\sqrt{L}$,

where L is the length of the line.

It is therefore not strictly accurate to apply the theory of least squares directly to the results of a series of observations—as this theory assumes that the errors are as likely to be $+^{ve}$ or $-^{ve}$, and it is consequently only applicable to compensating errors.

Captain J. E. E. Craster, R.E.,[2] in an investigation of the errors which occur in ordinary chaining, obtained the following results from the measurements of 40 different surveyors :

TABLE I

No. of Lines Chained	Length of Line between the Limits	Mean	Mean Error from Trigonometrical Determination
	Chains	Chains	Links
27	70-80	75·3	10·21
32	80-90	85·4	9·5
23	90-100	94·3	10·8
34	100-110	105·1	11·53
21	110-120	114·7	13·8
30	120-130	124·1	13·6

Assuming the partial error which is cumulative as d links per chain, and that which is compensating as f links per chain, the total

[1] See Chapter XIX. [2] *Engineering*, July 7, 1911.

cumulative error in a length of n chains will be $d \, . \, n$ links, and the compensating error in the same distance will be $f \, . \sqrt{n}$ links.

The resultant error x, of which these partial errors are the components, will then be

$$x = \sqrt{d^2 n^2 + f^2 n} \quad . \quad . \quad . \quad (15)$$

and this is the equation of an hyperbola.

Substituting the values in column 4 for x, and those in column 3 for n, and weighting the equations according to the number of lines in column 1, Captain Craster determined

$$d = \pm 0{\cdot}108067 \text{ links per chain,}$$
$$\text{and} \quad = \pm 0{\cdot}375374 \text{ links per chain.}$$

These results show the mean error, as determined by experiment, for lines between 70 and 130 chains in length, but they should not, of course, be applied to lines beyond these limits without verification.

Formula (15) thus shows the error which may be expected, *on the average*, upon lines over rough ground and between 70 and 130 chains in length.

Many of the individual errors, however, will exceed this amount, and it is desirable in practice to state some limit of error the exceeding of which will entail the rejection of the measurement as unsatisfactory. This limit is known as the

Permissible Error.—On the Ordnance Survey the value given to this, *i.e.* the greatest allowable deviation of the measured length from the length as determined trigonometrically (*vide* Chapter XIII), was

1 in 1000 for larger plans than those to the $\frac{1}{2500}$ scale.
1 in 500 for the $\frac{1}{2500}$ scale and smaller maps.

The former value, *i.e.* 1 in 1000, is generally adopted for ordinary chain survey plans. It follows, then, that as the *maximum* error is limited to 1 in 1000, the probable error in any single measurement is much less than this amount.

It should be observed that this form of expression for the permissible error assumes the error to be proportional to the length of the line, which is not strictly true, as some of the errors are compensating.

According to this assumption, the *mean* errors obtained in the six sets of observations in Table I were 1 in 737, 1 in 900, 1 in 873, 1 in 911, 1 in 830, and 1 in 912 respectively, while the mean error of the total set of observations was about 1 in 867.

With a steel band, or long steel tape, greater accuracy can be obtained, and the limiting error might be, say,

1 in 2000 for ordinary steel-band measurements,
or 1 in 5000 if the horizontality is judged by the eye, a constant pull exerted, and rough corrections for temperature applied after the tape has been accurately standardised at a definite temperature.

For very good pacing, the limit of error would be perhaps 1 in 100.

The accuracy obtainable on very precise base-line measurements is mentioned in Chapter XIV.

Errors in Areas.—The effect of errors in the linear measurements upon the accuracy of a calculated area may be found by differentiation as follows:

(i) In the case of a rectangle where S = area and a and b are the two sides.

Then
$$S = a\,b. \qquad (16)$$

Let δS_1 be the small error produced in S due to a small error δa in a, and δS_2 that due to δb in b.

By differentiation
$$\delta S_1 = b\,.\,\delta a$$

or
$$\frac{\delta S_1}{S} = \frac{\delta a}{a}. \qquad (17)$$

Similarly
$$\frac{\delta S_2}{S} = \frac{\delta b}{b}. \qquad (18)$$

If δa and δb are *very small* actual errors, the total fractional error $\frac{\delta S}{S}$ due to the two causes $= \pm\frac{\delta a}{a} \pm \frac{\delta b}{b}$, according to the signs of δa and δb respectively.

If $\pm\ \delta a$ and $\pm\,\delta b$ are the probable errors in a and b, then the probable error δS in S is given by
$$\frac{\delta S}{S} = \pm\sqrt{\left(\frac{\delta a}{a}\right)^2 + \left(\frac{\delta b}{b}\right)^2} \qquad (19)$$

or
$$\delta S = \pm\sqrt{(b\,.\,\delta a)^2 + (a\,.\,\delta b)^2}.$$

If δa and δb are proportional to a and b respectively, then $\frac{\delta a}{a}$ and $\frac{\delta b}{b}$ are constants, and the fractional error $\frac{\delta S}{S}$ is unaltered by any variation in the ratio of a to b.

For example, if the p.e. in the linear dimensions is $\pm$ 1 in 2000, *i.e.* if $\frac{\delta a}{a} = \frac{\delta b}{b} = \frac{1}{2000}$, then the p.e. in the area is
$$\frac{\delta S}{S} = \pm\sqrt{\left(\frac{1}{2000}\right)^2 + \left(\frac{1}{2000}\right)^2} = \pm\frac{\sqrt{2}}{2000}.$$

If, however, δa and δb are proportional to $\sqrt{a}$ and $\sqrt{b}$ respectively, as is usually assumed, then if $\delta a/\sqrt{a} = \delta b/\sqrt{b} = k$ (say), $\frac{\delta S}{S} = \pm k\sqrt{\frac{a+b}{ab}}$, which is a minimum when $a = b$, *i.e.* when the rectangle is a square.

Similarly in the case of a triangle, the area of which is derived by Method 1 (p. 30), *i.e.*
$$S = \tfrac{1}{2}\,bh \qquad (20)$$

then
$$\frac{\delta S}{S} = \pm\sqrt{\left(\frac{\delta b}{b}\right)^2 + \left(\frac{\delta h}{h}\right)^2}. \qquad (21)$$

(ii) When the area of a triangle is found by Method 2 (p. 31), *i.e.*

$$S=\tfrac{1}{2}\,b\,c\sin A. \qquad (22)$$

Let δS_1, δS_2, and δS_3 be the errors produced in S by the small errors δb, δc, and δA in b, c, and A respectively.

By differentiation $\delta S_1=\tfrac{1}{2}\,\delta b\,c\sin A$,

$$\frac{\delta S_1}{S}=\frac{\delta b}{b}. \qquad (23)$$

Similarly

$$\frac{\delta S_2}{S}=\frac{\delta c}{c}. \qquad (24)$$

Also $\delta S_3=\tfrac{1}{2}\,b.c\cos A\,.\,\delta A$,

$$\therefore \frac{\delta S_3}{S}=\cot A.\delta A, \qquad (25)$$

and this is a minimum when the angle A is 90°.

If $\pm\delta b$, $\pm\delta c$, $\pm\delta A$ are probable errors in b, c, and A respectively, the p.e. δS in S is given by the formula

$$\frac{\delta S}{S}=\pm\sqrt{\left(\frac{\delta b}{b}\right)^2+\left(\frac{\delta c}{c}\right)^2+(\cot A.\delta A)^2}. \qquad (26)$$

If δb and δc are proportional to $\sqrt{b}$ and $\sqrt{c}$, the fractional error $\frac{\delta S}{S}$ will be a minimum when $b=c$, and $A=90°$, *i.e.* when the triangle is a right-angled isosceles triangle—or half a square.

If δb and δc are proportional to b and c respectively, then $\frac{\delta S}{S}$ is independent of the ratio of b to c, but is a minimum when A is 90°.

Example.—If in the example on p. 31 the p.e. in the linear measurements was ± 1 in 1000, and in the angle A $\pm 30'$, *i.e.* $\pm\cdot 00873$ radians, the p.e. in the area of ABC

$$=\pm S\sqrt{\left(\frac{1}{1000}\right)^2+\left(\frac{1}{1000}\right)^2+(\cdot 074\times\cdot 00873)^2}$$
$$=\pm 4{\cdot}225\sqrt{\cdot 00002417}$$
$$=\pm\ \cdot 005 \text{ acre nearly.}$$

(iii) The case in which the three sides of a triangle are known is given in Example 6 at the end of the chapter.

(iv) When a chain is subject to a constant fractional error e, *e.g.* when the apparent length L of the chain is actually $(1+e)L$, then

$$\frac{\text{The actual area A}}{\text{The apparent area } A_1}=\frac{(1+e)^2L^2}{L^2}=1+2e \text{ nearly,}$$
$$\text{or } A=(1+2e)A_1.$$

Thus if the chain is x per cent long (short), the true area is $2x$ per cent (approx.) greater (less) than the computed area. (See Example 5, p. 47.)

EXAMPLES

1. (U. of L.) The following is an extract from a field book. Draw the station lines, writing on each its number, and point out which lines act as check lines. Do the measurements check ?

	256/3,	00/4	(541)	
	00/6,	575/5	(00)	
			Line 8	
		357/5	(512)	
486/2,	289/4,	635/6	(00)	
			Line 7	
	486/2,	289/4	(635)	
		575/5	(00)	
			Line 6	
			(575)	
			(357)	
	517/1,	00/2	(00)	
			Line 5	
		00/1	(810)	
		486/2	(289)	
		256/3	(00)	
			Line 4	
			(256)	
		658/2	(00)	
			Line 3	
			(658)	
			(486)	
		517/1	(00)	
			Line 2	
			(517)	
			(00)	
			Line 1	

2. (U. of B.) The following perpendicular offsets were taken at 50 ft. intervals from a chain line to an irregular boundary:

10·6, 15·4, 20·2, 21·3, 18·7, 16·4, 20·4, 25·8, 30·6, 20·8, and 17·4 feet.

Calculate the area in square yards enclosed between the chain line, the irregular boundary, and the first and last offsets.

3. The following figures give the values in links of offsets taken from the chain line of a survey to the hedge. Calculate the area included between the chain line and the hedge by the application of

(1) Simpson's Rule.
(2) The Trapezoidal Rule.
(3) The average ordinate rule.

Offsets	15	8	6	9	11	11·5	10	12	10
Distance	0	25	50	100	150	200	250	275	300

4. (I.C.E.) The following perpendicular offsets were taken at consecutive intervals of 20 ft. from a straight line to a wavy boundary:

9, 15, 12·3, 17, 5·2, 9·4, 7.

Find the area between the straight line and the boundary by Simpson's Rule.

5. (I.C.E.) Distinguish between "compensating" and "cumulative" errors in chaining.

A field was measured with a chain 0·3 of a link too long. The area thus found was 30 acres. What is the true area?

6. Find what would be the probable proportional error in the area of a triangle determined from the length of its sides, if the p.e.'s in these sides were $\pm\delta a, \pm\delta b, \pm\delta c$ respectively.

Apply your result to find the p.e. in area in the triangle ABC (Example 3, p. 31), if the p.e. in chainage was ± 1 in 1000.

7. (U. of L.) A surveyor finds that he can lay out the direction of an offset by eye in the field with a maximum error of 5° from the true perpendicular. With what degree of accuracy must he measure the lengths of the offset in order that the error on the paper from this source may not exceed that from the angular error?

Under these conditions, if the scale of the plan is to be 200 feet to 1 inch, about what must be the maximum length of offset in order that the displacement of a point on the paper, from the two sources of error combined, may not exceed 0·01 inch?

8. The sides of a quadrilateral are AB = 4·23, BC = 6·902, CD = 3·568, DA = 4·65, and the diagonal AC = 6·156 chains. Calculate

(*a*) The area of the quadrilateral ABCD.
(*b*) The angles at A and at C.
(*c*) The length of the diagonal BD.

Assuming the p.e. in the linear measurements to be ± 1 in 2000, find

(*d*) The p.e. in the area.
(*e*) The p.e. in the angles at A and at C.
(*f*) The p.e. in the length of the diagonal BD.

9. The plan of an old survey plotted to a scale of 41·66 ft. to 1 inch was found to have shrunk so that a line originally 10 inches long was only 9·86 inches. There was also a note stating that the chain used was 0·2 link too long.

If the area of the plan given by a planimeter was 34·76 sq. inches, what was approximately the correct area of the survey?

CHAPTER II

OPTICS AND MAGNETISM

OPTICS

WHEN a ray of light travelling in a medium A meets the surface of another medium B, a portion of the ray may penetrate into B, another portion may be absorbed, and the remainder is thrown back from the common surface into the first medium.

The portion which is thrown back is said to be reflected, and it is found experimentally that

(i) The angle of incidence is equal to the angle of reflection, and
(ii) The incident and reflected rays and the normal to the reflecting surface at the point of contact all lie in one plane.

Thus suppose CD (Fig. 2.1) represents a plane mirror or other polished surface, and F any object, such as a ranging rod. Rays of reflected light are thrown off from F in all directions, but only those which strike the mirror at O, so that the angle of incidence FON is equal to the angle of reflection EON, are reflected to the eye at E. The image of F appears to be at F_1 on the normal through F, and as much behind as F is in front of the mirror.

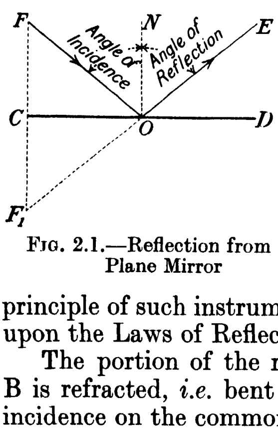


FIG. 2.1.—Reflection from a Plane Mirror

It will be seen later (p. 74) that the principle of such instruments as the sextant and optical square depends upon the Laws of Reflection stated above.

The portion of the ray which penetrates into the second medium B is refracted, *i.e.* bent out of its original course, unless the angle of incidence on the common surface is zero, or unless the two media have the same " refractive index ".

The Laws of Refraction for isotropic media (*e.g.* glass) are :

(i) The incident and refracted rays lie in a plane containing the normal to the common surface.
(ii) For the same two media the ratio of the sine of the angle of incidence to that of the angle of refraction is a constant and is known as the " Index of Refraction " (μ).

In passing obliquely from one medium into another of greater density (*e.g.* from air into water or glass) the ray is bent nearer the

normal as in Fig. 2.2, while in the reverse case a ray passing into a less dense medium is bent farther away from the normal.

When, in this reverse case, the angle between the normal in the rarer medium and the emerging ray reaches 90°, the value of the angle between the normal and the incident ray is known as the " Critical Angle ".

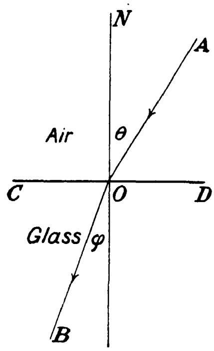


FIG. 2.2.—Refraction

If the angle between the normal and incident ray is greater than the critical angle, the ray does not emerge from the denser medium, but is totally reflected at the surface—the angle of reflection being equal to the angle of incidence.

This fact is taken advantage of by the employment of a right-angled isosceles prism as a reflector in the case of such instruments as a prismatic compass, or a line ranger or prism binoculars. Thus in Fig. 2.3, if ABC is the prism, and a ray from F meets the face AB at right angles, it continues in a straight line without refraction until it meets the hypotenuse, making an angle of 45° with the normal, when, instead of passing from the denser medium, glass, into the less dense, air, it is totally reflected, and passes at right angles out of the prism through the face BC to E. The advantage of such a reflector is that the reflecting surface does not become tarnished or covered with dust as would that of a plane mirror.

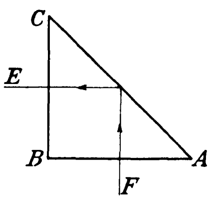


FIG. 2.3.—Prismatic Reflection

Figs. 4.10 and 4.11 illustrate the employment of prisms in the optical reading systems of a modern theodolite.

The value of the critical angle from water to air is about 48½°, and from glass to air is about 41° to 42°.

A simple lens is usually made of a single piece of glass having its two surfaces of approximately spherical form—one surface is sometimes plane, *i.e.* a portion of a sphere of infinite radius. Its properties depend upon the principles of refraction, and a few cases are here considered.

Types of simple lenses are shown in Fig. 2.4.

(*a*) = Double Convex	Converging.	(*d*) = Double Concave	Diverging.
(*b*) = Plano Convex		(*e*) = Plano Concave	
(*c*) = Concavo Convex		(*f*) = Convexo Concave	

(*c*) and (*f*) are also known as meniscus lenses.

It may be noted that converging lenses are thicker in the middle than at the edges, while diverging lenses are thinner at the middle than at the edges.

Principal Axis.—A lens is usually a solid of revolution, and its axis of revolution is termed the principal axis of the lens. When the two surfaces are spherical, this axis joins the two centres of curvature-

Principal or Solar Focus.—A ray of light passing through a lens is refracted according to the Laws of Refraction.

Those rays which enter an *ideal* converging lens in a direction parallel to the principal axis are refracted to pass through one point on the principal axis, known as the Principal Focus (F, Fig. 2.5). In the case of a diverging lens, similar rays, after passing through, diverge

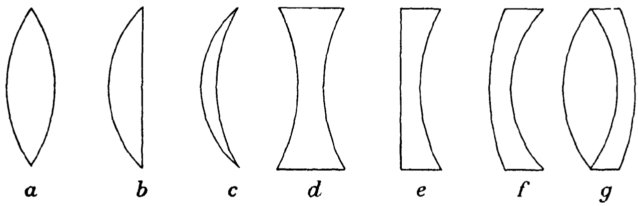


FIG. 2.4.—Lenses

in such a manner that if their directions are produced backwards they meet in a point F_1 (Fig. 2.6), which is the principal focus for this type of lens.

It will be noted that the rays actually pass through the principal focus in the case of a converging lens, but only *appear* to do so in the case of a diverging lens; the former is thus called a " real " focus, while the latter is a " virtual " focus.

Points corresponding with F and F_1 are found on the opposite sides of the lens for rays travelling in the opposite direction. The distance

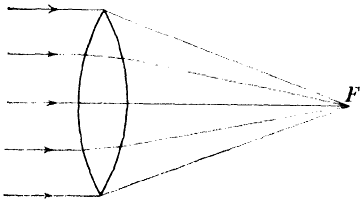


FIG. 2.5.—Converging Lens

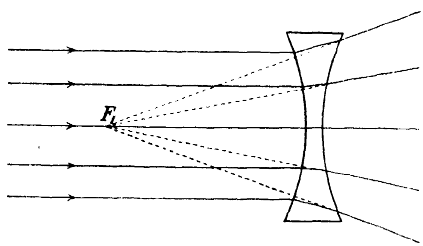


FIG. 2.6.—Diverging Lens

from the lens to the principal focus is known as the focal length, principal focal length, or solar focal length, and its magnitude depends upon the refractive index of the glass and the radii of curvature of the faces.

In exactly the same manner that a ray entering a lens parallel to the principal axis is refracted to pass through the principal focus F, so a ray which emerges from a double convex lens, parallel to the principal axis, must have previously passed through the principal focus F (Fig. 2.5) on the opposite side of the lens, while from a diverging lens the entering rays, which if produced would pass through a virtual focus F_1, emerge parallel to the axis.

Optical Centre.—Let O and O_1 (Fig. 2.7) be the two centres of curvature of a lens, and OT and O_1T_1 any two parallel radii. Draw TA and T_1A_1 tangential to the surfaces at T and T_1. Then if BTT_1B_1 represents the path of a ray through T and T_1, as TA and T_1A_1 are parallel and TT_1 therefore makes an equal angle with each, the inclinations of BT and B_1T_1 to the normals will consequently be equal, and BT and B_1T_1 will be parallel.

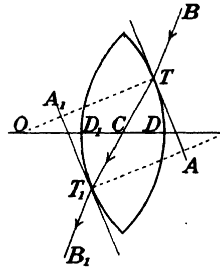


FIG. 2.7
Optical Centre

The point C at which TT_1 cuts the principal axis is known as the optical centre of the lens, and it is seen from the two similar triangles OTC and O_1T_1C that

$$\frac{OC}{O_1C}=\frac{OT}{O_1T_1}, \quad \text{or} \quad \frac{OT-OC}{O_1T_1-O_1C}=\frac{CD}{CD_1}=\frac{OT}{O_1T_1}=\frac{OC}{O_1C}$$

i.e. the point C divides the line joining the centres of curvature in the ratio of the radii of curvature, and divides the thickness of the lens itself in the same ratio.

All emergent rays, after passing through C, have their directions parallel to their original directions, but to simplify graphical constructions the thickness of the lens is generally neglected, and each ray

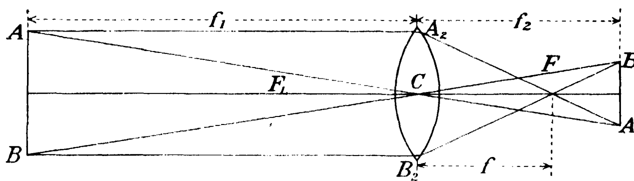


FIG. 2.8

which passes through C is continued in a straight line without any deviation.

In the case of a meniscus lens the optical centre lies outside the lens, its distance from the surfaces being still directly proportional to the radii of curvature ; while in the case of a plano-convex or plano-concave lens, one of the radii being infinite, the optical centre lies on the curved surface.

Conjugate Foci and Images.—In an ideal converging lens all rays from any distant point A which pass through the lens are so refracted as to pass through another point A_1 (Fig. 2.8). In a diverging lens, the direction of the emergent rays when produced backwards should pass through a point A_1.

In the first case, then, a real image of A is formed at A_1, while in the second case, as the rays do not actually pass through A_1, a virtual image is said to be formed. The two points A and A_1 are said to be conjugate foci. Let AB (Fig. 2.8) be a portion of a distant object such as a ranging rod or levelling staff : then *all* the rays from A which pass through the lens should converge to a point A_1 on the opposite side of the lens.

The position of A_1 may be found by considering the path of two particular rays. (1) The ray from A which passes through the optical centre C of the lens may be assumed to continue in the same straight

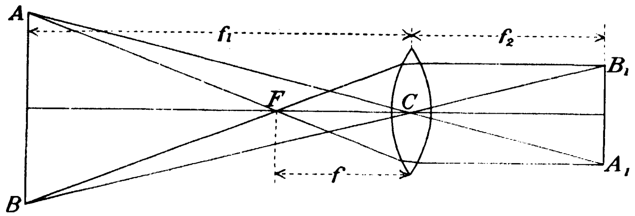


FIG. 2.9

line AC without deflection, so that A_1 lies upon AC produced. (2) The ray from A parallel to the principal axis is refracted through the principal focus F, so that A_1 also lies on A_2F produced, and the intersection of AC and A_2F fixes the position of A_1 as in Fig. 2.8.

Similarly the position of the image B_1 of any other point B may be found. The image in this case is real and inverted, but if the object is nearer to the lens than the principal focal length, *i.e.* if AB lies between F_1 and C, the image will be virtual, erect, and magnified (Fig. 2.10), and can be seen by the eye placed on the opposite side of the lens.

The point A_1 is found by producing FA_2 and CA backwards to intersect at A_1—the rays not actually passing through A_1; B_1 is found in a similar manner.

The image is also virtual in the case of a divergent lens.

From Fig. 2.8 it is evident that the greater the distance of AB from the lens C the flatter becomes the line ACA_1, and the nearer to F is the

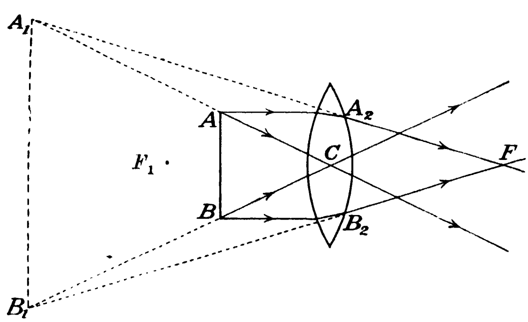


FIG. 2.10

point A_1 on the line A_2F, until, when AB is a very great distance away, its image is formed *at* F. The image of a point in the sun, for instance, is formed as nearly as possible at F—all the rays being practically parallel, so that F is sometimes known as the " Solar Focus ".

Let CF, the principal focal length, be denoted by f, and the conjugate

focal lengths, *i.e.* the distances of AB and A_1B_1 respectively from the lens, be f_1 and f_2 (Fig. 2.8.).

Then from the similar triangles ABC and A_1B_1C,

$$\frac{AB}{A_1B_1}=\frac{f_1}{f_2}, \quad . \quad . \quad . \quad . \quad (1)$$

and from the similar triangles A_2B_2F and A_1B_1F,

$$\frac{A_2B_2}{A_1B_1}=\frac{f}{f_2-f}, \quad . \quad . \quad . \quad (2)$$

and as AA_2 and BB_2 are each parallel to the principal axis of the lens,

$$A_2B_2=AB.$$

$\therefore$ from (1) and (2) $\quad \frac{AB}{A_1B_1}=\frac{f_1}{f_2}=\frac{f}{f_2-f},$

or by cross multiplication $f_1f_2-f_1f=f_2f,$
and dividing each term by ff_1f_2,

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}. \quad . \quad . \quad . \quad (3)$$

The same result may be deduced from Fig. 2.9.

The relative size of the object and its image evidently depends upon the magnitude of the conjugate focal lengths f_1 and f_2, the relationship being expressed by equation (1). In the particular case, when AB = A_1B_1, it follows that $f_1=f_2$, and hence by substitution in (3) that

$$f_1=f_2=2f. \quad . \quad . \quad . \quad (4)$$

That is to say, if the object is at a distance equal to $2f$ from the lens, the resulting image will be equal in size to the object itself; if the object is at a greater distance then $2f$ from the lens, the image becomes smaller as f_1 increases, until when f_1 approaches an infinitely great distance the size of the image diminishes and approaches a point at F in the limit; if f_1 is less than $2f$, the size of the image is greater than the object, and is real until $f_1=f$, when f_2 becomes infinitely large: for values of f_1 less than f the image is virtual, as already explained (Fig. 2.10).

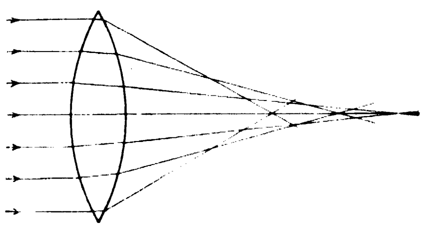


Fig. 2.11.—Spherical Aberration

Aberration.—If the lens is bounded by truly spherical surfaces, the rays from a given point are not all collected exactly at one point, but those through the outer portions of the glass fall short as in Fig. 2.11. This defect is known as **Spherical Aberration**, and is remedied by "figuring" or modifying the spherical shape of the lens by hand.

A " stop " may be used in a photographic camera to eliminate the outside rays and so render the image more sharply defined ; the time of exposure necessary to photograph the image upon a sensitised plate is longer, however, in this case, as the image is not so bright as before.

Chromatic Aberration.—White light is made up of coloured rays—red, orange, yellow, green, blue, indigo, and violet—and of these the violet is refracted most and the red least, so that when white light passes through a simple lens these coloured rays will each have a different focus, as in Fig. 2.12, the violet being the nearer the lens. This is a defect in a lens, as, instead of one definite image being formed of an object, a number of images are formed, those nearer the lens having a preponderance of violet with a red edging, while those more remote have a preponderance of red with the violet on the outside.

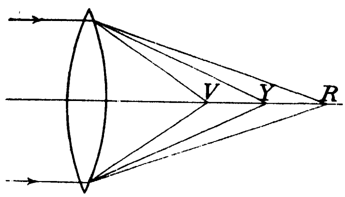


FIG. 2.12.—Chromatic Aberration

To correct this dispersion a compound lens is used as in Fig. 2.4, *g*, the double convex portion being made of crown glass and the meniscus of flint glass. As these two materials possess different properties of refraction, etc., the dispersion of the former can be largely neutralised by the latter without at the same time altogether neutralising that deviation of the rays which of course is essential for the formation of an image. The blue and orange rays are brought to the same focus in practice, and the combination of the two glasses is known as an " achromatic lens ".

Astigmatism (Fig. 2.13) is a defect which causes oblique pencils of light to focus on two straight lines a small distance apart, and at right angles to each other, instead of at one point. The nearest approach to a point focus is between these two focal lines where a " circle of least confusion " is formed. Often the various circles of least confusion do not lie in one plane, but on a surface which is concave towards the lens. This is an important defect—particularly for photographic work—as the whole field cannot be simultaneously focused on to a flat plate.

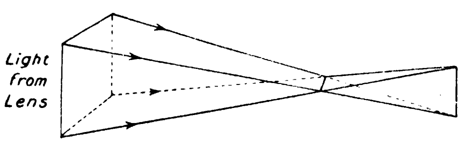


FIG. 2.13.—Astigmatism

The Telescope.—The telescope used for surveying instruments consists of a metal tube containing an object-glass at one extremity and an eye-piece at the other. The object-glass, as in Fig. 2.8, produces a real inverted image in the tube, which the eye-piece magnifies as in Fig. 2.10. The eye-piece, instead of being a simple lens, is more usually a combination, the two chief types being :

(1) The Ramsden eye-piece, which is composed of two plano-

convex lenses as in Fig. 2.14, *a*. The distance between them is two-thirds of the focal length of either.

(2) The Huygenian eye-piece, which is composed of two plano-

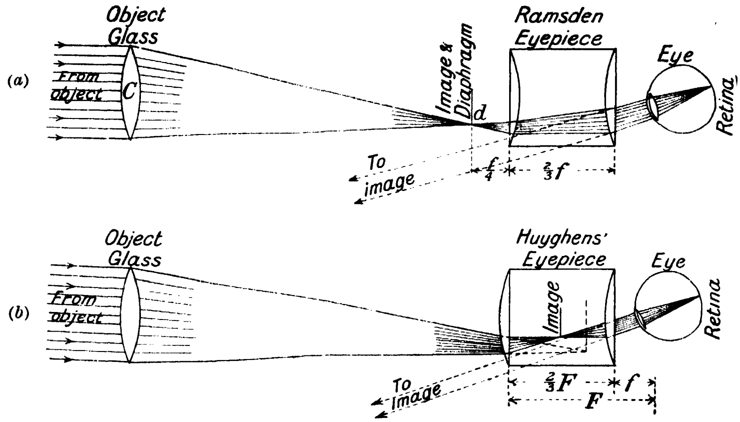


FIG. 2.14.—(*a*) Ramsden eye-piece. (*b*) Huygens' (or Huyghens') eye-piece).

convex lenses as in Fig. 2.14, *b*. The distance between them is two-thirds of the focal length (F) of the larger, and twice the focal length (f) of the smaller.

The former is the more generally employed, as it gives a flat field of view, and though the chromatic aberration is slightly more, its spherical aberration is less than that of the Huygenian. It is known as a positive eye-piece, for the inverted image which is formed by the object-glass appears still inverted to the observer.

The diaphragm is a brass or gun-metal ring fitting inside the telescope tube, and attached to it by means of either two or four capstan-headed screws which are capable of adjustment.

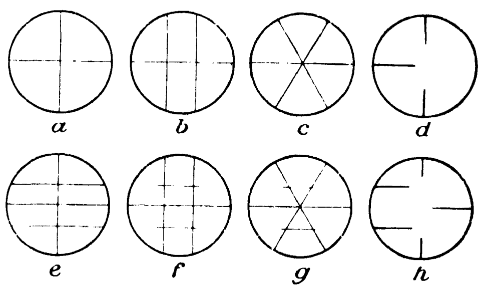


FIG. 2.15.—Types of Diaphragms

The ring carries the "cross-hairs", which may be arranged in several ways, as shown in Fig. 2.15, and may be formed

(1) By fine lines scratched upon glass;
(2) By fine spiders' webs stretched across the frame; or
(3) By fine metal points, generally of platinum-iridium.

An objection to the first of these methods is that the glass is apt to become dimmed through condensed moisture, and it also cuts out a certain proportion of light, but it is not so fragile as are the spiders'

webs. The metal points are preferred by many, as they are not very easily broken or displaced, and can be seen very distinctly; but graduations on a staff cannot be read as quickly with these, as more lateral adjustment than is necessary with a plain hair-line is required to get coincidence with the end of the metal point.

The diaphragm is placed in such a position that the cross-hairs may be made to coincide with the image formed at *d* by the object-glass (Fig. 2.14, *a*), and the distance of this (f_2) from C is a variable, depending upon the distance (f_1) of the object AB from C, as explained above.

The distance from the real inverted image to the eye-piece depends upon the eyesight of the person observing, and in order that the magnified vertical image shall be clearly seen, the distance of this from the eye should be the same as would be necessary to obtain the best view of an object by direct vision without the aid of a magnifying glass (cf. Fig. 2.10). It is necessary, therefore, that the distances of the diaphragm, both from the object-glass and from the eye-piece, shall be adjustable.

Generally the object-glass is fixed into one end of the telescope tube proper, while into the other end slides a close-fitting tube, worked by means of a rack and pinion from a large milled-headed screw on the outside of the telescope. To the sliding tube the diaphragm is attached by capstan-headed screws, and the eye-piece, which slides into the end of it, can be adjusted outwards or inwards with the fingers through a small distance.

Sometimes the object-glass is attached to a sliding inner tube worked by a rack and pinion, while the diaphragm is fixed to the main telescope tube.

In focusing the telescope on to an object, the eye-piece is adjusted with the fingers until the cross-webs are seen quite distinctly, and they then require little attention during the remainder of the day. The image of the object is then brought clearly into view by means of the large milled-headed screw on the side of the telescope, and should be accurately brought to the same focus as the hair-lines. When in this position the cross-hairs should appear stationary, *i.e.* should not move relatively to the image when the eye is moved upwards or downwards so as to look through a different part of the eye-piece lens. The distance from the object-glass to the diaphragm when focused to a distant object gives the focal length of the object-glass, and this it is which defines the "size" of the telescope, *e.g.* 10″, 12″, 14″, etc.

The imaginary line joining the intersection of the webs of the diaphragm to the optical centre of the object-glass and the continuation of this is known as the "Line of Collimation" of the telescope.

The *Internal Focusing Telescope* has now been extensively adopted by the makers of standard surveying instruments. As explained below, the focusing is affected by the movement of an auxiliary negative lens (see, for instance, Fig. 6.7) between the object glass and the diaphragm, which are fixed at a constant distance apart.

Consequently, the overall length of the telescope is unaltered by the movement of the draw-tube, the balance of the telescope is less affected, and the bubble less liable to be displaced, than with external

focusing. One incidental result is that the risk of breaking the parallel plate bubble-tube or the glass cover to a compass-box, when transitting a theodolite telescope, is eliminated. The movement of the negative lens is less than the movement of the draw-tube in the external focusing telescope and the wear on the rack and pinion is less; hence the line of collimation is less likely to be affected by focusing, particularly as any displacement of the line of collimation is appreciably less than the displacement of the negative lens itself. Other advantages of the arrangement are that as the draw-tube is not exposed to the weather, oxidation is less likely to occur, and the telescope can be made practically dust and water proof.

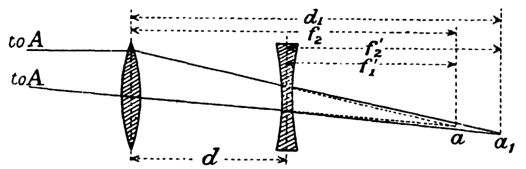


FIG. 2.16

Disadvantages are that the internal lens necessitates a certain loss of light, and for this reason some makers use a slightly larger object glass to compensate for it: under some conditions the negative lens may be liable to become bedewed; the cost is probably a little greater than with external focusing.

The principle underlying the method of focusing with a negative lens may be seen from the diagram in Fig. 2.16, where the two lenses are represented by single glasses. Let A represent a point upon the staff, from which two rays only are shown. These rays, together with others, would, in the absence of the negative lens, come to a common focus at a, where an image of A would be formed. The effect of the additional lens is to tend to diverge the rays; this causes them to produce the image of A at a_1 instead of at a, and it will be seen from the following analysis that by suitably altering the distance between the two lenses the point a_1 may be made to fall on a fixed plane, viz. that of the diaphragm wires.

Let f be the principal focal length of the object-glass,
f' be the principal focal length of the negative lens,
f_1 be the distance from the object-glass to the staff,
f_2 be the distance from the object-glass to the image which would be formed were the negative lens absent,
f_1' be the distance from the negative lens to the image which would be formed were the negative lens absent,
f_2' be the distance from the negative lens to the image actually formed at a_1,
d_1 be the distance from the object-glass to the image actually formed at a_1,
d be the distance between the two lenses.

Then from equation (3) $\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}$ and $f_2=\frac{ff_1}{f_1-f}$, . . (5)

$$\frac{1}{f'}=\frac{1}{f_1'}-\frac{1}{f_2'}, \quad f'=\frac{f_1'f_2'}{f_2'-f_1'}. \quad . \quad (6)$$

Substituting $f_1' = (f_2 - d)$ and $f_2' = d_1 - d$,

$$f' = \frac{(f_2 - d)\,(d_1 - d)}{d_1 - f_2}. \quad . \quad . \quad . \quad (7)$$

From (5) and (7) $\left(d_1 - \frac{ff_1}{f_1 - f}\right) f' = \left(\frac{ff_1}{f_1 - f} - d\right)(d_1 - d)$, whence

$$d^2(f_1 - f) - d\{ff_1 + d_1(f_1 - f)\} - d_1f_1f' + d_1ff_1 + d_1ff' + ff_1f' = 0. \quad (8)$$

This equation, a quadratic expressing the value of d in terms of f, f', d_1, and f_1, is referred to on p. 250.

Here f and f' are constants, these being the principal focal lengths of the two lenses, and the remaining quantities are d_1 and f_1.

Usually, as in the Zeiss instrument (p. 168), d_1, the distance between the object-glass and the cross-hairs, is kept a constant value, consequently as f_1, the distance to the staff, varies, d must be altered by means of the milled-headed focusing screw until equation (8) is satisfied.

Magnetism

The word " magnet " dates from very early times, being originally applied to pieces of lodestone or " magnetic iron ore " (Fe_3O_4), found in Magnesia in Asia Minor, and in other districts.

It was discovered that this substance was endowed with the power of attracting pieces of iron and steel, and also that it possessed the peculiar property that, when freely suspended, it adopted a definite position in a direction approximately north and south. Natural magnets are comparatively rare, however, but certain substances which are capable of being magnetized—particularly iron and steel—may be converted into artificial magnets.

The end of the magnet which is directed towards the north is known as the north pole, or north-seeking pole, of the magnet; similarly the opposite end is termed the south or south-seeking pole.

Observation shows that there is a repulsive force exerted between the " like " poles, and an attractive force between the " unlike " poles of two magnets, so that the north pole of one magnet repels the north pole and attracts the south pole of another, the intensity of the force varying inversely as the square of the distance between the poles concerned.

The earth may be considered as a huge natural magnet having its poles situated near the north and south geographical poles respectively; consequently the polarity of the north magnetic pole is of the opposite sign to that of the north-seeking end of an ordinary magnet, and corresponds with that of the south-seeking end.

The Compass.—There are two general types of compass, (1) the circular box compass and (2) the trough compass.

Sometimes the circular box type is fitted with a graduated disc below the level of the magnetic " needle "—*e.g.* in the ordinary small pocket compass—and sometimes with a flat circular ring scale on the

same level and as nearly as possible in contact with, without touching the ends of the needle—*e.g.* in a miner's dial.

The trough compass (Fig. 2.17), which is in the shape of a long narrow rectangular box or trough, has at each end of the box a small

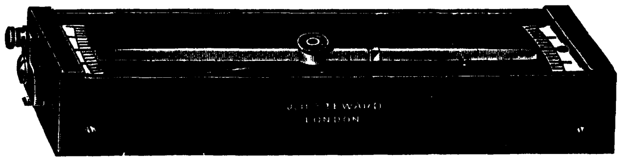


FIG. 2.17.—Trough Compass

flat curved scale, level with the needle and graduated to about 5° only on each side of the centre line.

The magnet is in the form of a thin steel " needle ", supported at the centre of its length on a hardened steel pivot, which fits into a socket consisting of an agate or other hard stone cup mounted in brass. From this pivot the magnet should be raised when the instrument is not in use, otherwise undue wear results in the development of a considerable amount of friction which is undesirable.

The needle is of rectangular cross-section and has the longer dimension sometimes vertical and sometimes horizontal.

The former or edge-bar type is particularly suitable when the scale is level with the extremity of the needle, as there is then no difficulty in estimating the reading, since the sharp vertical edge of the magnet may be almost in contact with the scale (Figs. 2.17 and 2.18, *c*).

The flat needle is more frequently used for the disc type of scale and it is then lozenge-shaped in plan (Fig. 2.18, *a*). In this case the point of the needle cannot be brought into such close contact with the graduations, as the magnet must be sufficiently raised above the card to enable it to swing freely even when the instrument is not held exactly level, *i.e.* when the plane of the instrument is not exactly parallel to the longitudinal axis of the needle. There is consequently considerable liability to error due to parallax, and an incorrect reading may be obtained unless the eye is vertically above the needle. If a plane mirror is fixed below the needle, parallax may be avoided by moving the eye into such a position that the reflection of the needle is exactly covered by the needle itself; this ensures that the eye is correctly placed. Though frequently adopted for scientific apparatus, this simple device is not commonly found in surveying instruments.

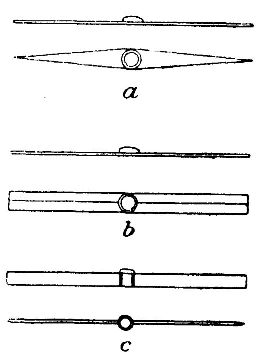


FIG. 2.18
Compass Needles

The edge-bar needle is used occasionally for disc scales in lieu of the lozenge-shaped flat type.

For the raised scales, too, a flat needle is sometimes employed, but then it is generally rectangular in plan, and the reading observed against an engraved mark as in Fig. 2.18, *b*.

Fig. 2.19 shows a section through an improved tubular magnetic compass, readable to within 5 minutes of arc or less, and made by Messrs. Cooke, Troughton and Simms. It is suitable for use as a theodolite fitting (see p. 86).

Declination.—The horizontal direction adopted by the longitudinal axis of the needle at any position—provided there is no local attraction due to the near proximity of magnetic substances—is known as the magnetic meridian, and the angle this line makes with the true or geographical meridian of the place is known as the magnetic " declination " or magnetic " variation ". The distribution of the earth's

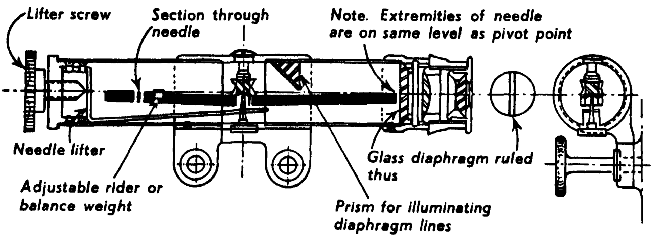


FIG. 2.19.—Tubular Compass

magnetism is by no means regular, consequently lines of equal magnetic declination—known as **Isogonic** lines—do not form complete great circles as do the geographical meridians, but radiating from the north and south magnetic regions they follow irregular paths, which are shown on such charts as those prepared by the Admiralty for the use of mariners and others.

There are two lines of zero declination—or **Agonic** lines—*i.e.* the loci of points at which the magnetic meridian coincides with the geographical meridian. One runs from the north magnetic pole through Canada and the east of the United States, and through Brazil to the south ; the other runs south from the north magnetic pole, through the Gulf of Bosnia, across Europe to the Aegean Sea ; it skirts the Red Sea and then turns sharply towards Hyderabad. It next makes, towards the north, a large almost-closed loop known as the Siberian Oval, on the east coast of Asia, before proceeding across Sumatra and Western Australia to the south magnetic pole. The Isogonic lines over Britain run a little to the E. of N.

In 1925 the work of recording earth magnetism was transferred from Greenwich to Abinger, near Dorking. In that year the magnetic declination was 13°-10′ W. at Greenwich and 13°-23′ W. at Abinger, and a difference of 13′ to 14′ between the two has been found in later years.

The values of the magnetic elements are published by the Admiralty and may be found in *Whitaker's Almanack.*

The magnetic declination at Abinger in 1945 was about 10°-00′ W. and in 1946 probably about 9°-51′, decreasing about 9′ each year. Obviously the magnetic declination at various places on the earth's surface can have any value, particularly in the polar regions.

Near masses of magnetic rock, such as basalt, the local variation of the needle may be very marked, and in such districts the compass becomes quite worthless as a surveying instrument.

It may also be noted here that the magnetic needle may be seriously deflected from its normal position when in the proximity of iron or steel rails, railings, lamp-posts, gas and water pipes, girders, etc., while iron buttons, steel-rimmed spectacles, keys, and other small articles about a surveyor's person may appreciably deflect a sensitive needle.

Inclination or Dip.—If a symmetrical, evenly balanced piece of steel be magnetised and supported at its centre of gravity in such a manner that it is free to rotate in a vertical plane in the magnetic meridian, it will be observed that at most positions on the earth's surface the position of equilibrium of the magnetic needle so formed is not in a horizontal plane, but in a plane inclined at a definite angle to the horizontal. This angle is known as the " Dip " or " Inclination " of the needle. In the northern hemisphere the north end of the needle, and in the southern hemisphere the south end of the needle, is usually deflected downwards.

Lines, known as **Isoclinic** lines, may be drawn on a map to show the loci of places having the same value of dip. These lines, which are more or less irregular, encircle the earth in a similar manner to the geographical parallels of latitude.

The line along which there is no dip is in the vicinity of the Equator; it is known as the Magnetic Equator, and its representation on the chart as the **Aclinic** line.

In England the value of the angle of dip is about 67° (at Greenwich), while at the magnetic poles the needle would be vertical and the dip 90°.

Though of importance for scientific purposes, the *amount* of the " dip " of the needle in various places is seldom measured by a Surveyor. It affects him, however, in so much that a needle which is correctly balanced for one latitude is incorrectly balanced for other regions, and in consequence of this will no longer float in a horizontal plane, and the bearing on the central pivot is unsatisfactory. To remedy this, and to counteract the vertical components of the magnetic forces which are tending to dip the needle, a small sliding sleeve-weight is fitted to the needle. By moving this towards or away from the pivot, the balance may be adjusted to suit varying conditions. (See, for example, Fig. 2.19.)

In order to weight that end of the needle which tends to be raised resort is occasionally made to sealing-wax.

Changes in Magnetic Declination.—Neither the dip nor the declination of the compass has a constant value at a particular place—both are subject to variations. It is the changes in the latter, however,

which are of most importance to a Surveyor, and these may be classed under four heads, viz. :

(*a*) Secular variations.
(*b*) Annual variations.
(*c*) Diurnal variations.
(*d*) Irregular variations.

(*a*) **Secular Variations.**—The magnetic poles are continually altering their positions relatively to the geographical poles, and records over a large number of years show that, due to this cause, the mean declination of a place alters in a more or less regular manner from year to year.

For instance, near Greenwich the earliest records show that in 1580 the declination was about 11°-15′ E. Since that date the declination diminished until in 1657 it was nil, *i.e.* the magnetic and geographical meridians at Greenwich were coincident.

The declination then became westerly, and gradually increased in amount until in 1818 the maximum variation of 24°-38′ W. was reached.

Since 1818 the declination has been gradually decreasing again until in 1918 the mean value was about 14°-25′ W.

The annual rate of change since 1880 has varied between 4′ and 11′, the average at Abinger for the last ten years being about 9′ W. declination decreasing.

In a similar manner corresponding changes, though of different magnitudes, occur at other places upon the earth's surface.

It is obvious from the above account that the direction of the magnetic meridian as represented upon colliery and other plans should be corrected periodically ; otherwise, if data taken with reference to the present magnetic meridian are to be added to plans drawn some years previously, corrections must be applied to all the observations, or the data referred to the true meridian on each occasion.

(*b*) **Annual Variations.**—The term is restricted to those variations in declination which have a yearly period ; their amplitude is small, being little over 2′ in Britain.

(*c*) **Diurnal Variations.**—The extent of the diurnal variation is affected by several factors, such as

(1) The locality—*i.e.* it usually increases as the magnetic poles are approached and the Equator departed from.
(2) The season of the year—being considerably more in summer than in winter ; for this reason, and as the rate of change is much more excessive during the day than during the night (see below), sunlight is supposed to have some influence upon the magnetic elements.
(3) The amount of diurnal variation also changes from year to year.

The rate of change during the twenty-four hours is also variable, the needle being moderately steady during the night. In this country

the declination decreases slightly in the early morning until about 8 A.M., when it attains its minimum value. The needle then moves slowly backwards towards the west, passing through its mean position about 10 A.M. and reaching its most westerly deviation about 2 P.M. It then more slowly oscillates back towards its mean position, which it passes in the evening about 6 to 7 P.M. The extent of the variation in England is roughly 12′ in the summer and 7′ in the winter, though it varies with the phase of the sunspot period.

(*d*) **Irregular Disturbances**, due to what are known as "magnetic storms", cause displacements of the needle through amounts varying in extent up to 1° or 2°; they are considered to be in some way due to solar influence and are often coincident with such phenomena as the "aurora borealis", earthquakes, etc. They are probably associated in some way with sunspots.

CHAPTER III

INSTRUMENTS

THE **vernier** is a device which enables measurements to be taken to a finer degree of accuracy than would be possible by direct estimation of the divisions on the main or primary scale of an instrument, and it is particularly useful when the further subdivision of this scale would give graduations too closely engraved to be easily distinguished. It consists of a small scale which moves with its graduated edge in conjunction with the graduated edge of the primary scale, and its principle may be understood from the following.

Direct Vernier.—If it is required to read to $\frac{1}{n}$th part of a graduation on the main scale, the vernier scale is divided into n equal parts, and has a total length equal to $(n-1)$ primary divisions, so that

$$n \text{ vernier divisions} = n - 1 \text{ primary divisions,}$$

$$\text{and } 1 \text{ vernier division} = \frac{n-1}{n} = \left(1 - \frac{1}{n}\right) \text{ primary divisions,}$$

i.e. the difference between a vernier and a primary scale division is $\frac{1}{n}$th the latter.

In the direct vernier both scales are graduated in the same direction, *i.e.* both from left to right or both *vice versa*, so that if the zero of the vernier coincides with the zero of the primary scale, then

the 1st vernier division will lag $\frac{1}{n}$th of a primary division behind the 1st primary graduation,

the 2nd ,, ,, ,, $\frac{2}{n}$ths of a primary division behind the 2nd primary graduation,

and the nth ,, ,, ,, $\frac{n}{n}$ths of a primary division behind the nth primary graduation,

i.e. will coincide with the $(n-1)$th graduation.

Thus if the vernier is moved forward $\frac{1}{n}$th of a primary division, the 1st vernier graduation will coincide with a primary graduation,

and if it is moved forward $\frac{p}{n}$ths of a primary division, the pth vernier division will coincide with a primary graduation.

If the vernier is in such a position that its zero lies between the mth and $(m+1)$th division on the primary, and the pth vernier division coincides with some line on the main scale, the reading is

$$\left(m + \frac{p}{n}\right) \text{ primary divisions.}$$

It should be noted that it is quite immaterial with *which* primary graduation the pth vernier division happens to coincide; the only graduation on the primary to be noted is that last passed by the zero of the vernier (*i.e.* the mth), and the only vernier division to be noted (the pth) is that which coincides with some one—no matter which—primary division.

In Fig. 3.1 the reading on the primary is 29°-30′ (or $29\frac{1}{2}$°) + 13′ (*i.e.* $\frac{13}{30}$ of $\frac{1}{2}$°) on the vernier, giving a complete reading of 29°-43′.

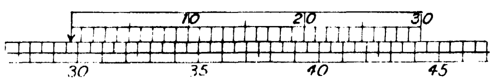


FIG. 3.1—Direct Vernier, reading 29°-43′

Retrograde Vernier.—On this type of vernier n vernier divisions are equal to $(n+1)$ primary divisions, so that 1 vernier division $=\left(\frac{n+1}{n}\right)=\left(1+\frac{1}{n}\right)$ primary division; and the difference is $\frac{1}{n}$th of a primary division as before.

The scales, however, are graduated in *opposite* directions, one from left to right and the other from right to left as in Fig. 3.2, so that if the two zeros coincide, the 1st vernier graduation will lag $\frac{1}{n}$th of a primary division behind a primary graduation, the 2nd will be $\frac{2}{n}$ths behind another graduation, and so on. When the vernier zero is moved forward $\frac{1}{n}$th of a primary division, as in the direct type, the 1st graduation of the vernier will be brought into coincidence with a primary division; and when moved forward $\frac{p}{n}$th of a primary division, the pth graduation will be found to coincide with a primary division.

The advantage of a retrograde vernier is that the graduations are not quite as fine as those of a direct vernier; but the latter is generally employed, as it is, perhaps, rather more simple to read. In Fig. 3.2 the reading given by the primary scale is 3·2 inches, while, as coincidence is obtained on the vernier at ·04, this gives a total reading of 3·24 inches.

The Double Vernier.—When a simple vernier—either direct or retrograde—is used, readings can only be taken in one direction on the scale, unless for each reading the fraction of a primary division,

as indicated by the vernier for forward motion, is subtracted from unity for motion in the reverse direction.

For example, in the case of an instrument the primary scale of which is divided into half degrees, subdivided by means of the vernier to half minutes, if the bearing of a line is 229°-43½′ from left to right, the vernier will read 13½′, while the primary scale will read 229°-30′ for clockwise motion, giving a bearing of 229°-43½′. For counter-clockwise motion—if the observation is made from right to left—the primary scale would read 130°. The vernier reading would be 13½′ as before, and this must be subtracted from 30′—(the value of a primary division) giving 16½′ to be added to the 130°—as the bearing from right to left is 130°-16½′.

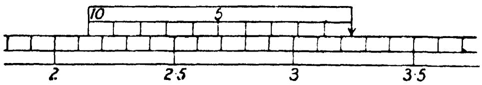


Fig. 3.2.—Retrograde Vernier, reading 3·24 inches

This calculation may be avoided by placing two simple verniers end to end, forming one scale, with the zero in the centre, and using one of these for clockwise motion and the other for counter-clockwise. The combination, which is double the length of an ordinary vernier, is known as a double vernier. The graduations of most theodolites are figured in a clockwise direction only, and fitted with single direct verniers.

A modification of a double vernier, sometimes found on a dial or circumferenter, is shown in Fig. 3.3 where the vernier is at the zero position, and in Fig. 3.4 where the reading is 70°-24′ to the right or 289°-36′ to the left. It will be noticed that the length of the scale is the same as that for an ordinary direct vernier, and as 10 vernier divisions = 9½ primary divisions on each side of the zero, or 20 vernier divisions = 19 primary divisions, the difference is $\frac{1}{20}$th of a primary division; therefore as each primary division represents 1°, the vernier divisions each represent 3 minutes, *i.e.* $\frac{1}{20}$th of a degree.

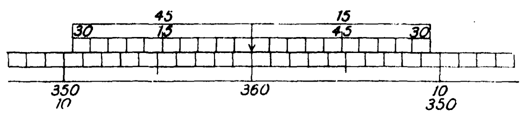


Fig. 3.3—Double Vernier, reading to 3′

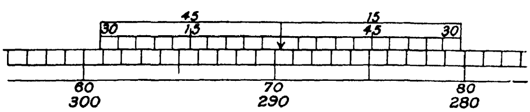


Fig. 3.4.—Double Vernier, reading 70°-24′ or 289°-36′

For motion to the right/left the vernier is read from 0 to 30 at the right/left extremity, and then from 30 at the left/right extremity to the 60 (*i.e.* zero) in the centre.

The Micrometer Microscope.—The micrometer microscope is a device which enables a measurement to be taken to a still finer degree of accuracy than that obtainable with a vernier, and many

theodolites, tacheometers, omnimeters, etc., are now fitted with this appliance.

The micrometer tube consists of an object-glass, an eye-piece, and a diaphragm which is capable of delicately controlled movement at right angles to the longitudinal axis of the tube. Let f_1 be the distance of the object-glass from the graduated limb f_2 the distance of the object-glass from the image which is formed inside the tube, and f the solar focal length of the object-glass lens.

Then, as explained on p. 53, Chapter II, if f_1 has a value between f and $2f$, f_2 is greater than f_1 and a magnified image is obtained, the magnification being expressed by the ratio $\frac{f_2}{f_1}$ or $\frac{f}{f_1 - f}$.

The diaphragm consists of an adjustable slide carrying two hair-lines spaced a very small distance apart. The motion of the slide is controlled by a finely threaded screw which is rotated with the fingers by a milled head on the outside of the tube. Adjacent to the milled head and formed out of the same piece of metal is a disc or drum of larger diameter, having its outer edge accurately graduated so that any portion of a revolution of the screw may be estimated by noting the reading indicated on this by a fixed mark on the frame of the instrument.

On applying the eye to the eye-piece the graduations of the primary scale should be seen quite distinctly, and by correctly focusing the eye-piece any possible error due to parallax is eliminated, as explained on p. 56, by bringing the hair-lines into the same plane as the image, and thus preventing any relative movement of the hair-lines and the image as the eye is moved to look through different parts of the eye-piece lens.

On the central diameter of the field of view is seen a fixed hair-line, or a small V notch, while the two fine hair-lines of the diaphragm are seen parallel to the graduations of the primary scale.

To test the adjustment of the micrometer, the two hair-lines are moved by the rotation of the milled head until they are so situated that, as nearly as can be judged, one of the primary graduations accurately bisects the narrow intervening space.

The reading of the graduated disc (which should be zero when adjustment is made to the fixed hair-line or notch) is then noted, and the screw rotated until some other primary graduation bisects the interval, when the reading on the drum should be the same as before, the screw having made one or more complete revolutions. This operation is known as "taking a run", and several trials should be made to obtain a mean result.

If it is found that one revolution—or some other whole number of revolutions—of the drum does not move the hair-lines through the exact distance represented by a primary graduation, the micrometer is in need of adjustment.

But as the distance moved by the hair-lines in one revolution is determined by the pitch of the screw, and therefore cannot be altered, the adjustment must be made by altering the size of the image of the

primary graduations as seen through the eye-piece, and this can be done by moving the microscope bodily towards or away from the graduated main limb of the instrument.

In other words, the magnitude of f_1 is altered, and this entails a corresponding alteration in f_2 and in the size of the image, as obviously the actual size of the object, *i.e.* of a primary graduation, is a fixed quantity.

Thus, if the image graduations are too large, f_1 is increased ; and if too small f_1 is decreased. This is a permanent adjustment and is effected by unclamping the two capstan-headed screws or other arrangements which attach the microscope to the main instrument frame. The alteration in f_1 and consequently in f_2 necessitates a fresh adjustment for parallax, which is easily effected by focusing the eye-piece.

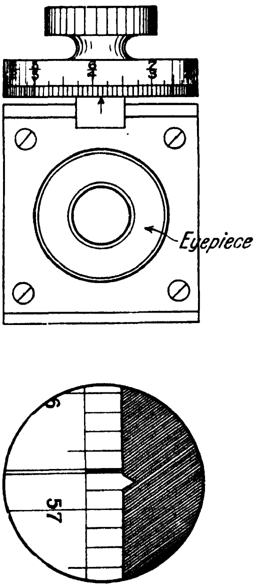


FIG. 3.5.—Micrometer, reading 56°-46′-08″

To use a micrometer microscope.—As a preliminary to the measurement of an angle AOB, say, with a micrometer theodolite, the drum-head reading is adjusted to zero, bringing the hair-lines to the (approximate) centre of the V notch. The zero of the primary scale (or other selected graduation mark) is then made to bisect the space between the two hair-lines, the adjustment being completed with the appropriate clamp and tangent screw arrangement of the theodolite.

The telescope is sighted to A, with the micrometer thus reading zero. It is then directed to B as explained later, and the angular movement is recorded on the scale. In the example in Fig. 3.5 the primary scale is graduated to degrees, subdivided into 10-minute intervals, and the main scale reading as indicated by the V notch is between 56°-40′ and 56°-50′. The micrometer enables such intervals as that between the 40′ and the fixed index mark represented by the V notch, to be measured.

To do this, the drum-head is turned until the 40′ graduation mark is intercepted between the two movable hair-lines in the figure. Now one complete revolution of the drum-head corresponds to 10′ on the primary scale. The edge of the drum is therefore divided into 5-second intervals, which can be further subdivided by estimation. The reading in Fig. 3.5 is 6′-08″ on the drum. The total angle is therefore 56°-40′ plus 06′-08″, *i.e.* 56°-46′-08″.

It will be noted that had the hair-lines been moved so as to be intercepted by the 56°-50′ graduation, the drum-head reading would still be 06′-08″, and so it would whatever graduation were intercepted, provided the micrometer were in perfect adjustment. The angle is, however, always the main-scale reading just short of the index mark plus the drum-head reading.

The particular instrument shown in Fig. 3.5 is graduated in both directions, which is sometimes a convenience; but more frequently only one set of figures are engraved on the drum. It may be noted that no attempt is made to use the notch except as an approximate index: when setting to zero as described above, all that is required is that the hair-lines shall be sufficiently near the apex of the V to ensure that no confusion can arise: the hair-lines need not *exactly* coincide with the apex, but a good correspondence is desirable.

The **clinometer**, of which instrument there are numerous patterns upon the market, is a simple device for measuring vertical angles, and is particularly useful for determining the slope of the ground when chaining, so that the slope measurement may be reduced to its horizontal equivalent without resort being made to " stepping ".

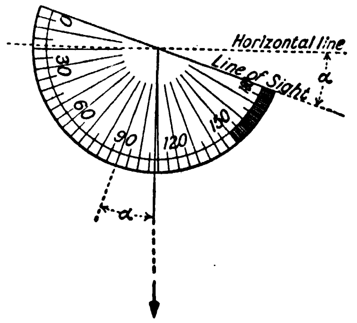
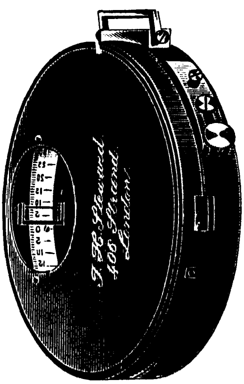


FIG. 3.6.—Simple Clinometer

A crude form may be constructed by suspending a plumb-line in front of an ordinary semicircular or rectangular protractor (Fig. 3.6). A vane is fixed on a rod or staff at the height of the observer's eye, and this is held vertically at the top / bottom of the bank, the inclination of which is required. The observer then stands at the bottom / top of the bank and sights along the top edge of the protractor to the vane, thus making the line of sight parallel to the surface of the ground. The intercept between the 90° graduation and the plumb-line gives the required angle of inclination as in Fig. 3.6.


FIG. 3.7.—Mirror Clinometer

A handy form of mirror clinometer by Steward is shown in Fig. 3.7. The eye is applied to a small hole in the edge of the circular box, and the staff vane or other object is sighted through the rectangular aperture in the opposite edge. This aperture is clearly seen in the figure. Thus the line of sight is again parallel to the ground, but the graduated scale being weighted, swings freely to its normal position, and its reflection in a small mirror can be seen from the eye aperture. At the same time a small indicator bar attached to the outer case is also reflected in the mirror, and it is this reflection which is made to intersect the vane. The intersection of this reflection with that of the scale gives the reading required—the scale graduations for angles of elevation being coloured red, and those for angles of depression, black.

By means of the slide shown above the square aperture in the figure the swinging scale can be thrown off its bearings when the instrument is not in use.

The **hand level**, one form of which, by Stanley, is shown in Fig. 3.8, is a very compact little instrument. It is preferably held against a ranging or other rod to steady it, and at a definite height above the ground. It is sometimes used, in conjunction with a levelling staff, to ascertain the levels of points on short cross-sections, to afford data either for the calculation of quantities, for the location of contour lines, or for other purposes.

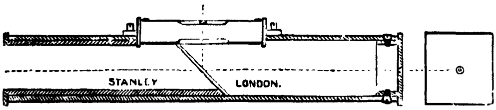


FIG. 3.8.—Hand Level

It consists of a small metal tube of rectangular cross-section to which a spirit-level is attached. The bubble is reflected through an aperture in the top of the tube on to a mirror, which is inside the tube near its centre and inclined at 45° to the axis.

If the mirror occupies the horizontal half of the cross-section its horizontal edge, together with the pin-hole of the eye-piece, defines the line of collimation of the instrument; and the reflection of the bubble must appear to intersect this at the time an observation is being made.

If the mirror occupies a vertical half of the cross-section, the horizontal line which the bubble must appear to intersect, and which defines the line of collimation, is marked either by a horizontal wire or thread, or by the edge of a diaphragm which half closes the end of the tube remote from the eye-piece.

The **Abney level** (Fig. 3.9) consists of a telescopic tube and reflecting level. The telescope is sighted on to a vane as in the case of a clinometer; and the bubble (the reflection of which can be seen in the mirror while an observation is being made, as with the hand level described above) is brought to the centre of its run and intersected by the line of collimation by means of the rack adjustment.

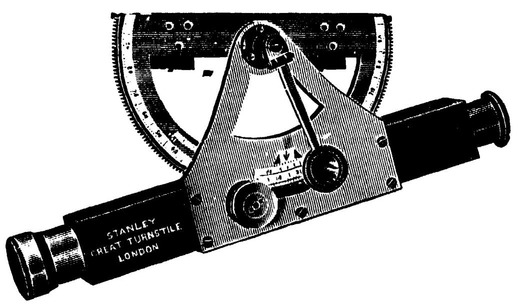


FIG. 3.9.—The Abney Level

The bubble tube is therefore adjusted to the horizontal position while the telescope is parallel to the slope of the ground; and the inclination may accordingly be read from the scale by the aid of the vernier and magnifying glass. On the instrument in Fig. 3.9 the angle may be read to single minutes, but such precision can rarely be attained in the observations.

The instrument is exceedingly handy for cross-sectioning purposes, and for determining—as with a clinometer—the slope of the ground during ordinary chaining operations.

The adjustment of the instrument may be tested by sighting to any well-defined distant object and observing the angle of elevation or depression recorded by the vernier. The instrument is then inverted and the same object again intersected, the bubble being brought to the centre of its run as for the first observation. Evidently the angle of elevation or depression will now be indicated on the opposite side of the zero of the scale. Thus suppose the true angle is $\theta°$ elevation, and the instrument has an index error of $a°$, so that the first reading $\theta_1 = \theta - a$, while the second reading $\theta_2 = \theta + a$.

The mean of these two values is $\frac{\theta_1 + \theta_2}{2} = \frac{(\theta - a) + (\theta + a)}{2} = \theta$, *i.e.* the correct value, and $\theta_1 - \theta = a$ is the amount of the index error.

The correct inclination of a piece of ground may be determined with a faulty instrument, or its index error ascertained by the following alternative method, which is also applicable to the clinometer and other similar instruments.

At the top of the slope is fixed a vertical rod, and attached to it is a horizontal vane at a height h above the ground. At the foot of the slope the instrument is held at the same height h above the ground and steadied against another vertical rod.

The angle of elevation is read as θ_1.

The instrument is next held at the top of the slope and the angle of depression θ_2 observed in the same way to the vane on a staff held at the foot of the slope.

So that, as before, if there is an index error a, the true value of the inclination $\theta = \theta_1 - a$ or $\theta_2 + a$, *i.e.*

$$\theta = \frac{\theta_1 + \theta_2}{2}, \text{ the mean of the two observed values,}$$

while
$$a = \theta_1 - \theta \text{ or } \frac{\theta_1 - \theta_2}{2}.$$

The **prismatic compass** (Figs. 3.10 and 3.11, by J. H. Steward) consists of a shallow cylindrical metal box of $2\frac{1}{2}$ inches to 6 inches diameter, in the centre of which, on an agate bearing, is balanced a circular disc, graduated to single degrees, to 30′ or even to 20′, and attached to a magnetic needle. To prevent undue wear, the disc can be raised off its bearing when not in use, and a small knob is also provided for damping the oscillations and quickly bringing the needle to rest. At one edge of the box, and carrying a hair-line sight, is a hinged frame that may be folded down over the dial when not required, while diametrically opposite to this is a prism, which may also fold over on the outside edge of the box. The graduations of the scale on the dial are reflected from the hypotenusal side of this prism to the eye, and are slightly magnified owing to the shorter sides of the prism being made a little convex (Fig. 2.3, Chapter II).

In some forms of instrument, by means of a spring situated near

the hinge of the hair-line frame, the needle is automatically raised from its pivot whenever the frame is folded over the dial.

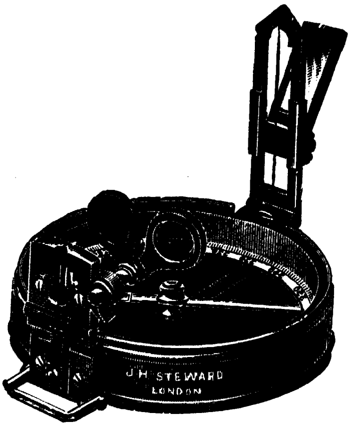


FIG. 3.10.—Prismatic Compass

The top of the box may be covered with a glass lid, so that the whole dial is visible, as in Fig. 3.10, or a metal cover may be provided over all except the small portion under the prism.

Another type is of watch form (Fig. 3.11).

Sometimes dark glasses are provided for use when sighting to a luminous object, and sometimes also a mirror—which can be adjusted to any angle in a vertical plane—is fitted to slide on the frame which carries the hair-line sight.

This enables an object of considerable elevation or depression to be sighted by reflection when it would otherwise be difficult to intersect, because, as the needle must be allowed to swing quite freely, the instrument cannot be tilted to any great extent.

A tripod, also, is sometimes provided, but generally the compass is simply held in the hand as nearly as possible over the station-point at which a bearing is required.

To determine the bearing of a line AB from the station A, the eye is applied to the slit in the prism-holder and the prism raised or lowered in its slide until the dial graduations which are reflected in the prism are clearly seen.

The hair-line is directed to the object B and the disc allowed to swing freely into the meridian, the box being held as horizontally as possible over the station-point A. The object and the graduations can be seen simultaneously and the reading with which the hair-line appears to coincide gives the bearing of AB.

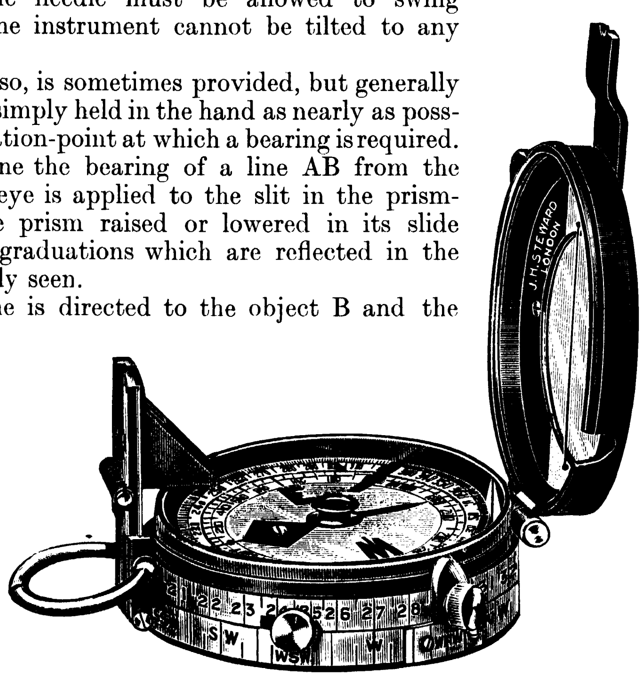


FIG. 3.11.—The "Verner" Luminous Prismatic Compass

When the bearing of AB is due magnetic north, the reading under

the prism should be 360°, so that in consequence the 360° graduation is at the south end of the needle; similarly when AB lies due east, the prism, which would be on the western side of the dial during the observation, should be over the 90° graduation of the disc, and the graduations are therefore as shown in Fig. 3.11.

For rough traverses or preliminary work, and the filling in of detail on topographical surveys, the instrument is very useful, though not as accurate as a theodolite.

In a traverse the bearing of each line is obtained directly from the magnetic meridian, with the result that errors of bearing do not accumulate. The presence of iron or other magnetic substances near a station-point may seriously affect a reading, but if the bearing of each line be observed twice—once from either end—any error due to local attraction may be detected and its amount estimated (see Loose Needle Traverse, p. 128).

For military purposes the prismatic compass has been much used—both for sketching and for night marching. The dial is treated either (1) with a luminous paint or (2) with a radium compound. With the former and more usual type (1) it is necessary to expose the dial for some considerable time to bright sunlight or to the flash of burning magnesium ribbon in order that the dial may be luminous at night-time; but the radium compounds of the latter type preserve their luminosity for an indefinite period and do not need to be excited by a bright light in this way. The glass cover of a military compass (Fig. 3.11) is capable of rotation, and has marked upon it a luminous direction bar or line. During the daytime, to prepare for the night operations, the dial is allowed to swing freely, and the line of sight directed along the route of the march which it is proposed to follow, while the luminous direction line on the glass is turned to lie immediately over the north point of the needle. At night then, if the instrument be so turned that the luminous bar lies over the north point, the sights will be aligned in the required direction, and a succession of short distances can thereby be set out to guide the march.

The **box sextant** (Fig. 3.12) is a very handy and compact little instrument which is used for the measurement of angles. It consists of a cylindrical box of about 3 inches diameter, and—when not in use—is only 1½ inches deep. The lid may be attached to the bottom of the instrument when it serves as a handle, as shown in Fig. 3.12. The small telescope is fixed in position by means of a screw on the top face of the box, and it can be easily removed. Generally the telescope is unnecessary and the eye is applied to a " pin-hole " in the movable slide which covers

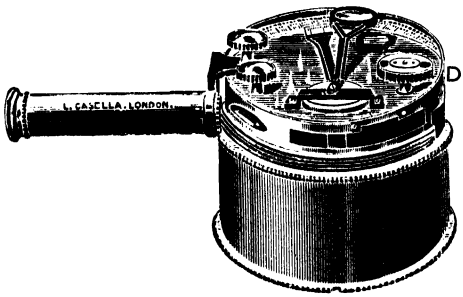


FIG. 3.12.—The Box Sextant

the opening through which the telescope is otherwise directed. In the interior, opposite this aperture, is the " horizon " glass, the upper portion of which is silvered and the lower portion plain.

The second mirror—known as the " index " glass—is attached to a toothed segment which gears with a pinion capable of being turned by means of a large milled-headed screw D on the top of the instrument. Attached to the same axis as this index glass, but on the outside of the box, is an arm carrying a vernier, which moves over a scale graduated on silver to about 140° and reading to single minutes. An adjustable microscope is attached to the top, and a pair of coloured glasses are provided in the interior for use when observations are being made to the sun or other bright object.

To use the instrument, it is held in the right hand, directly over the station-point A at which the angle is to be measured. Under normal conditions the left-hand object B is then sighted by direct vision through the lower or clear portion of the " horizon " glass, and the milled-headed screw D turned until at the same time the image of the right-hand object C is viewed in the silvered upper portion of the glass. When the two objects B and C apparently coincide, the reading indicated by the vernier on the top should give the value of the angle BAC.

It may be proved that the angle between the mirrors is half the angle between the objects, so that the scale on the top, though graduated to say 140°, is really an arc of only 70°.

Let HR and IN (Fig. 3.13) represent the horizon and index glasses respectively, and A the position of the eye.

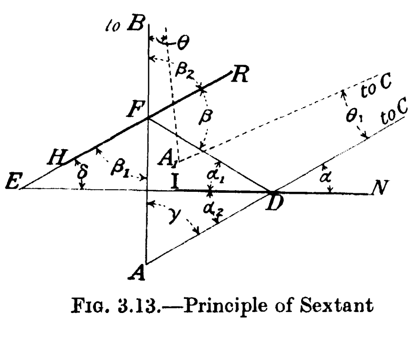


FIG. 3.13.—Principle of Sextant

The pole B is seen through the lower portion of HR, AB being the line of sight, while the image of C, after reflection from the index glass at D, is seen at F in the same line AB.

Let CD be produced to meet BF at an angle γ at A; and let RH meet NI produced at an angle δ at E.

The complement α of the angle of incidence of the ray CD on the mirror IN is equal to the complement α_1 of the angle of reflection; and the complement β of the angle of incidence on the mirror RH is equal to the complement β_1 of the angle of reflection.

Also the angle AFH (β_1) is equal to the vertically opposite angle BFR (β_2); and similarly the angle ADE (α_2) $=\alpha$, *i.e.*

$$\alpha = \alpha_1 = \alpha_2 \quad . \quad . \quad . \quad (1)$$

and

$$\beta = \beta_1 = \beta_2. \quad . \quad . \quad . \quad (2)$$

But the exterior angle (β) of the triangle EDF is equal to the sum of the interior and opposite angles, *i.e.*

$$\beta = \delta + \alpha_1,$$

or from (1) multiplying by 2

$$2\beta = 2\delta + 2\alpha. \qquad . \qquad . \qquad . \qquad (3)$$

Also the exterior angle BFD of the triangle ADF is equal to the sum of the interior and opposite angles, *i.e.*

$$(\beta_2 + \beta) = (\alpha_1 + \alpha_2) + \gamma,$$

or from (1) and (2) $\qquad 2\beta = 2\alpha + \gamma. \qquad . \qquad . \qquad . \qquad (4)$

Therefore from (3) and (4) $\quad 2\delta + 2\alpha = 2\alpha + \gamma,$

$$\left.\begin{array}{ll}\text{or} & 2\delta = \gamma \\ \text{or} & \delta = \dfrac{\gamma}{2}\end{array}\right\}, \qquad . \qquad . \qquad . \qquad (5)$$

i.e. the angle between the mirrors (δ) is half the angle between the rays (γ).

It will be seen in Fig. 3.13 that it has been assumed that the point of intersection, A of CD and BF, is the position of the eye, and that it coincides with the station-point at which the angle subtended by BC is required. The eye may, however, be at any point along FA and would not necessarily lie at the intersection of BF and CD.

Generally, also, the station-point would be at some position A_1, immediately below the instrument itself, so that an error is introduced due to the observation of the angle BAC instead of the angle BA_1C.

The difference between the observed angle BAC and the correct value of the angle BA_1C is the sum or difference of the two angles ABA_1 (θ) and ACA_1 (θ_1), according to the side of AB on which A_1 falls ; and it is just as likely that it shall fall upon one side as the other.

But the displacement (r) of A_1 from the line of direct sight AB is likely to be small compared with that (r_1) from the reflected ray CA, as A lies altogether outside of the instrument in the latter direction.

Hence as $\theta = r/BA_1$ and $\theta_1 = r_1/CA_1$ approximately, it is evident that the value of the error ($\theta_1 \pm \theta$) is chiefly dependent upon θ_1, and for this to be small, CA_1 should be large. So that when an angle is to be measured between two objects, one of which is near and the other more distant, it is advantageous to sight to the former directly, and to view the latter by reflection ; and to do this the instrument should be inverted if necessary.

With an instrument of 3 inches diameter, taking the displacement of A_1 perpendicular to CA as 1 inch say, the value of θ is $1\frac{1}{2}$ mins. when CA_1 is 200 ft., and $14\frac{1}{2}$ mins. when CA_1 is 20 ft.

When the angle BAC is greater than the range of the scale, or even when approaching its limit of 140°, the method of measurement adopted is to subdivide the angle into two portions, by means of any arbitrary intermediate station, marked by a ranging rod or a well-defined permanent feature in the distance, *e.g.* a spire.

The value of each partial angle is observed in the usual manner and the results added.

When the two stations B and C are at considerably different altitudes, the horizontal projection of the angle BAC cannot be easily

determined with this instrument, as, if the box is tilted, the angle on an oblique plane is obtained.

The value of this inclined angle may be measured and reduced to the horizontal if desired, but for all ordinary surveying purposes the instrument is held in a strictly horizontal position and the horizontal angles determined directly.

The adjustments of the instrument are two in number :

(1) The horizon glass should be at right angles to the plane of the instrument, as is the index glass, which is permanently fixed so by the makers.

(2) The vernier should read zero when the image of a distant object, in the upper portion of the horizon glass, coincides with the same object as seen by direct vision in the lower portion, the two mirrors then being parallel.

To test these adjustments, the vernier is set to the zero of the scale and a distant object sighted.

If the adjustments are both correct, the portion of the image seen in the upper silvered portion of the glass should form an exact continuation of that portion of the object seen by direct vision in the lower portion of the glass, *i.e.* the object and its reflection should coincide exactly, so that if the sun is sighted, for instance, one perfect orb should be obtained, part by reflection and part by direct sight.

Any vertical displacement is due to the horizon glass not being perpendicular to the plane of the instrument, and this is remedied by means of the key which is provided for the purpose, and which, when not in use, screws on to the top of the box. The key is applied to the two square-headed screws which are situated near the end of the graduated scale on the top of the instrument and attached to the carrier of the horizon glass.

Any horizontal displacement is due to the fact that the horizon glass is not parallel to the index glass (in plan), and this may be remedied by applying the key to a specially provided square-headed screw in the side of the box, and by this means rotating the horizon glass into its correct position.

Sometimes instead of correcting this adjustment, the index error is ascertained by noting the position of the vernier on the scale, after the lateral displacement is eliminated by the rotation of the milled-headed screw D.

If a spire is being observed, an error in the first adjustment would produce an effect as shown in Fig. 3.14 (*a*) or (*b*).

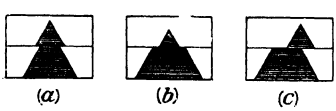


FIG. 3.14

While an error in the second adjustment would cause a displacement such as that in Fig. 3.14 (*c*).

Care must be taken when testing these adjustments that a *near* object s not sighted, for if coincidence is then obtained the mirrors will not be parallel.

Thus in Fig. 3.15, using the same notation as in Fig. 3.13, if the image and object appear to coincide at F, BDFA represents the path of the reflected ray, and BFA that of the direct ray.

Let BFA make an angle β_2 with HR and of ϕ with NI, and let the angle ABD be γ.

Then $\alpha = \alpha_1$ and $\beta = \beta_1 = \beta_2$ as in Fig. 3.13,

the external angle	$\alpha = \phi + \gamma$	(6)
and the external angle	$(\beta_2 + \beta) = \phi + \alpha_1$,	(7)
so that from (6) and (7)	$2\beta = 2\alpha - \gamma$.	(8)

The condition for the mirrors to be parallel is that α_1 shall be equal to β (*i.e.* the alternate angles made by FD crossing HR and IN (Euclid, I. 29)), and this is only satisfied when B is sufficiently distant for γ to be negligible.

By comparison with Fig. 3.13 and equation (5) it will be seen that, if produced, FR and IN would meet, but in the opposite direction to the case there shown, and the included angle $\delta = \frac{\gamma}{2}$.

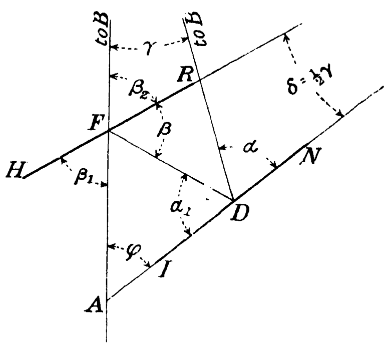


Fig. 3.15

The reading on the scale therefore should be $-\gamma$, in order that correct values should be obtained for observations to more distant objects, *i.e.* if the scale is now adjusted to zero, the values of angles taken in general field work will be too great by an amount γ.

If AB is very small, *e.g.* 20 ft. and the distance of D from the direction of AB is, say, 1 inch, the value of γ will be about 14 mins. ; if AB is 100 ft., γ will be 3 mins. ; and if 1000 ft., γ will be 18 secs.

There are several other important sources of error, among which may be mentioned :

(i) That due to the centre of rotation of the vernier arm, not coinciding exactly with that of the index mirror (compare p. 112) ; or

(ii) Not coinciding with the centre about which the graduated scale arc is described.

(iii) Imperfections in the graduations themselves (compare p. 117).

(iv) Want of rigidity, etc., in parts of the instrument.

The **nautical sextant**, which is similar in principle to the box sextant, is shown as made by J. H. Steward in Fig. 3.16. The silver arc is of large radius and the vernier, which is provided with a clamp and tangent screw, may read to 10 seconds. The horizon glass is opposite the telescope, and the index glass rotates with the arm to which the vernier is attached.

To test if the index glass has been correctly placed perpendicular to the plane of the instrument, the vernier may be set roughly at 45° to 60° on the arc, when the observer glances obliquely in the index glass mirror from beyond the pivot side of the sextant until he sees in it the reflection of the graduated main arc.

If now on moving the index arm to and fro the silver arc and its reflection appear to form one continuous curve, the adjustment is

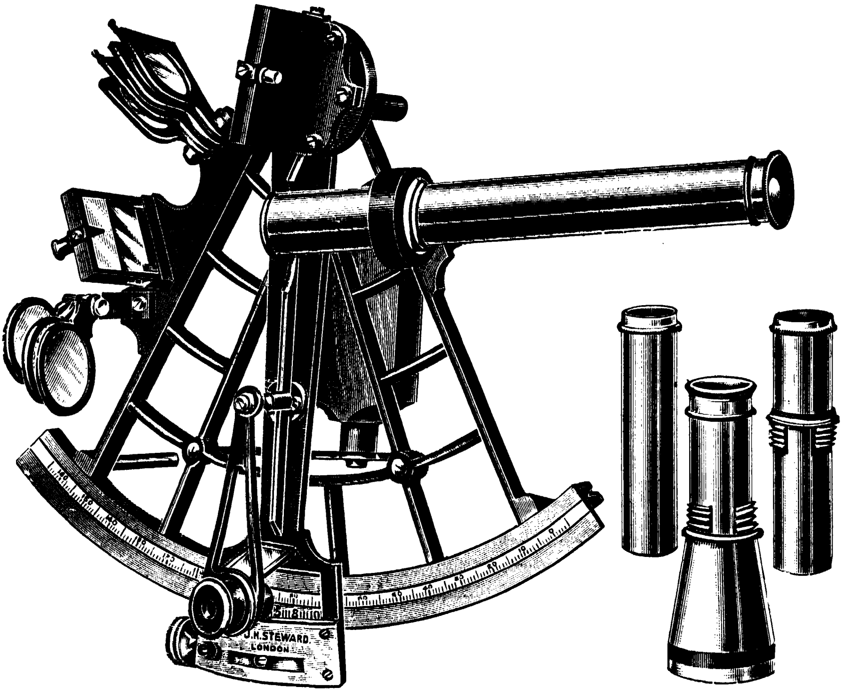


Fig. 3.16.—Nautical Sextant

correct. If not, and if means of adjustment are not provided, the instrument should be returned to the makers for adjustment.

The horizon glass is provided with special adjusting screws, and the adjustments are similar to those of the box sextant.

There is frequently an extra adjustment to ensure that the axis of the telescope is parallel to the plane of the instrument.

To test this, the eye-piece is focused clearly, and rotated in its socket until the two hair-lines are parallel to the plane of the instrument, so that when measuring a horizontal angle the hair-lines are horizontal.

The angle between two distant objects not less than about 90° apart is observed in the usual manner by obtaining coincidence between one of the objects and the image of the other. If this coincidence is obtained upon the lower (say) of the hair-lines, then on slightly tilting

the plane of the instrument to bring the objects on to the second hair-line they should still appear to coincide.

If there is a divergence—which means that the observed value of the angle would be different for coincidence obtained upon the different hair-lines, the telescope must be adjusted by means of the specially provided opposing screws, which attach the telescope support to the frame of the instrument.

A method of determining the index error by observations upon the sun is sometimes adopted. The angle subtended by the diameter of the sun is measured several times upon the ordinary graduated arc of the instrument, and the mean result—say α'—adopted. The angle is then measured an equal number of times upon the " arc of excess ", which is a small graduated length of arc below the zero, and in the direct continuation of the ordinary scale. Let the result in this case be α_1'.

The angle subtended by the diameter of the sun will be equal to $\left(\frac{\alpha+\alpha_1}{2}\right)'$, and a mean value [1] for this is about 32′-02″·36.

The index error is equal to $\left(\frac{\alpha-\alpha_1}{2}\right)'$; this must be subtracted from the sextant readings if α is greater than α_1, and added if α is less than α_1. (*Vide* Example 3, p. 82.)

Artificial Horizon.—The altitude of a celestial body is determined at sea by measuring the vertical angle upwards from the visible horizon, and afterwards applying a correction for dip, the magnitude of which depends upon the height of the observer above sea-level.

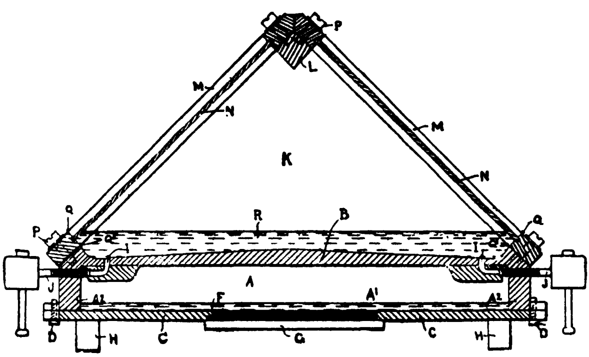


FIG. 3.17.—" Shadbolt " Patent Artificial Horizon

On land this method is obviously impossible, so that an " artificial horizon " is sometimes employed. It consists essentially of a trough containing mercury, and is sometimes protected by a roof of two glass plates inclined at 45°. The surface of the liquid thus automatically adjusts itself so as always to form a horizontal mirror. Fig. 3.17 gives a section through Steward's " Shadbolt " type, which dispenses with a separate mercury reservoir.

The angle observed is that between the object S and its reflection S_1 in the mercury, and it is double the required angle of altitude.

Thus let AB (Fig. 3.18) be the surface of the liquid ; S the star or

[1] Vide *Nautical Almanac.*

other object whose altitude is required; S_1 its reflection; and E the position of the eye.

The observed angle is SES_1, while the true altitude is the angle SEA_1 or the angle SS_1A—these angles being practically equal on account of the great distance away of S.

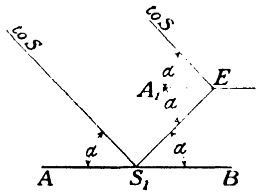


Fig. 3.18.—Artificial Horizon

But the complement of the angle of incidence, *i.e.* $\angle SS_1A$ = the complement of the angle of reflection, *i.e.* $\angle BS_1E$, and $\angle BS_1E = \angle S_1EA_1$, because AB and EA_1 are both horizontal.

Therefore $\angle SS_1A = \angle S_1EA_1 = \frac{1}{2}\angle SES_1$, *i.e.* the required altitude is half the observed angle SES_1.

The various corrections to be applied in the case of astronomical observations will be considered later.

The **dial**, of which two types by Davis & Son are shown in Figs. 3.19 and 3.20, consists of a large compass box usually of 5 to 6 inches diameter. It contains a long magnetic needle balanced upon a pivot in the centre, and from which it may be raised when not in use by means of a small lever.

Below the needle is a fixed circular disc, divided into quadrants, and graduated in tens of degrees in both directions from the north and south respectively. On the same level as the extremities of the needle and inside the box is another scale, divided into single degrees continuously from 0° to 360° in a clockwise direction; to subdivide this, a vernier is attached to the inside edge of the box. Sometimes the vernier is used in conjunction with a scale upon the outside of the box as illustrated. This is very convenient when the head-room is limited.

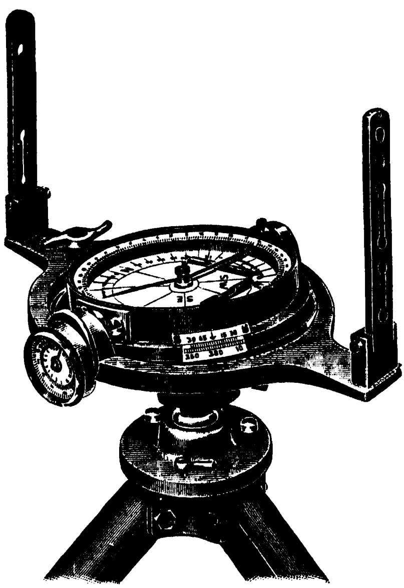


Fig. 3.19.—Standard Hedley Dial

The tripod head is fitted with a ball and socket joint, and with clamping screws for fixing in any position the axis to which the head of the instrument is attached. The whole head can be rotated about this axis, and clamped.

Also, by means of the large milled-headed racking screw on the under side of the box, the frame of the instrument, carrying the sights and telescope (if any), and the outer part of the box carrying the vernier, can be rotated relatively to the inner part of the instrument, which is attached to the central axis, and carries the needle and scales.

By means of a clamp—corresponding with the upper parallel plate clamp of a theodolite—this relative motion can be prevented when necessary, and the vernier and scale fixed relatively to each other.

To enable the vernier to be adjusted quickly to 360°, even when it may be difficult or impossible to see the scales, a small plug is provided to fit into a corresponding hole in the bottom of the box.

On the frame of the instrument or inside the dial are two spirit-levels at right angles to each other, and at the extremities of the arms are hair-line sights, which may be folded down when not in use. These are shown in Fig. 3.19 which represents the "Davis-Derby" pattern of a standard Hedley dial. Usually in each is (1) a narrow vertical slit; (2) a wider opening down the centre of which is a vertical hair-line; (3) a small circular aperture; and (4) a larger circular aperture provided with a horizontal and a vertical cross-hair. Those openings *without* hair-lines in the one sight are placed opposite the openings *with* hair-lines in the other sight, and the eye is applied to the former for sighting purposes—the long aperture being used for movements in azimuth and the smaller for vertical angles (dip, etc.).

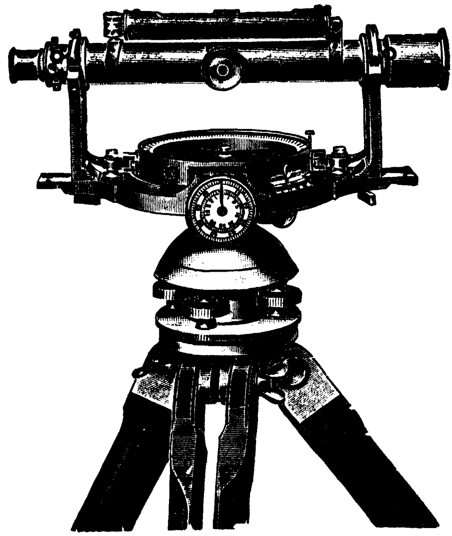


FIG. 3.20.—Davis's improved Hedley Dial fitted with Hoffman Patent Joint, and Four Levelling Screws

In the design shown in Fig. 3.20, a telescope in upright supports has been substituted for the simple sights, the two arrangements being interchangeable. As in this instrument, the telescope may have internal focusing and be provided with stadia lines in the diaphragm.

In Figs. 3.19 and 3.20 a circular attachment is fixed on the side of the dial to indicate vertical angles, while in other forms these angles are shown by a pointer which travels over a graduated horizontal arc inside the compass box.

Sometimes a semicircular attachment is used in place of the circular box shown in the figures, and sometimes the rotating frame is made to indicate vertical angles upon a narrow folding graduated arc which may be erected over the dial when required.

It will be noticed that on the disc below the needle the N.S.E.W. symbols are arranged as in Fig. 3.21.

Thus if the frame carrying the vernier and sights is clamped to

the inner portion of the instrument carrying the scales, etc., then if the line of sight is directed towards the magnetic north, when the zero of the vernier and scale are coincident, the N. end of the needles will lie over the N. graduation of the disc. When the line of sight is turned through an angle a towards the west, say, the needle of course remains fixed in direction, but the disc is carried with the sights through the angle a in a counter-clockwise direction.

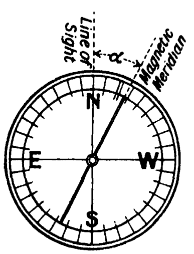


Fig. 3.21

It is thus evident from Fig. 3.21 that if the value of this angle is to be shown *below the needle*, it is necessary that the E. and W. symbols shall be reversed from their usual positions on a map.

EXAMPLES

1. (I.C.E.) Show how to construct the following verniers ·
   (1) To read to 10″ on a limb divided to 10 minutes;
   (2) To read to 20″ on a limb divided to 15 minutes;
   (3) The arc of a sextant is divided to 10 minutes. If 119 of these divisions are taken for the length of the vernier, into how many divisions must the vernier be divided in order to read to (*a*) 5 seconds, (*b*) 10 seconds ?

2. The angle ABC as measured with a sextant is found to be 75°-37′. What is the horizontal value to the nearest minute of this angle, if the horizontal projection of AB is 150 ft. and that of BC 250 ft. ? The instrument at B is 10 ft. above the point sighted at A and 5 ft. below that sighted at C.

What would be the displacement of C from its true position relative to AB if the measured angle ABC were plotted instead of its horizontal projection ?

3. (I.C.E.) The following angular measurements of the sun's diameter have been made " on " and " off " the arc with a nautical sextant :

On the Arc	Off the Arc
35′-30″	29′- 5″
35′-20″	29′-15″
35′-25″	29′-10″

What is (*a*) the angle subtended by the sun's diameter ; (*b*) the index error of the sextant ?

State whether this error must be added to, or subtracted from, readings in order to obtain true angles.

4. (U. of B.) The angle BAC was measured in the ordinary manner with a theodolite, and the reading found to be 70°-30′. The reduced level of the instrument axis at A was 57·5 feet O.D., and that of the observed points at B and C 67·5 ft. and 42·5 ft. O.D. respectively. The length of AB was 100 yds., and that of AC 200 yds.

What result would have been obtained had a sextant been employed instead of a theodolite ?

CHAPTER IV

THE THEODOLITE

THE **theodolite**, of which the " Transit " is the main type, is the most accurate instrument used by Surveyors for the measurement either of horizontal or of vertical angles.

The Transit Theodolite.—One type of transit theodolite as made by Stanley is shown in Fig. 4.1, and another model made by Reynold is shown in Fig. 4.2.

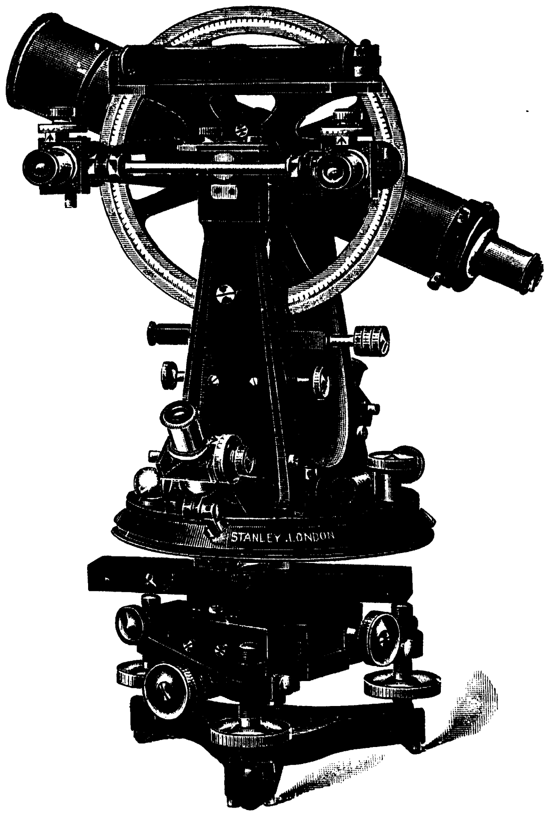


FIG. 4.1.—Stanley's Micrometer Transit Theodolite

The instruments of different makers and the different models of each maker vary in many of their details, but the essential parts are the same in each. Fig. 4.3 refers to an early pattern of instrument but it shows the principal parts very clearly. The telescope is fitted with an object-glass, eye-piece, and diaphragm as described in Chapter II, and is supported by means of an axis at right angles to its length, on two upright A frames. Rigidly attached to the same axis is a graduated vertical circle; and the telescope—carrying this with it—is capable of making a complete revolution in the vertical plane, the movement through a half-revolution being known as " transitting ".

The horizontal axis can also be lifted from its bearings and replaced end for end in the A's, so that the vertical circle may be on the right

(face right) or on the left of the telescope (face left). If the same object is sighted and the A's are not moved, it is obvious that in one case the telescope is upside down.

The T frame (Fig. 4.4), which carries at the extremities of its horizontal limbs the two verniers for reading the vertical circle scale, is centred on the transverse horizontal axis of the telescope. It is prevented from rotating with the telescope and vertical circle by means of the two screws (H, Figs. 4.3 and 4.4) at the lower extremity of the vertical limb, which bear on a nib projecting from the lower cross-piece of the A support.

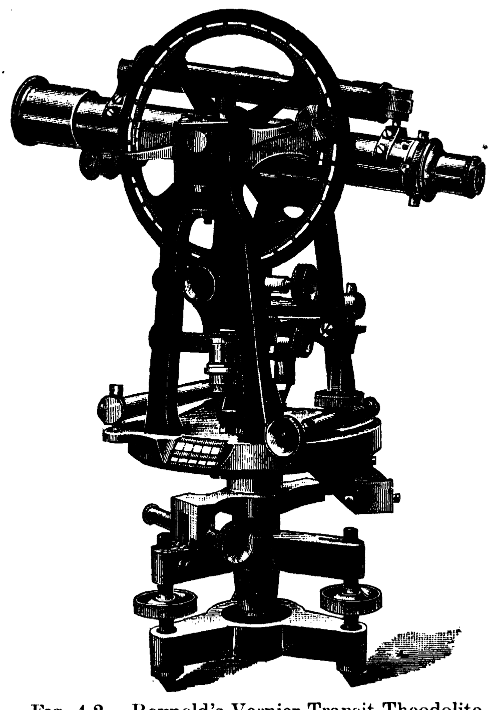


Fig. 4.2.—Reynold's Vernier Transit Theodolite

Fixed to the top of the T frame as in Fig. 4.1, or to the telescope itself as in Fig. 4.2, is a long sensitive bubble tube.

The A frames stand upon a circular flat plate whose edge is fitted with two verniers 180° apart, or with three verniers 120° apart. This vernier plate is rigidly attached to the central axis Z of the instrument (Fig. 4.3). In many modern instruments the A (or U) frames are in one piece with the axis Z.

The verniers, which may read to, say, 20 seconds on a 6-inch or to 1 minute on a 4-inch or 5-inch instrument, are provided with small adjustable microscopes to facilitate accurate reading.

Sometimes micrometer microscopes are provided as in Fig. 4.1 (see p. 66, Chapter III), when a 6-inch instrument may read to 5 seconds, and a 5-inch instrument to 10 seconds. Further subdivision is possible by estimation.

Beneath the vernier plate is the scale plate S, the edge of which is graduated on silver and protected by a sheath from the upper plate, except where the verniers are in contact with the scale. It is the diameter of this plate which defines the size of the theodolite, *i.e.* 4-inch, 6-inch, 8-inch, etc. The scale plate is supported by a hollow axis which fits accurately around the central axis, and rests upon the boss of the upper parallel base plate L (Fig. 4.3).

The lower parallel plate (N) may be screwed or otherwise fixed on to the tripod head, and in the older forms of instrument is so shaped

as to form a ball-and-socket joint with the lower ends of the two axes above mentioned.

In the newer models, such as Figs. 4.1 and 4.2, the two plates are

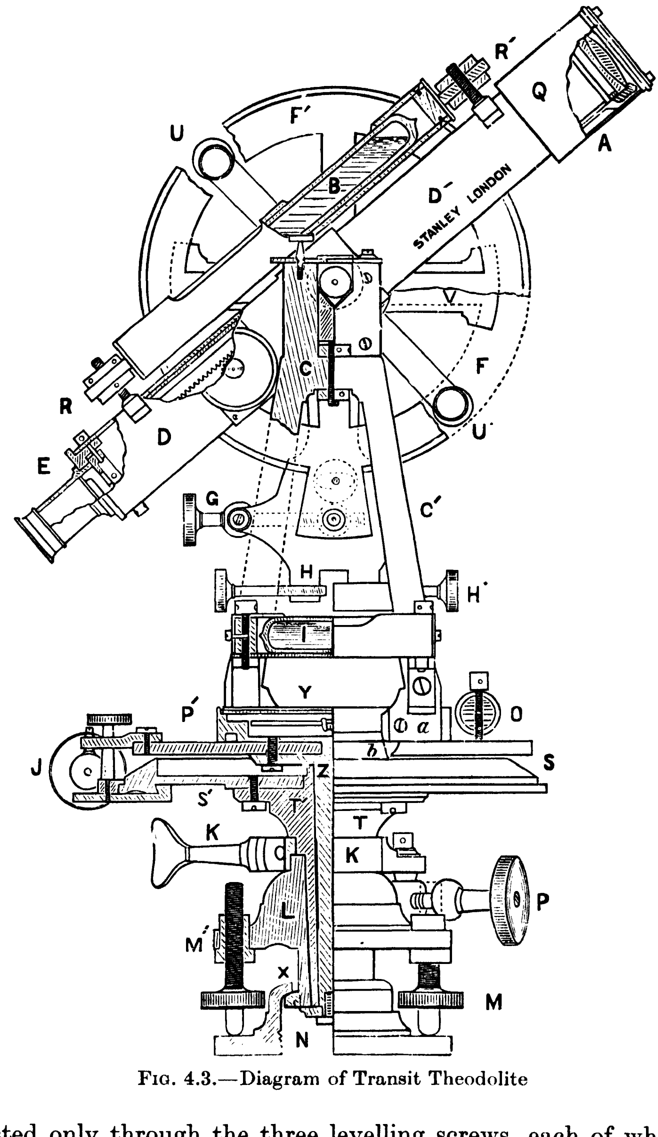


FIG. 4.3.—Diagram of Transit Theodolite

connected only through the three levelling screws, each of which has a ball-and-socket joint at its foot.

The lower plate has a central aperture through which a plumb-bob may be suspended from a hook on the bottom of the central axis.

The upper parallel plate is supported by means of three or four

"levelling" screws, and rests on the lower parallel plate, relative to which any horizontal movement of rotation is prevented: in a four-screw instrument, by a stop which fits round the foot of one of the screws, and in a three-screw instrument by the three ball-and-socket joints above mentioned.

The tangent screw P (Fig. 4.3) connects the boss which is shown suspended from the band K to a boss (not shown) fixed on the plate L. K is thus fixed relatively to L, unless the distance between the bosses is altered by screwing or unscrewing P.

When the band K is loosened, the support T may rotate inside it relatively to L, but if the key K is tightened the band grips the support T (not L), so that there can then be no movement of T and S relatively to L except through the tangent screw P.

The "clamp and tangent screw" arrangement connecting the scale and vernier plates at J is similar. Thus:

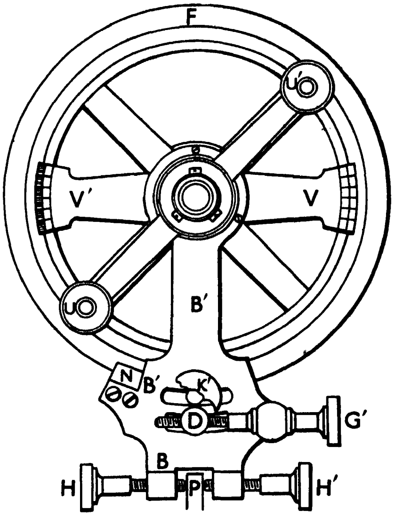


FIG. 4.4

(i) The vernier plate, carrying the A frames, telescope, etc., may rotate alone about the inner axis Z, the scale plate (S) being clamped to L and held stationary meanwhile; or

(ii) The vernier and scale plates, being clamped together, may rotate simultaneously about the axes T and Z.[1]

The instrument is fitted with a compass, sometimes of the circular box type, when it is placed between the feet of the A frames; and sometimes of the trough type attached to the underside of the scale plate as in Figs. 4.1 and 4.2.

To enable the instrument to be quickly levelled, two bubble tubes are fixed at right angles to each other upon the upper surface of the vernier plate.

For underground or night-work one arm of the horizontal axis is sometimes made hollow, to allow of the illumination of the diaphragm webs, from a small lamp carried on a bracket attached to the A frame.

The "**Tavistock**" **Transit Theodolite**[2] made by Messrs. Cooke, Troughton and Simms, originated in a conference held at Tavistock, in 1926, to draw up a specification of a first-class instrument embodying

[1] Other clamp-and-tangent screw arrangements are shown in Figs. 4.3, 4.4, 4.5. See *Surveying Instruments* (Stanley).

[2] "The Tavistock Theodolite", *Engineering,* May 29, 1931; "The Tavistock Theodolite", *The Geographical Journal,* May 1931; "The 'Tavistock' Transit Theodolite", *The Canadian Surveyor,* Oct. 1930.

modern improvements. Representatives of the War Office, Admiralty, R.E. Board, Ordnance Survey, etc., and the better-known English instrument-making firms collaborated.

The general view of this instrument, which weighs only 11·75 lbs., is shown in Fig. 4.5, and a vertical section in Fig. 4.6. It resembles the continental types such as Zeiss[1] and Wild in several respects. The

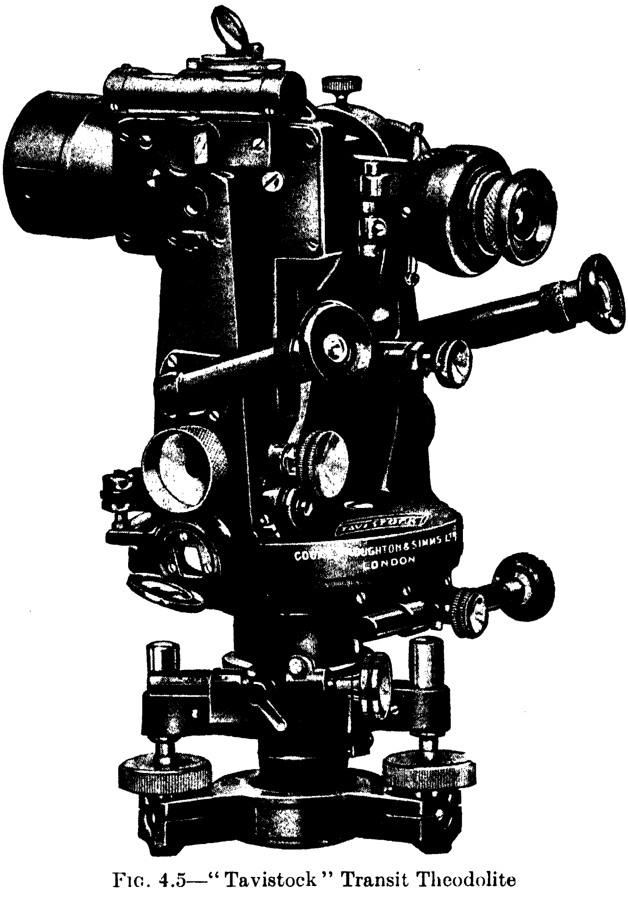


FIG. 4.5—"Tavistock" Transit Theodolite

horizontal and vertical circles are of glass, $3\frac{1}{2}$ in. and $2\frac{3}{4}$ in. in diameter respectively. Each is divided to 20 minutes on the underside of the glass, and this is silvered: each is read direct to 1 second by special micrometer-reading devices. The reading eye-pieces for the horizontal and vertical circles are shown on the right and left, respectively, of Fig. 4.6, and the milled heads for operating the circle micrometers are mounted immediately below. The eye-pieces are pivoted and can be swung to any convenient reading position.

[1] "The Zeiss Universal Theodolite", *Engineering*, Dec. 6, 1929.

The diagrams Figs. 4.7 and 4.8 are typical views of the circle divisions and the fine reading scales as seen through these eye-pieces.

The fiducial line is represented by the thick line in the small middle opening (3), the fine lines on either side of it represent graduation marks

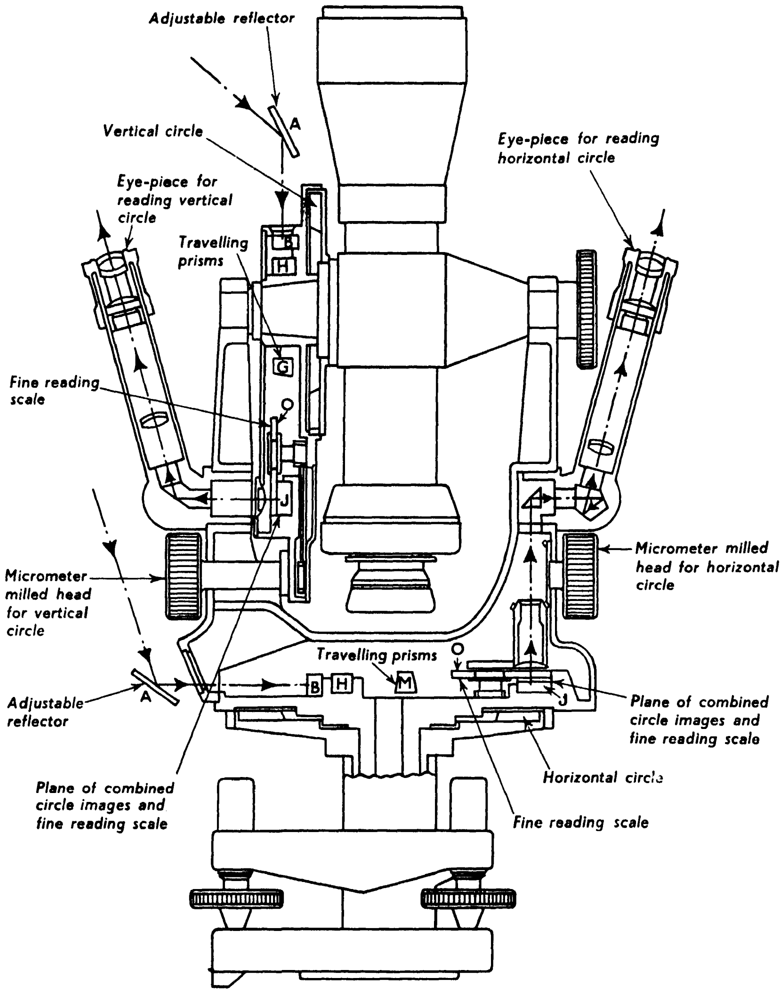


FIG. 4.6.—Tavistock Theodolite Part Section

180° apart on the scale. To " set the micrometer " the appropriate milled head is rotated, and these two fine lines, which remain a definite distance apart, are moved laterally until they lie symmetrically about the fiducial line. In this way, the two readings on opposite sides of the graduated circle are automatically averaged, and centering errors are eliminated.

The telescope, which transits at both ends, has internal focusing, and the optical system is anallactic : the glass diaphragm has stadia lines in addition to the cross-lines : it can readily be removed and replaced by a " spare " without affecting the collimation. The spirit-levels have the following sensitivity (see p. 203) : plate, 40 secs. per 2 mm. run ; altitude, 20 secs. per 2 mm. run. The latter level is provided with a prism reader, and the bubble is observed from the eye-piece end of the telescope.

The optical system is described in the literature issued by the makers of the instrument, and the following account is based on it.

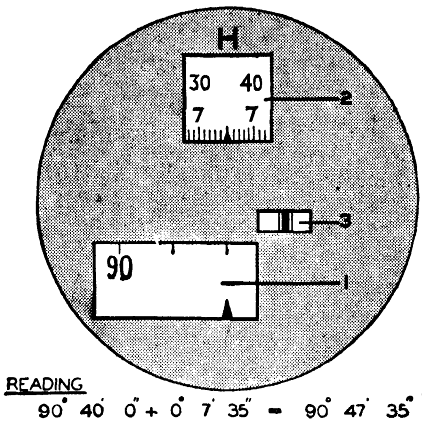


FIG. 4.7

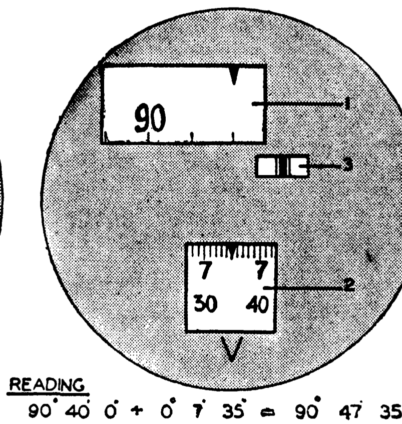


FIG. 4.8

We are indebted also to Messrs. Cooke, Troughton and Simms for permission to reproduce the relevant diagrams and illustrations.

In Fig. 4.10 the scales to be read are at F, on the two opposite sides of the graduated horizontal circle. The eye is above the plane of the paper at J and the source of illumination is at A on the opposite side of the instrument to J.

Rays of light from the bright sky or an artificial source, are reflected by the adjustable mirror A, into the instrument. Half are reflected to the right and half to the left, by the double reflecting prism B, and from B into the prisms C. At C, the rays falling on the sloping face D are deflected downwards through the small opening E on to the scale F.

After reflection upwards from the silvered surface, the rays again meet the sloping face D and are reflected towards G. They are reflected through 90° at G, and a further 90° at H, into the field prisms J. At J the rays from the two opposite sides of the instrument are reflected inwards, *i.e.* brought towards one another, and then reflected upwards to the eye of the observer. The observer then sees the two openings E illuminated by the light from A, with the lines and figures of the scale appearing as dark marks against the lighter background.

Objectives K are introduced where shown, and these form the image of the graduations at E, at the focal plane L of the prisms J.

The images from the two sides of the instrument are brought together and appear as the straight scale in Fig. 4.9. The graduations on both halves appear to increase numerically from left to right as in the figure. The two scales are not quite coincident, *i.e.* the 91° graduation does not quite coincide with the 271° graduation in Fig. 4.9, but there is a gap between them sufficiently wide for the fixed index line to be accommodated between them as shown in the small window. This fiducial line is at the junction between the two prisms at J. The main scale as shown is divided into degrees and 20 minute intervals. The overlap between the scales remains constant, but unless the readings have been set at an even 20′ division, the two graduations will not lie evenly about the fiducial mark—in fact only one line might be visible

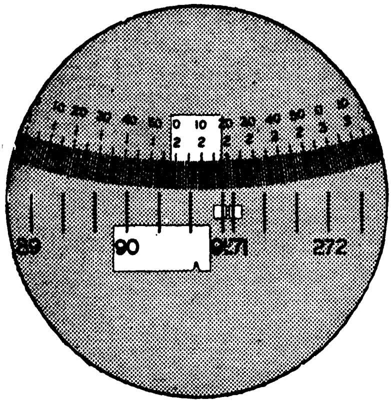


FIG. 4.9.—Tavistock Theodolite. Reader's Apertures

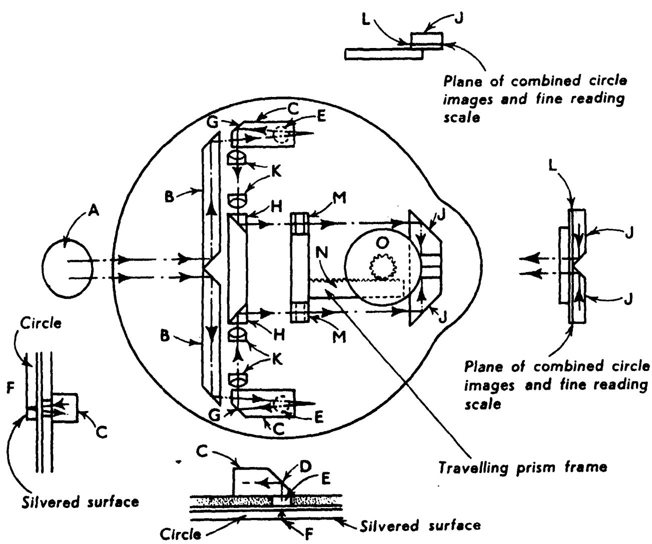


FIG. 4.10.—Tavistock Theodolite. Optical System—Horizontal Circle

in the window. Rotation of the micrometer milled head will cause the images of both scales to move as one in the same direction, so that the gap can be made to straddle the index line: the index mark in the large

opening (1) will then coincide with a main scale division. In Fig. 4.7 the main scale reading is 90°-40′ and the two graduations can be seen in the little window (3), equally spaced about the index line.

The movement necessary to set the image of the main scale into this position is recorded in the top window (2). This represents a portion of the fine-reading glass scale seen at O in Figs. 4.10 and 4.11, a piece of which is represented by the curved scale in Fig. 4.9. As already mentioned, this scale is divided into minute divisions, each subdivided

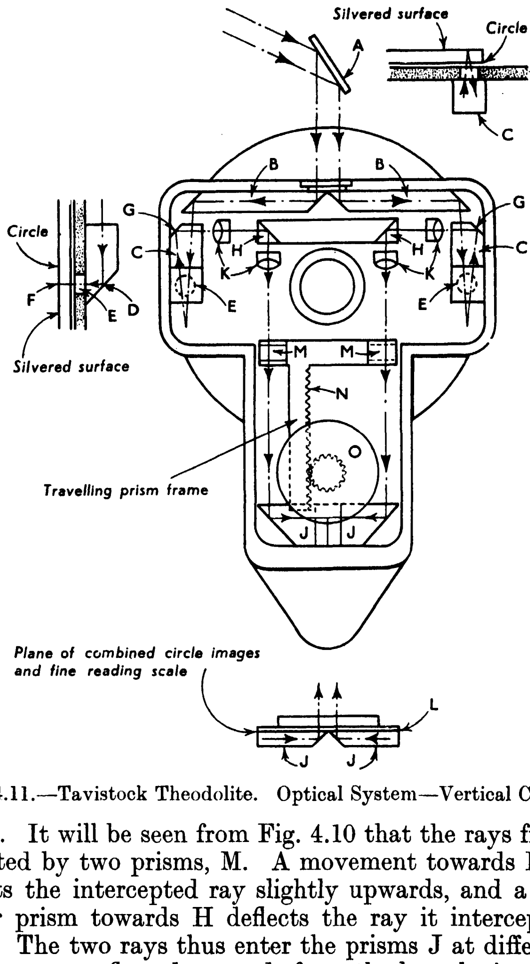


FIG. 4.11.—Tavistock Theodolite. Optical System—Vertical Circle

into seconds. It will be seen from Fig. 4.10 that the rays from H to J are intercepted by two prisms, M. A movement towards H of one of these deflects the intercepted ray slightly upwards, and a movement of the other prism towards H deflects the ray it intercepts slightly downwards. The two rays thus enter the prisms J at different levels, and, when they are reflected upwards from the last sloping face of the prisms J, one is moved laterally towards the junction between the two prisms, and the other away from it. As the junction lies between them, both rays move in the same direction, and by equal amounts.

The range of movement of M is one inch, and this is sufficient to move the rays through 20 minutes on the main scale. The fine-reading

circle is geared to the frame N carrying the prisms M, and graduated to subdivide the 20-minute interval into minutes and seconds. The movement in Fig. 4.7 corresponds to 7′-35″. It will be noted that the scales are masked except for the small portions actually required to be seen.

The optical system for the vertical circle is shown in Fig. 4.11: its arrangement can readily be followed from the diagram.

The **plain or Y theodolite** and the **Everest theodolite** have both been superseded by the Transit. There are many such instruments still in use, however, the Everest having been at one time very popular in India, so that a few notes have been included on them. They differ from the transit chiefly in respect of the fact that the telescope cannot be turned through a complete circle in a vertical plane, *i.e.* cannot be transitted. Instead of the complete vertical graduated circle of the transit, the Y has only what corresponds to the lower semicircle of it, and the Everest, two quadrants, one towards the eye-piece and one towards the object glass end of the telescope. In the Y instrument the telescope is supported in bearings from which it can be lifted, and reversed, or rotated about its longitudinal axis, just as it can in the Y level described on p. 163. As in both instruments, the A supports are lower than in a transit, the instruments are possibly somewhat more rigid and compact, though not so generally useful as the Transit.

Setting up.—Before measuring an angle or ranging a line with a theodolite, the instrument must be set up accurately over the station-point and the vertical axis placed in a truly vertical position. A plumb-bob is suspended from a point at the head of the tripod, immediately below the centre of the instrument, or from the lower extremity of the axis itself under the upper parallel plate, and the instrument is moved until the plumb-bob hangs exactly over the station-point as marked, probably by the head of a nail driven into a wooden peg.

To do this by moving the tripod legs, and at the same time to keep the parallel plates approximately horizontal, often requires some patience, especially on sloping or awkward ground. Consequently several arrangements have been devised to facilitate the task.

One type of mechanical centering substage is shown in Fig. 4.1, where linear movements in each of two directions at right angles can be made by manipulating the two milled-headed screws. In other devices the movements are rotational about fixed centres, and sometimes the movement is affected merely by the sliding of one surface freely over the others until they are in the desired position, and then clamping. Sometimes the centering device is above the footscrews, as in Fig. 4.1, sometimes below the footscrews, but in the base of the instrument, and sometimes incorporated in the head of the tripod.

The next operation is to level the instrument so that the vertical axis shall be truly vertical. Most of the modern instruments are fitted with three levelling screws, though four screws are still preferred by some Surveyors. The " tribrach " arrangement, however, causes less strain upon the axis and is quicker to operate.

The head of the instrument is turned about either of its axes until one of the bubble tubes (*e*) on the vernier plate lies with its longitudinal axis parallel to one pair of screws (*b* and *c* say); the other bubble (*f*) will then lie over the third screw (*a*) or be parallel to the line joining *a* to the mid-point of *bc* as in Fig. 4.12.

In the case of a four-screw base each of the two levels (*e* and *f*) would be parallel to the line joining a pair of opposite screws (*i.e.* *bc* and *ad*) as in Fig. 4.13.

The two screws *b* and *c* are rotated in opposite directions in order to tilt the upper plate in one direction or the other and bring the bubble *e* to the centre of its run.

Similarly the bubble *f* is brought to the centre of its run by rotating the single screw *a* in the tribrach or the two screws *a* and *d* in the four-screw arrangement.

If the levels are in correct adjustment and at right angles to the vertical axis, each bubble will now remain in the centre of its run

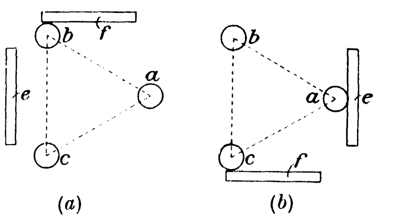


FIG. 4.12

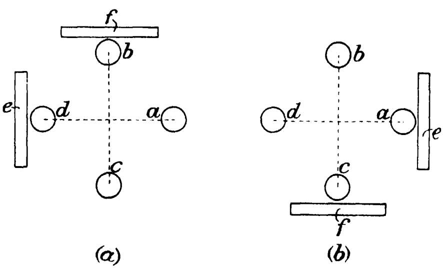


FIG. 4.13

during a whole revolution of the instrument about this axis, the vertical axis will be truly vertical, and the instrument ready for use.

PERMANENT ADJUSTMENTS

The permanent adjustments for a theodolite are:

(1) Adjustment of the parallel plate bubble tubes.
(2) Adjustment of the horizontal or transverse axis of the telescope.
(3) Adjustment of the line of collimation vertically.
(4) Adjustment of the line of collimation laterally.
(5) Adjustment of the bubble tube on the telescope or T frame.
(6) Determination of the index error of the vertical circle.

(1) **Adjustment of the Parallel Plate Levels.**—These levels should both lie in a plane at right angles to the central axis of the instrument, and hence when this axis is truly vertical each bubble should be in the centre of its run and remain there during a complete revolution.

If the bubble *e* is brought to the centre of its run, as already explained, while over the screws *b*, *c* (Fig. 4.12), and if the longitudinal

axis of the bubble makes an angle towards b, say of $(90-a)^\circ$, with the plane containing the central axis of the instrument and the centre of the bubble tube, then this central axis evidently will not be vertical (unless $a=0$), but will be inclined at a° to the vertical (Fig. 4.14, a).

On rotating the instrument through 180° in azimuth, the *axis* of course remains in the same position as before, but the bubble tube is now inclined to it at an angle of $(90-a)^\circ$ on the opposite side, *i.e.* towards c, or $(90-2a)^\circ$ to the vertical; that is, the axis of the bubble is no longer horizontal, but is inclined at $2a^\circ$ to the horizontal as shown in Fig. 4.14, b.

This tilting of the tube naturally causes the bubble to move away from the centre of its run towards b, through $2n$ (say) of the graduations marked on the glass.

If the parallel plate screws b and c are now manipulated so as to reduce the deviation of the bubble to n divisions, the inclination of the bubble axis will be reduced to a to the horizontal, and at the same time the central axis of the instrument will be tilted through the angle a into the vertical position.

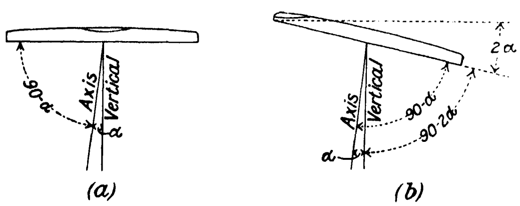


FIG. 4.14

To adjust the bubble e, it should now be brought back through the remaining n divisions to the centre of its run by means of the two capstan-headed screws which attach the tube to the upper surface of the vernier plate—the end nearer b being lowered and that nearer c raised.

Then in case the instrument may have been slightly disturbed during this process, the operation should be repeated until it is found by trial that the bubble remains undeflected for a complete revolution.

The bubble f is adjusted in a similar manner.

It should here be noticed that, without actually altering the adjustment of the bubbles, the verticality of the central axis may be assured at any time by bringing the bubble to the centre of its run when parallel to two screws b and c, reversing end for end, and halving the total deviation of the bubble, *e.g.* $\frac{2n}{2}$ divisions, by the parallel plate screws b and c; the bubble should then remain with an eccentricity of n divisions *towards the same end of the tube* during any portion of a revolution, though it would appear, at one time, n divisions out of centre towards b, and in the reverse position n divisions towards c.

In fact it is generally preferable to set up the instrument in this way, using the large bubble connected with the telescope, as this is much more sensitive. The procedure is as follows: the vertical arc being clamped at zero, the instrument is rotated until the bubble is parallel to the screws b and c (say), when it is brought to the centre of its run, the screws b and c being used for this purpose.

The instrument is then turned through 180° in azimuth so that the bubble tube again lies parallel to *bc*. If the deviation of the bubble from the centre is now $2n$ divisions, n divisions are taken up by means of the screws HH′ (Fig. 4.3); or, if vertical angles are not to be measured and the bubble tube is attached to the telescope, by the vertical circle tangent screw G. The remaining displacement of n divisions is corrected by the screws *b* and *c*.

The bubble is then turned through 90° in azimuth until it is over the remaining parallel plate screw or screws, and (if there is any deviation) by these only brought to the centre of its run.

The bubble should now "traverse", *i.e.* remain in the centre of its run for a complete revolution in azimuth. If not, this procedure must be repeated until the result is satisfactory. The vertical axis will then be truly vertical, and the smaller bubbles on the vernier plate can, if required, be brought to their central positions and permanently adjusted by the capstan-headed screws already mentioned.

(2) **Adjustment of the horizontal transverse axis** of the telescope, which should be exactly at right angles to the vertical axis.

(*a*) The first method of testing this adjustment is by means of a striding bubble, *i.e.* a sensitive tube carried on two legs, each of which is notched at the foot in order to fit the axis trunnions.

As a preliminary, the truth of this appliance should be tested thus: Set up the theodolite with its vertical axis truly vertical, uncover the supports at the ends of the horizontal axis, and place the level astride the telescope so that the notches fit and rest on the trunnions; bring the bubble to the centre of its run by means of the parallel plate screws, and then reverse end for end without disturbing the instrument. The bubble should still remain in the central position: if not, half the deviation may be adjusted exactly as in the first adjustment, by means of the parallel plate screws, and the remainder by the capstan-headed screw provided on one of the legs of the striding bubble for this purpose. On account of the sensitiveness of the bubble this operation may be tedious and require several trials, so that, instead, it may be advisable merely to note the number of divisions deviation, and to allow for this when using the appliance.

For instance, if on reversal the bubble deflects through $2n$ divisions towards one of its extremities (a' say), the horizontal transverse axis of the theodolite will be truly horizontal when the deviation of the bubble is reduced to n divisions towards a'. Consequently the bubble may be adjusted to this position (*i.e.* n divisions eccentricity), instead of to the central position, and the actual correction of the striding bubble fitting need not be attempted.

To proceed to test the adjustment of the theodolite axis the striding bubble is set astride the telescope as before and brought to its ascertained normal position by means of the parallel plate screws. The instrument is then rotated about its vertical axis through 180° in azimuth, so that the positions of the A frames or other vertical supports are reversed. If the horizontal transverse axis is at right angles to the vertical axis of rotation of the instrument, the bubble will remain

in its normal position for any position of the supports. If not, and the bubble moves through $2m$ divisions from the normal position, half this deviation (*i.e.* m divisions) is corrected by means of the parallel plate screws and half by means of the capstan-headed or other screws (Fig. 4.3) provided under one end of the horizontal axis at the head of the A supports. The operation should be repeated until the adjustment is perfect.

(*b*) Another method of testing this adjustment, known as the Spire Test, is as follows:

The instrument is set up near a spire or other elevated object; the ends of the horizontal axis are unclamped in their supports, and the telescope is directed to the top of the spire. Both the clamps connected with the horizontal movements of the instrument are tightened, and by means of one or other of the tangent screw arrangements the cross-hairs are made to coincide exactly with the point of the lightning conductor or with some other well-defined point.



FIG. 4.15
Spire Test

The telescope is then depressed, and the observer at the instrument directs an assistant to place a mark on the wall near its foot or to insert a thin arrow in the ground so as to appear to coincide with the cross-hairs.

The telescope is then lifted from its supports and the instrument rotated through 180° in azimuth (*i.e.* horizontally) until the positions of the two supports are reversed. The telescope is replaced in its bearings so as to be in the same position relative to the spire, etc., as before, each end of its axis now resting in the opposite support of the instrument to that in the previous case. Coincidence is obtained with the elevated point and the telescope depressed. The cross-hairs should again cut the same point marked near the foot of the spire. If not, a second point should be marked to coincide in this case.

The conditions, it will be seen, are the same for each observation, except that the positions of the supports have been reversed for the second case; the discrepancy, if any, is thus due to a difference in height of these supports.

If the support R, originally on the right, is exactly at the same level as that on the left, L, then, when the line of collimation is at right angles to the horizontal axis, the trace of this on the face of the tower would be a vertical line, and a point p_1 vertically below the spire point p would be obtained (Fig. 4.15). When the right support R is the higher, a point p_2 would be obtained lying to the right of p_1, as the trace on the tower would be tilted downwards to the right. Similarly when the supports are reversed, the higher support R being on the left, the point p_3 is thrown to the left of p_1.

If the line of collimation is *not* exactly at right angles to the horizontal axis (see adjustment (4)), the same reasoning still holds (as the position of the telescope relative to the spire is unaltered), except that the trace on the tower would be hyperbolic instead of a straight

line, and p_1 would not necessarily lie exactly below p. In any case, the point p_1—which would be the point obtained were the supports of equal heights—will lie midway between p_2 and p_3.

To correct any inequality in the heights, the support R is lowered or the support L raised by means of the screws (C, Fig. 4.3) provided for this purpose near the top of the A frames, until the cross-hairs appear to have moved about a quarter of the distance from p_3 to p_2.

The effect of this will be to throw the trace of the line of collimation from pp_3 to some new steeper position $p'p_3'$, cutting the trace pp_3 at the height of the instrument from the ground ; if the alteration is of the correct amount, $p'p_3'$ will be vertical.

Now the telescope is again elevated, and the cross-hairs redirected to p by means of either of the vertical axis tangent screws, so that the line $p'p_3'$ is moved laterally until p' coincides with p.

On depressing the telescope the cross-hairs should coincide with p_1 midway between p_2 and p_3 ; if not, a further correction must be applied to the adjusting screws until the desired result is obtained.

This *test* can be carried out on any type of theodolite, but usually the adjustment cannot be made on an instrument of the Y pattern, and in this case it is less necessary as the telescope is not designed to work through large vertical angles.

Similarly some Transit instruments are not provided with any means of adjustment, the argument being that if the A frames are made very substantial the axis can only get out of adjustment by extra wear upon one of the bearings, and this would necessarily be small, and best left to the instrument maker to rectify.

(*c*) If adjustment (4) is correct, the spire test may be carried out by reversing the supports (*i.e.* turning the instrument through 180° in azimuth) and then swinging the telescope of a Transit instrument through 180° vertically instead of lifting it out of its bearings. In the second observation the telescope would then be upside down, and so would not be in exactly the same position with reference to the tower as during the first observation. This, however, would be immaterial, provided that the line of collimation is in correct adjustment.

In Fig. 4.15 the distance p_2p_3 is, of course, very much exaggerated—the actual value seldom exceeding an inch or two.

In lieu of using arrows or of marking the tower wall to fix the points p_1 and p_3, an ordinary levelling staff may be placed horizontally on the ground at the foot of the wall, when the readings on the scale as the telescope is depressed each time may be noted.

(3) **Vertical Adjustment of the Line of Collimation.**—The object of this adjustment is to place the intersection of the cross-hairs on the horizontal diameter of the telescope, while the fourth adjustment is to place it on the vertical diameter.

It may be difficult to complete the one adjustment without affecting the other, so that this adjustment is best done first as it is of least importance. It can be rechecked at any time.

For a Y theodolite a very simple method, which may be applied indoors, is convenient. The instrument is set up and levelled ; the

vertical circle clamped with the telescope bubble at the centre of its run and the telescope itself resting lightly in its bearings with the clips unfastened. A reading is taken on to a levelling staff held at a convenient distance away, or a mark p_1 is made on a wall or other surface to coincide with the intersection of the cross-hairs. The telescope is then lifted carefully from its bearings and inverted, when the cross-hairs should again coincide with the same mark p_1 as before. If not, a new mark p_2 is made, and the diaphragm raised or lowered by means of the capstan-headed screws until the hair-lines fall exactly midway between p_1 and p_2, because it is obvious that if the line of collimation is inclined at, say α, to the longitudinal axis in the first position, it will be equally inclined at α in the opposite direction after inverting the telescope, and hence the axis will lie midway between these two positions.

Sometimes in the case of a Y instrument the tests for adjustments (3) and (4) may be carried out simultaneously by focusing the cross-hairs on to a point p on a wall, and rotating the telescope in its supports about its longitudinal axis. If the line of collimation coincides with the longitudinal axis, the hair-lines will continue to cut the marked point during the whole revolution, otherwise the intersection point will appear to describe a circle having the point p on its circumference.

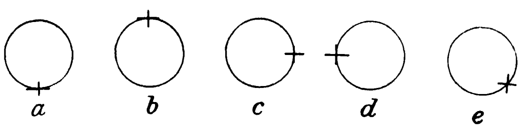


FIG. 4.16

For instance, if in that position of the telescope when the point is first bisected the diaphragm is too low, the circle will appear to be described above the point p (Fig. 4.16, *a*) ; if the diaphragm is too high, the circle will be below (Fig. 4.16, *b*) ; if the diaphragm is too much to the right, the circle will be apparently on the left; and *vice versa* the circle will be on the right if the cross-hairs are on the left. In a case where both adjustments (3) and (4) are incorrect the circle will be set obliquely, as for example in Fig. 4.16, *e*, when the cross-hairs are low and to the right for the normal position of the telescope. The adjustments are made by means of the capstan-headed screws of the diaphragm as before explained.

With a transit theodolite it will suffice to proceed as follows : set up the instrument and observe a finely graduated vertical scale at a convenient short distance away—indoors if desired. Note the least and greatest readings that can be seen in the field of view. The mean of these two readings may be considered as the axial reading, and the diaphragm should be moved upwards and downwards as required until the horizontal hair-line or axial stadia point gives this reading. As before, the top and bottom diaphragm capstan-headed screws are used for the adjustment.

(4) **Lateral Adjustment of the Line of Collimation.**—The line of collimation of the telescope should be at right angles to the horizontal transverse axis, in order that the surface described by it during a

revolution of the telescope will be a plane. If the line of collimation is not perpendicular to the horizontal axis, it will describe the surface of a cone[1] as it revolves, and its trace upon a level stretch of ground will be hyperbolic (see also Fig. 4.27, p. 111).

(*a*) To test the accuracy of this adjustment, the instrument is set up as already described, the telescope clips uncovered to allow it to rest freely in its supports, the two lower clamps tightened, and exact coincidence obtained by means of one of the tangent screws with some point p_1 (Fig. 4.17), *e.g.* an arrow placed at a convenient distance away—say 50 ft.

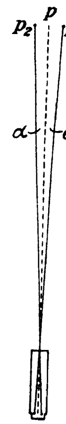


FIG. 4.17

The telescope is lifted as gently as possible out of its bearings and replaced upside down. If the line of collimation lies in the axis of the telescope and at right angles to the axis of rotation, then the cross-wires should again cut the point p_1; if not, another arrow, p_2, is placed near p_1 to coincide with their intersection.

The true perpendicular to the horizontal axis then lies through a point p midway between p_1 and p_2, because if the cross-hairs in the first case lie in the telescope a little to the left of the longitudinal axis, the point p_1 will be thrown too much to the right, and as, when the telescope is turned upside down, the cross-hairs will be on the opposite side of the longitudinal axis, the point p_2 will be thrown by an equal amount too much to the left.

To complete the adjustment, the diaphragm is moved laterally by means of the capstan-headed screws in the sides of the telescope until the cross-wires appear to coincide with p—the point midway between p_1 and p_2.

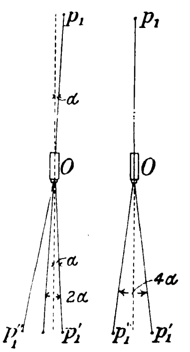


FIG. 4.18

As in adjustment (2), a horizontal levelling staff affords a very convenient means of marking the points p_1 and p_2, and dispenses with the need for an assistant.

The above method may be adopted with any type of theodolite, but as it is difficult to avoid disturbing the instrument while inverting the telescope, the following method (Fig. 4.18) is often applied to a Transit.

(*b*) The instrument is set up and levelled as before; the cross-hairs directed to some well-defined point p_1; the telescope transitted, *i.e.* swung over vertically; and another point p_1' fixed to coincide with the intersection of the webs. The telescope is now lifted from its supports and replaced upside down (*i.e.* face right instead of face left, or *vice versa*), and directed towards p_1, when the cross-hairs should again fall exactly on this point. If they do not—through any accidental disturbance of the instrument, or through want of adjustment of the diaphragm—they are redirected on to p_1 by means of either of the horizontal tangent screws. The telescope is transitted

[1] On account of focusing, the surface of revolution will not be a true cone, but will be approximately so.

once more and directed towards p_1' with which the webs should coincide. If not, a new point p_1'' is marked—or noted on a horizontal levelling staff.

If p_1, p_2, p_1', p_1'' (Figs. 4.17-4.18) are approximately equidistant from the instrument, the intercept between p_1' and p_1'' will be double that between p_1 and p_2, so that this test is more delicate than that previously described; and any disturbance of the instrument in inverting the telescope in its supports is not so likely to affect the result, as adjustment is made on to p_1 after this operation.

Thus if in the first case Op_1 is inclined at a to the right of the true perpendicular to the horizontal axis, Op_1' will be inclined at a to the same side when the telescope is transitted (*i.e.* to the left when looking towards p_1'), so that the angle between $p_1'O$ and p_1O produced (where O represents the position of the instrument) is $2a$.

Similarly $p_1''O$ makes an angle of $2a$ on the opposite side of p_1O produced, so that the angle $p_1'Op_1''$ is $4a$, *i.e.* four times the error in the line of collimation.

To correct, therefore, while the instrument is still directed towards p_1'', the horizontal capstan-headed screws of the diaphragm are manipulated until *one-quarter* of the deviation $p_1'p_1''$ is allowed for.

The whole operation is then repeated until the adjustment is found to be correct.

(*c*) As a modification of method (*b*) the position of the supports may be reversed (*i.e.* the instrument rotated through 180° in azimuth) and the telescope transitted, instead of lifting it out of its supports, when intersecting p_1 the second time.

No inaccuracy would result in methods (*a*) or (*b*) from a lack of adjustment of the horizontal axis (2); nor in method (*c*), provided that p_1 and p_1' are not on very different levels.

(5) **Adjustment of the Telescope Bubble.**—The object of this adjustment is to place the axis of the bubble parallel to the line of collimation, and consequently parallel or approximately parallel to the longitudinal axis of the telescope.

On a moderately level site, two pegs, A and B, are driven in the ground at a convenient distance apart—say 200 feet—and the instrument set up equidistant from them, *e.g.* exactly midway between them (Fig. 4.19). If the bubble tube is on the vernier arm, and not on the telescope itself, the reading on the vertical circle vernier is set at zero, and the bubble is brought to the centre of its run by means of the clipping screws HH′ of the vernier arm. If the bubble tube is fixed to the telescope, the vertical circle tangent screw may be employed if preferred, instead of the clipping screws: in this case the vernier reading is not kept at zero.

An assistant is directed to hold an ordinary levelling staff (p. 175) on each of the pegs A and B in turn, and the readings of the axial web at these points are taken—the bubble being accurately adjusted to the centre of its run for each reading. The difference between these readings (*c*A and *c*B, Fig. 4.19) will give the true difference in level between the two pegs, as any errors affecting the one reading will equally affect

the other, *e.g.* in Fig. 4.19, if the line of collimation is inclined upwards both readings will be too large by the amount *ec*, where O*e* represents a level line through the instrument and O*c* the line of collimation.

The two pegs may be driven in to the same level if desired ; this is not necessary nor is it always convenient.

The instrument is now set up near one of the end pegs, *e.g.* at D (Fig. 4.20), and readings taken on the two pegs, A and B—the bubble again being in the centre of its run for each, and the vernier of the vertical circle reading zero if the bubble tube is attached to the T frame.

If the intersection of the cross-hairs lies upon the longitudinal axis

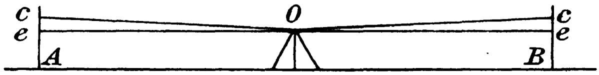


FIG. 4.19

of the telescope, the line of collimation will be coincident with this axis, and provided that the draw-tube moves as it should, parallel to the axis, will not be affected by focusing. If, in addition, the axis of the telescope is parallel to the axis of the bubble, the line of collimation OA′B′ will be horizontal for each of the readings A′A and B′B on A

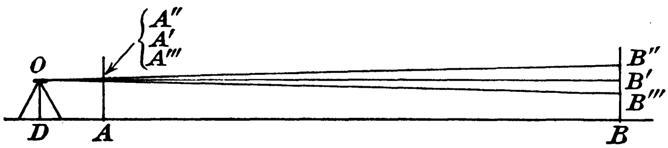


FIG. 4.20

and B, so that the differences in the readings should agree with the true differences in level already obtained.

If the longitudinal axis is *not* parallel with the axis of the bubble, the line of collimation will be inclined to the horizontal, when the bubble is in the centre of its run, and the angle of inclination will be constant for the two readings.

If the line of collimation is tilted upwards so that the readings are A″B″ (Fig. 4.20), the apparent difference in level is B″B − A″A, instead of B′B − A′A, *i.e.* instead of *c*B − *c*A (Fig. 4.19). The peg B is thus apparently too low relative to A, and the eye-piece end of the telescope must be raised.

Similarly, if B‴B − A‴A is less than *c*B − *c*A, the line of collimation is tilted downwards, and the eye-piece end of the telescope must be lowered by means of the parallel plate screws—or by means of the vertical arc clipping screws HH′—until the correct difference of level is obtained from the altered readings. The reading on the nearer peg A will alter only very slightly, so that, if the staff at B is observed while

the inclination of the telescope is being altered, this correct difference is quickly obtained in one or two trials, and the line of collimation is then horizontal and—as the diaphragm screws are unaltered—still in the centre of the telescope.

The next operation is to make the axis of the bubble parallel to this by adjusting the capstan-headed screws which attach the tube to the telescope, or to the top of the T frame which carries the verniers, so as to bring the bubble to the centre of its run.

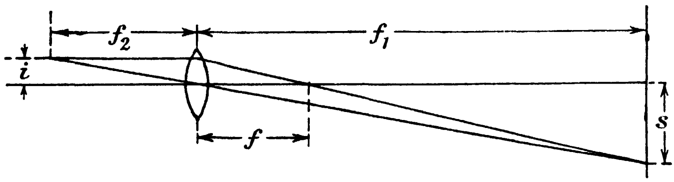


Fig. 4.21

It is of interest to examine the case where the axial cross-hairs are not exactly on the optical axis of the telescope. The notation used is that adopted in Chapter II, *e.g.* Figs. 2.8 or 2.9.

Suppose that, during focusing, the cross-hairs move linearly along a line inclined at θ to the optical axis and that the deviation from this axis is i when the diaphragm is at a distance f_2 from the object glass (Fig. 4.21).

Then the line of travel of the hair-line cuts the optical axis at a from the object glass, whence

$$i=(a-f_2)\tan\theta=c(a-f_2)\text{ say.}$$

Now, if the intercept on the staff between the reading given by the cross-hairs and the optical axis is s, it is seen from the geometry of Fig. 4.22 (see also Chapter II)

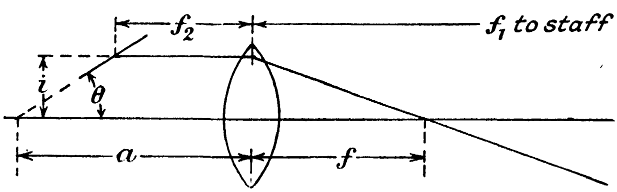


Fig. 4.22

$$s=i(f_1-f)/f,$$

substituting for i

$$s=c(f_1-f)(a-f_2)/f. \quad . \quad . \quad . \quad (1)$$

But $\qquad 1/f=1/f_1+1/f_2,$ (equation 3, p. 53),

i.e. $\qquad f_2=ff_1/(f_1-f).$

Substitute in (1), and

$$s=\frac{c(f_1-f)}{f}\left(a-\frac{ff_1}{f_1-f}\right)$$

$$=ca\frac{(f_1-f)}{f}-cf_1$$

or

$$s=cf_1(a/f-1)-ca. \qquad . \qquad . \qquad . \qquad (2)$$

This is the equation of a straight line, as c, a and f are constants; it cuts the optical axis at a point in front of the object glass (*i.e.* where $s=0$) at a distance $f_1=f\,.\,a/(a-f)$, and the angle α it makes with the optical axis is $\tan^{-1} c(a/f-1)$.

In the special case when $\theta=0$, *i.e.* when i is constant,

$$s=\frac{f_1-f}{f}\,.\,i, \qquad . \qquad . \qquad . \qquad (3)$$

and this gives the intersection point at which $s=0$ at $f_1=f$.

Consider now two rays—(1) that from the cross-hair, through the lens centre at O, to the staff, and (2) that from the cross-hairs parallel to the optical axis, through the outside focus, to the staff. In the general case, dealt with above, the inclination of each will alter as f_1, f_2 and i alter. But the staff readings at which they intersect will be collinear and appear to lie on a straight line represented by equation (2). In the special case where $\theta=0$ the first ray only will change in direction due to focusing, and the second ray mentioned, *i.e.* that through the outside focus, will coincide with the imaginary line (equation 3) which is an " apparent " line of collimation. Obviously the two-peg test, if carried out as described, will result in the bubble axis being adjusted parallel to this imaginary line passing through or near the outside focus and will be quite satisfactory.

[NOTE.—In Gravatt's three-peg test, three pegs A, B, C, are driven in the ground and spaced so that AB = BC = 100 to 150 ft., say. The instrument is set up equidistant from A and B and the relative levels of these pegs determined as in the two-peg test. The relative levels of B and C are next determined in the same manner, the instrument being set up equidistant from them. Finally the instrument is set up at a point D, near A, and staff readings taken on the three pegs A, B and C in turn. The apparent levels are deduced from these readings. These apparent levels will differ from the true levels previously determined, unless the instrument is in adjustment. The analysis given above indicates, however, that whether the cross-hairs of the telescope are on the optical axis or not, the error between C and A should be double the error between B and A, provided AB = BC. The adjustment, which is specified, is that if the error between C and A is *not* double that between B and A, the top and bottom diaphragm screws should be adjusted until this condition is satisfied. The test is now discarded, as its usefulness is very doubtful. If the error between C and A was found not to be double that between B and A, and this conclusion was confirmed by re-testing, the inference would probably be that the motion of the draw-tube was irregular; attempts to minimize the effects by trial and error manipulation of the diaphragm screws seem unlikely to be profitable.]

For a Y instrument where the telescope can be lifted from its supports, and where the bubble tube is attached to the telescope, the following alternative method may be adopted.

The line of collimation having been adjusted, the telescope, resting lightly and unclipped in its supports, is turned until it lies parallel to the two base screws *b* and *c* (Figs. 4.12 or 4.13), and by means of these screws the bubble is brought to the centre of its run, while the vernier of the vertical circle is approximately at zero. The telescope is then carefully lifted from its bearings and replaced end for end, still being parallel to *bc*. The bubble should still be in the central position ; if not, half the deviation is corrected by means of the parallel plate screws *bc* (or the vertical circle tangent screw), and half by means of the capstan-headed screws attaching the level tube to the telescope (compare Fig. 4.14).

The level tube may also require a little lateral adjustment in order to bring its axis parallel to the longitudinal axis of rotation of the telescope. If so, the error can be detected by turning the telescope slightly about this axis and noting whether the bubble keeps to the central position. If not, on some instruments, the adjustment may be made by screws capable of horizontal motion at the ends of the level tube.

(6) **Determination of the Index Error of the Vertical Circle.**—When the bubble of the telescope is in the centre of its run, and therefore when the line of collimation is horizontal, the vernier of the vertical circle should indicate zero. If it does not do so, the value of the reading, which is the " index error ", is noted, and, as the case may be, added to or subtracted from all angles of elevation or depression observed with the instrument.

In a Transit instrument the vernier can be clamped to zero, and the telescope afterwards brought to the horizontal position by the clipping screws H and H′ (Figs. 4.3 and 4.4), which bear on opposite sides of the lug P upon the A frame. There should therefore be no index error with a Transit instrument.

To measure a simple horizontal angle BAC with a theodolite, the instrument is accurately set up and levelled so that the plumb-bob hangs over the point A. The parallel plates are unclamped and the zero of the vernier brought to the zero of the primary scale, exact coincidence being obtained, after reclamping, by means of the fine adjustment tangent screw.

The lower clamp having been loosened, the instrument is turned about the outer axis until the telescope is directed towards the left-hand object B, when the clamp is again tightened, and exact coincidence obtained with the cross-hairs, by means of the *lower* tangent screw. The telescope is thus directed towards B while the vernier still reads zero.

The next operation is to unclamp the upper screw and rotate the instrument about the inner axis until the telescope is pointing towards C, then reclamp and get exact coincidence by the use of the upper tangent screw. During this movement the lower clamp and tangent screw are untouched, and hence the value of the angle BAC is recorded by the vernier, which is read by means of one of the microscopes.

It should be noted that any movement of the telescope in a vertical

plane does not affect the reading on the horizontal scale, so that it is merely the value of the horizontal projection of the angle BAC which is given by the vernier reading, and this is what is usually required for surveying purposes.

The box sextant, on the other hand, cannot be held exactly level if B and C are at very different altitudes, and consequently the instrument measures the actual value of the angle instead of its horizontal projection.

For Repetition and Reiteration see p. 118 and p. 119.

To measure the angle of elevation or depression of an object C, the theodolite is set up and levelled as before, and the vernier of the vertical arc accurately adjusted to zero, by means of the vertical circle clamp and tangent screw. The bubble is brought to the centre of its run by means of the screws H and H_1 (Fig. 4.4), thus making the line of collimation horizontal while the vernier reads zero.

In the case of a Y or Everest theodolite, or some types of Transit, the bubble is merely brought to the central position by the vertical arc clamp and tangent screws, and the reading of the vernier noted when this is completed; the index error is thus obtained if the zeros do not coincide.

The line of collimation having been fixed horizontal, the clamp is loosened and the telescope tilted about its transverse axis until it is directed to C, when it is again clamped and exact coincidence obtained by means of the tangent screw G (*not* the screws H and H_1). The reading of the vernier $\pm$ the index error, if any, then gives the angle of elevation or depression required.

To range a Line of considerable Length with a Theodolite.—Let AB be a portion of the line which it is required to produce beyond B. The instrument is set up, accurately centred over the point B, and levelled.

With a Transit instrument, the telescope is then directed towards A; both horizontal clamps are tightened, and the cross-hairs made to intersect A correctly by means of either of the horizontal tangent screws. The telescope is next transitted, *i.e.* revolved in a vertical plane, and a new position C located at the intersection of the cross-hairs on the opposite side of B to A. The position of C is first marked approximately by means of a wooden peg, after which the exact prolongation of AB is fixed by sighting to a nail, a pencil, or other object held against a white paper background, and this point is permanently marked on the head of the peg, probably by driving in a nail.

To ensure the accuracy of adjustment (4) (Fig. 4.18), the process may be repeated after reversing the positions of the supports by turning the instrument through 180° in azimuth and thus causing the telescope to be inverted when directed towards A. If the two points on the peg C are not coincident, the mean position is adopted.

To proceed with the line the theodolite is set up at C, the point B intersected, and the telescope again transitted to fix a fourth point D, and so on.

To avoid the accumulation of errors due to observation, etc., the distances apart of the stations B, C, D should be as great as is convenient without sacrificing distinctness of the object to be bisected.

With a Y theodolite the telescope may be lifted from its supports and replaced end for end instead of transitting; or the head of the instrument may be turned through 180° in azimuth. In this case the vernier of the horizontal circle must be noted when the telescope is directed to A, and 180° added to this reading to locate C.

For the other methods the reading of the horizontal circle is immaterial.

Errors.—The chief errors to which theodolite observations are liable will now be enumerated.

1. Inaccurate centering: *i.e.* the error due to the axis of the instrument A′ not being exactly above the station-point A, so that instead of the required angle BAC being measured, BA′C is observed.

The error is a "compensating" one, because for a maximum displacement of AA″, say, the axis may fall anywhere within the dotted circle (Fig. 4.23), which has its centre at A and a radius AA″.

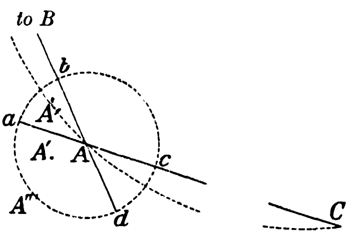


FIG. 4.23.—Error due to Defective Centering

For points within the sector *a*A*d* the error is the sum of the angles subtended at B and C by AA′, and is negative.

For points within the sector *b*A*c* the error is the sum of the angles subtended at B and C by AA′, and is positive.

For points within the segments *a*A*b* and *d*A*c* the error is the difference between the subtended angles at B and C, and may be positive or negative according to whether A′ falls within or without the arc of a circle which passes through BAC.

For any position of A′ on the arc, $\angle$BAC = $\angle$BA′C.

The subtended angles are inversely proportional to the lengths AB and AC for any specific displacement of A′, so that it is particularly important to centre the instrument accurately when the rays are short, otherwise considerable error may result.

For example, if AA′ = $\frac{1}{2}''$ at right angles to AC, and AC = 50 ft., the angle subtended at C = $\sin^{-1}\frac{1}{1200} = 2'\text{-}52''$, while if AC = 500 ft. the subtended angle is $\sin^{-1}\frac{1}{12000} = 17''$.

It is obvious, then, that for accurate work it is not sufficient to mark a station-point merely with a $1\frac{1}{2}''$ or $2''$ wooden peg—especially when the lengths of sight are short. A nail should be driven into the head of the peg to mark the exact point, and the instrument adjusted until the plumb bob is suspended as nearly as possible over this. If the hook from which the plumb-bob is suspended is not quite on the axis—particularly if it has been knocked and bent from its normal position—a small centering error may be introduced. For very long sights, *e.g.* in a large triangulation survey, ordinary centering errors are generally negligible.

In prolonging a line AB to C it is evident that the perpendicular displacement of C′ from its true position C, due to a perpendicular displacement of the instrument B′B from B, is expressed by the relationship $\frac{C'C}{AC}=\frac{B'B}{AB}$, any displacement parallel to AC being immaterial.

2. An error may arise due to the inaccurate bisection of the object which is being sighted. For instance, if the point B is marked by a ranging rod, the observer has to use his judgment as to the point where the cross-hairs bisect this. Bright sunlight on one face with dark shadow on the opposite face of the rod may lead to an error of this class. This error, again, is compensating, and its magnitude is inversely proportional to the length of the rays. For accurate work with short sights a plumb-bob suspended over the station-point may be sighted (Fig. 13.6).

3. When a station-point is permanently marked by a peg and temporarily for sighting purposes by a ranging rod by the peg, care must be taken that the rod is placed exactly in the correct line. It should also be exactly vertical, otherwise considerable error may arise, particularly if, owing to intervening obstacles, a point near the top of the rod has to be sighted. As a general rule the lowest point visible should be viewed.

When a plumb-line is used for sighting purposes care should be taken that it is not deflected due to wind and other causes. This error is generally "compensating", and its magnitude is inversely proportional to the length of the rays.

4. An error may also arise due to "parallax", *i.e.* to the cross-hairs not coinciding exactly with the image in the telescope. Under these circumstances, by moving the eye to different parts of the eye-piece lens the hair-lines can be made to move relatively to the image when it becomes very difficult to bisect the station-point correctly. The eye-piece should be adjusted until all possibility of error from this source is eliminated.

This error is "compensating", and more appreciable for distant objects.

5. If the "vertical" axis of the instrument is not truly vertical, errors are introduced into both horizontal and vertical angles. The nature of these errors may be appreciated from Figs. 4.24 and 4.25, though the exposition will be more clear after Chapter XV on Spherical Trigonometry has been read. Thus let O represent the intersection of the telescope transverse axis and the instrument vertical axis. Then OZ represents the vertical axis coinciding with the true vertical.

If the telescope is directed to A, it will swing in the vertical plane OZA; and if directed to S it will swing in the vertical plane OZSB. The bearing of S from A is the horizontal angle AOB, and this is the same as the angle at Z in the spherical triangle AZB. The altitude of S is the angle SOB and is represented by the arc BS if the sphere is considered to be of unit radius.

If, now, the "vertical" axis is displaced α from the vertical in the plane OAZ, in Fig. 4.25 OZ_1 represents the tilted axis and OZ the true

vertical. The horizontal plate has pivoted about the perpendicular axis HOH_1 and is represented by the tilted circle H_1A_1H. The angles Z_1OZ and A_1OA are each equal to α.

When directed towards A the telescope will swing in the same plane

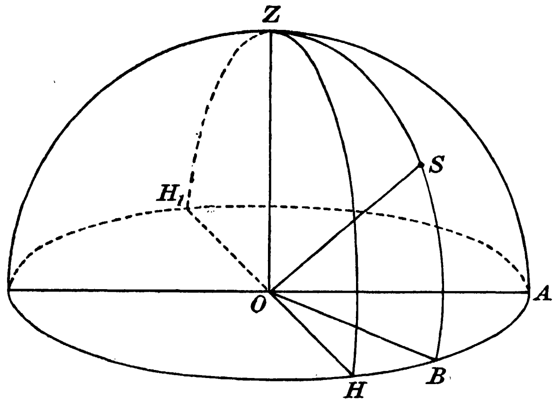


FIG. 4.24

as before, Z_1ZA_1AO, represented by the plane of the paper. When directed towards S, the telescope will swing in a plane Z_1SB_1O, this

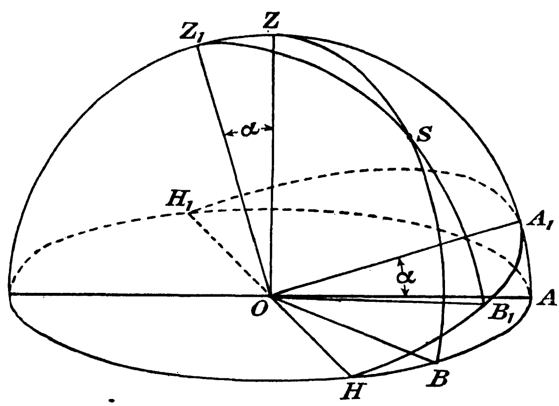


FIG. 4.25

plane containing the axis Z_1O, and hence being perpendicular to the inclined plane OA_1H.

The recorded bearing of S from A_1 (or A) is thus now represented by the angle A_1OB_1, *i.e.* by the angle Z_1.

The true bearing is Z and the recorded bearing is Z_1, and the difference may readily be calculated by the application of the formulae in the Appendix, applied to the spherical Z_1ZS. Thus, if the altitude of S is β, *i.e.* $\angle SOB$, then in the spherical triangle we have $Z_1Z = \alpha$, the

inclination of the axis; $ZS = 90 - \beta$, the co-altitude of S; the angle $Z'ZS$ at $Z = 180° - Z$ where Z is the bearing of S from A.

From formulae 7 and 8 (p. 544) we have

$$\tan \tfrac{1}{2}(Z_1 + S) = \frac{\cos \frac{1}{2}(ZS - Z_1Z)}{\cos \frac{1}{2}(ZS + Z_1Z)} \cot \tfrac{1}{2}(180 - Z),$$

$$\tan \tfrac{1}{2}(Z_1 - S) = \frac{\sin \frac{1}{2}(ZS - Z_1Z)}{\sin \frac{1}{2}(ZS + Z_1Z)} \cot \tfrac{1}{2}(180 - Z).$$

From these equations, the value of Z_1 for any specific values of Z, β, and α can be calculated.

Since the sum of the three angles of a spherical triangle exceeds 180° by an amount known as the spherical excess, e (p. 399), it follows that $Z_1 = Z + e - S$. But when the altitude of $S = 0$, *i.e.* when $\beta = 0$, then S is nearly equal to e, and is exactly equal to it when Z is 90°, *i.e.* $Z_1 = Z = 90°$ and there is no error at $Z = 0°$, 90°, 180°, and 270°. The maximum error is at about 45°, *i.e.* when $\tan^2 Z = \cos \alpha$ and $\tan^2 Z_1 = \sec \alpha$. As an example, if α, the tilt of the axis, is 1°, then $Z_1 = 45°\text{-}00'\text{-}16''$ nearly, when $Z = 45°$; and $Z_1 = 134°\text{-}59'\text{-}44''$ nearly, when $Z = 135°$, *i.e.* the error in azimuth at 45° is $Z_1 - Z = +16''$, and at 135° $Z_1 - Z = -16''$.

As the altitude of S increases, however, the angle S of the spherical triangle Z_1ZS increases while the spherical excess decreases. Hence the negative error $(Z_1 - Z)$ increases appreciably: in fact, it can be seen from Fig. 4.25 that when S approaches a position between Z and Z_1 the two angles may vary by nearly 180°. For less extreme altitudes, substitution in the equations given above shows that the maximum differences occur at intermediate values of Z, the approximate formula on p. 111 giving the maximum at $Z = 90°$. Thus, when $Z = 90°$, $\beta = 40°$, $\alpha = 1°$ then $Z_1 = 89°\text{-}09'\text{-}40''$ approx, and the error $Z_1 - Z = -50'\text{-}20''$. The angle S is, under these conditions, $1°\text{-}18'\text{-}19''$ approx.

In the above analysis only the angle between A, a point along the direction of maximum tilt, and an arbitrary point S has been investigated. In practice, the direction of OA will not be known and the surveyor will be concerned with the error in the horizontal angle between two stations, say S_1 and S_2: the error will depend upon the positions of S_1 and S_2 relative to the unknown A. The maximum error will be the maximum value of the sum of the errors in the two angles measured from OA, when S_1 and S_2 are on opposite sides of OA.

Obviously this source of error is important, and particular care must be exercised in levelling the instruments when horizontal angles between points at greatly different levels are to be measured, the long sensitive telescope bubble being then used with advantage.

The error in altitude as indicated in Fig. 4.25 being $B_1S - BS$, *i.e.* $SZ - SZ_1$, SZ_1 can be calculated from the spherical triangle SZ_1Z by application of the sine formula (1) on p. 544. If the sensitive telescope bubble is adjusted for each reading, the error in altitude can be calculated from the expression $\sin \beta_1 = \sin \beta \sec (\alpha \sin \theta)$, as explained below.

6. If the transverse axis of the telescope is not exactly at right

angles to the vertical axis (see adjustment (2)) the line of collimation will not swing in a vertical plane, and consequently angles of elevation and depression will not be measured in a vertical plane, but in an inclined plane.

Thus in Fig. 4.26 if BOB′, *i.e.* β, is the true angle of elevation of B, and α is the error in the perpendicularity of the horizontal axis to the vertical axis, then the recorded angle of elevation is BOB″ $=\beta_1$ where BB″ is the line traced out by the cross-wires upon a vertical plane, *i.e.* B″ will fall to the right of the vertical BB′ through B when the higher support is to the right, and *vice versa* the line of collimation will swing down to B‴ on the left if the higher support is on the left.

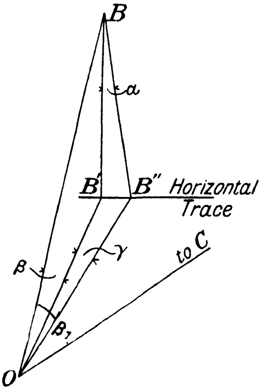


FIG. 4.26

Thus

$$\sin \beta_1 = \frac{BB''}{OB} \text{ or } \frac{BB'''}{OB} = \frac{BB' \sec \alpha}{OB} = \sin \beta \sec \alpha.$$

If $\alpha = 1'$ and $\beta = 45°$ the error is negligible, while if α is as much as $1°$ and $\beta = 45°$ the error is about 32″.

A more considerable error is introduced into a horizontal angle when the two points B and C, between which the angle is subtended, are at widely different levels. Thus, imagine the point B has an angle of elevation of β, while C lies in the horizontal plane through the instrument O.

The true horizontal angle is, then, B′OC, where B′ is vertically below B, the recorded angle is B″OC, and the error introduced is the angle B′OB″ ($=\gamma$). Then in Fig. 4.26, since OB′B″ $= 90°$ if B′BB″ is to equal α,

$$\tan \gamma = \frac{B'B''}{OB'} = \frac{BB' \tan \alpha}{OB'} = \tan \beta \tan \alpha,$$

or

$$\gamma = \alpha \tan \beta, \text{ nearly.}$$

So that if $\alpha = 1'$ and $\beta = 45°$, the error γ is equal to 1′, which is very appreciable.

If C has an angle of elevation δ, the error in the horizontal angle BOC is $\tan^{-1}(\tan \beta \tan \alpha) - \tan^{-1}(\tan \delta \tan \alpha)$, which becomes nil when $\beta = \delta$. If C has an angle of depression δ, the sign of this expression becomes +.

If the angle is measured twice, transitting the telescope and turning through 180° in azimuth between the observations, then the average of the two values will be correct and will eliminate this error.

It will be noted that if the " Vertical " axis of the instrument is inclined at α to the true vertical, as illustrated in Fig. 4.25, and if the transverse axis is horizontal when the telescope is directed towards A, this transverse axis will be tilted at α when the telescope is turned in azimuth through 90°, or at $\alpha \sin \theta$ when turned through $\theta°$ (*i.e.* $Z_1 = \theta$).

Hence the expression derived above for a tilted transverse axis may

be applied to the case of a tilted vertical axis by substituting $\alpha \sin \theta$ for α, *i.e.* the error in azimuth due to this cause $= \tan^{-1}[\tan(\alpha \sin \theta) \tan \beta]$

or $$\gamma = \alpha \sin \theta \tan \beta \text{ approximately,}$$

i.e. if $$\theta = 90°, \quad \alpha = 1°, \quad \beta = 40°,$$
$$\gamma = 50'\text{-}21'' \text{ approx.,}$$

which agrees closely with the result obtained by the fuller analysis. If $\theta = 45°$, $\alpha = 1°$, $\beta = 40°$, $\gamma = 35'\text{-}36''$, whereas the fuller analysis gives $Z - Z_1 = 34'\text{-}59''$.

If the sensitive bubble is accurately centered for each observation, then the error in altitude for any observation with a tilted vertical axis, is similarly obtained from the equation for a tilted transverse axis, *i.e.*

$$\sin \beta_1 = \sin \beta \sec (\alpha \sin \theta).$$

7. If the line of collimation is not perpendicular to the horizontal axis as described in adjustment (4), it will approximately trace out the surface of a cone which has its vertex in the telescope tube. The trace of the intersection of this cone with a vertical plane will be hyperbolic, and hence when the angle between two points B and C is being measured, if B is at a very different level to C, the observed value will not be correct.

Thus to take a simple case, suppose C is at the same level as the instrument axis, while B has a considerable altitude (Fig. 4.27).

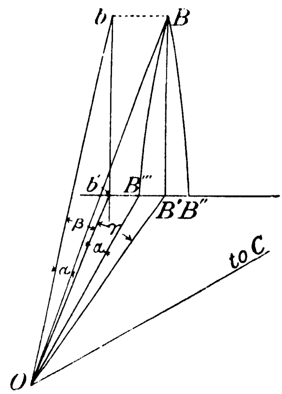


FIG. 4.27

Then the angle required is that between B′—a point vertically below B on the same level as C—and C. But if the cross-hairs, which are assumed to be a little on the left (say) of the axis, are focused on to B and the telescope then swung down, the hyperbola traced out will cut B′C in a point B″ to the right of B′, and the measured angle will be that between B and C, and therefore too small.

If, however, the telescope is inverted so as to bring the cross-hairs on the right of the axis, *i.e.* if the face of the instrument is changed, then too large a value from B‴ to C is obtained, and the average between the two observed angles will give the correct angle.

The magnitude of the induced error may be studied from Fig. 4.27.

Let O*b* be the direction of the telescope axis when the line of collimation OB is directed to B, so that the angle *b*OB is the error α in collimation.

Then if *b*′ and B′ be the projections of *b* and B, and BB‴ the path traced by the cross-hairs upon a vertical plane through *b* and B, when the telescope is rotated into the horizontal plane the error (α) in collimation is the angle *b*′OB‴, for the axial line of the telescope will pass through *b*′ vertically below *b*, while the line of collimation will pass through B‴, and the angle between the axial line and the line of

collimation is constant and equal to α, except for any differences caused by focusing.

The true horizontal angle between B and C is, therefore, B′OC where B′ is vertically below B and C is in the horizontal plane through the instrument, while the recorded angle is B‴OC, and the error is B′OB‴, *i.e.*

$$B'Ob' - B'''Ob' \text{ or } (\gamma - \alpha),$$

where $$\tan \gamma = \frac{B'b'}{Ob'} = \frac{Bb}{Ob'} = \frac{Bb}{Ob \cdot \cos \beta} = \frac{\tan \alpha}{\cos \beta}.$$

Usually $\gamma = \alpha \sec \beta$ is a close approximation.

If $\alpha = 1°$ and $\beta = 45°$, $\gamma = 1°\text{-}24'\text{-}09''$, and the error introduced into the horizontal angle BOC is 24′-09″. If $\alpha = 5'$ and $\beta = 10°$, $\gamma = 5'\text{-}5''$, and the error is 5″.

Similarly if C has an elevation of $\delta°$, the total error in the angle is

$$\tan^{-1}\left(\frac{\tan \alpha}{\cos \beta}\right) - \tan^{-1}\left(\frac{\tan \alpha}{\cos \delta}\right), \text{ or } \alpha(\sec \beta - \sec \delta).$$

This is evidently nil when $\delta = \beta$, *i.e.* when B and C are in the same horizontal plane, and when $\delta = -\beta$, *i.e.* when the angle of depression of C is equal to the angle of elevation of B.

For accurate work, the errors due to both sources 6 and 7 are eliminated by measuring the angle twice—once F.R. and once F.L., —and by taking the average as the true value. Pivots should not be changed between these observations, as, if this is done, for the reasons explained on p. 116, any pivotal error in the instrument affects the result. If desired, however, after completing the above two observations, the pivots may then be changed and another set of two F.R. and F.L. observations taken. The mean of these should again be correct, and should afford a check upon the first set.

No special advantage results from the change of pivots, and usually it is equally effective and more simple to take the additional pair of F.R. and F.L. readings upon a different part of the horizontal scale, without this change.

The effect and elimination of this error in the ranging out of a line is seen in the description of adjustment (4), p. 99.

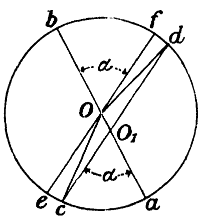


FIG. 4.28

8. If the line of collimation is displaced only in a vertical plane from the longitudinal axis of the telescope, no error should result if the telescope bubble has been adjusted as described on pp. 100-104.

9. An error may be introduced into horizontal angles as a result of imperfect or eccentric centering of the vernier plate. Thus in Fig. 4.28 let O be the true centre of the scale plate, while O_1 is the centre about which the vernier plate and the top of the instrument rotate, *i.e.* O_1 is the centre of the central axis, and that O of the hollow outer axis.

The effect of the error can perhaps be most clearly seen by considering the movement of the telescope and vernier from a position

aO_1Ob when the two centres are in line. Let the telescope be turned through an angle α to a position cO_1d, *i.e.* $\angle aO_1c = \angle dO_1b = \alpha$. Through O draw a line *eOf* parallel to cO_1d—so that $\angle aO_1c = \angle fOb = \alpha$. Then, as O is the true centre of the primary scale circle *acebfd*, the arcs *ae* and *bf* are each equal to a proportion $\frac{\alpha}{360}$ of the whole circumference—*i.e.* to α divisions of the scale.

But the reading of the vernier at *c* is the arc *ac*, and this is the value of the angle *aOc*, which is less than the angle *aOe* by the angle *cOe*.

Similarly if there is a vernier exactly on the opposite side of the plate, the reading of this will be denoted by the arc *bd* instead of by *bf*, and will thus be too large by the angle *dOf*. But as *ce* and *fd* are equal, the angles *dOf* and *cOe* are equal ($=\delta$ say).

Thus *ac* denotes an angle $\alpha - \delta$ and *bd* denotes an angle $\alpha + \delta$, so that the average of these gives the true value α.

The error from this source may—especially on an instrument that has had much wear and tear—be several minutes in magnitude when one vernier only is read, a maximum error of 1 minute being caused by an eccentricity of 0·00087 inch in a 6-inch diameter scale plate. For a good instrument the error should be much less—*i.e.* not more than 10″ to 20″ for a 6-inch theodolite.

10. In the description of the adjustment of the horizontal transverse axis of a theodolite on p. 95, it is tacitly assumed that there is no pivotal error in the instrument. Pivotal error is usually small, but it is almost impossible to prevent its occurrence, and, owing to wear of the pivots and their supports, it may, in many cases, become quite appreciable.

The following investigation illustrates the nature of this error, and explains how its presence may be detected. Usually the two pivots—one at each end of the transverse axis—rest in V supports at the top of the A frames. Let it be assumed that the diameters of the pivots are D and d; that the angles of the V supports are $2\phi_A$ and $2\phi_B$; that the angles of the V terminals of the striding bubble legs are $2\theta_1$ and $2\theta_2$; and that the lengths of the legs above the V terminals are h_1 and h_2 respectively.

Firstly, let the larger pivot D be supported in the frame A. Then, when the striding bubble itself has been adjusted, and the usual operations have been carried out as described in the text, we have

$$h_1 + \frac{D}{2} \operatorname{cosec} \theta_1 = h_2 + \frac{d}{2} \operatorname{cosec} \theta_2 + x \quad . \quad . \quad (1)$$

and

$$h_1 + \frac{d}{2} \operatorname{cosec} \theta_1 = h_2 + \frac{D}{2} \operatorname{cosec} \theta_2 - x, \quad . \quad . \quad (2)$$

where x is the difference in level of the two pivot centres, that of D being assumed the lower.

From (1) and (2) $x = \frac{(D-d)}{4} (\operatorname{cosec} \theta_1 + \operatorname{cosec} \theta_2)$,

and the inclination of the transverse axis is $\tan^{-1}\frac{x}{L} = \alpha$ (say), where L is the horizontal distance between the two striding bubble legs.

The striding bubble apparently indicates that the transverse axis is truly horizontal, and perpendicular to the vertical axis, and this would actually be so were there no pivotal error in the instrument. The fact that the axis is not horizontal, but inclined at α, is not indicated, and it can only be discovered if the test is repeated with changed pivots, *i.e.* with pivot D in the support B.

In the first case, with pivot D in the support A, the line joining the apices of the Vs is inclined to the axis of the pivots at an angle

$$\gamma_1 = \tan^{-1}\frac{D \operatorname{cosec} \phi_A - d \operatorname{cosec} \phi_B}{2L},$$

i.e. at $\alpha + \gamma_1$ to the horizontal.

On changing pivots, as the supports are not moved, this line remains unaltered, but the inclination of the transverse axis to this line joining the apices is now

$$\gamma_2 = \tan^{-1}\frac{D \operatorname{cosec} \phi_B - d \operatorname{cosec} \phi_A}{2L}.$$

The inclination of the transverse axis to the horizontal is therefore

$$\alpha + \gamma_1 + \gamma_2,$$

the vertical axis remaining truly vertical.

If the striding bubble is now applied, its axis will be inclined to the horizontal at an angle of

$$2\alpha + \gamma_1 + \gamma_2,$$

and this inclination will be indicated by the divergence of the bubble from its normal position.

Thus it is seen that if the Striding Bubble Test is applied to an instrument having a pivotal error—and only one position of the pivots is tested—there still remains, after adjustment, an error of α for the one position and an error of $(\alpha + \gamma_1 + \gamma_2)$ for the reverse position of the pivots. These errors cannot both be eliminated by the Surveyor. The Striding Bubble Test does not even enable him to eliminate the error of the transverse axis for one position of the pivots, *e.g.* for the position when pivot D is in support A. If half the deviation $(2\alpha + \gamma_1 + \gamma_2)$ of the bubble is corrected by means of the horizontal axis capstan screws, an error of $\pm \frac{\gamma_1 + \gamma_2}{2}$ still remains whether pivot D is in support A or in support B.

Consider now the Spire Test method of carrying out this adjustment. When the vertical axis of the instrument is truly vertical, let the inclination of the transverse axis be

α_1 to the horizontal, downwards towards the right, (*a*)

the larger pivot D being on the support A to the right.

The line joining the apices of the Vs will be inclined downwards towards the right at an angle of $\alpha_1 + \gamma_1$ to the horizontal as before,

where $$\gamma_1 = \tan^{-1}\frac{D \operatorname{cosec} \phi_A - d \operatorname{cosec} \phi_B}{2L}.$$

On changing pivots and also changing the position of the supports, the inclination of the axis becomes

$$\alpha_1 + \gamma_1 + \gamma_2, \text{ downwards towards the left,} \quad . \quad (b)$$

where $$\gamma_2 = \frac{D \operatorname{cosec} \phi_B - d \operatorname{cosec} \phi_A}{2L}.$$

If the telescope is now inverted so that the face of the instrument is changed from F.R. to F.L., the inclination of the axis to the horizontal will be

$$\alpha_1 \text{ to the left} \quad . \quad . \quad . \quad (c)$$

when pivot D is in support A, and A is to the left, and

$$\alpha_1 + \gamma_1 + \gamma_1 \text{ to the right} \quad . \quad . \quad . \quad (d)$$

when pivot D is in support A, and A is to the right.

Thus, if in each case the telescope is directed towards the tip of the spire and depressed as described on p. 96, the traces on the vertical plane will be four straight lines if there is no collimation error in the instrument, or four hyperbolas if the line of collimation is out of adjustment horizontally.

Let δ be the displacement of the points at the foot of the tower, due to an assumed collimation error.

The displacements of the points marked at the base of the tower will be

(*a*) H tan $\alpha_1 \pm \delta$ to the left (F.R., telescope normal, D in A),
(*b*) H tan $(\alpha_1 + \gamma_1 + \gamma_2) \mp \delta$ to the right (F.R., telescope normal, D in B),
(*c*) H tan $\alpha_1 \pm \delta$ to the right (F.L., telescope inverted, D in A),
(*d*) H tan $(\alpha_1 + \gamma_1 + \gamma_2) \mp \delta$ to the left (F.L., telescope inverted, D in B),

where H is the height to the tip of the spire.

The mean of the two positions (*a*) and (*c*) should coincide with the mean of the two positions (*b*) and (*d*), and this position should be vertically below the spire tip.

To make the line of collimation traverse through this point as well as through the tip of the spire necessitates the adjustment both of the transverse axis and of the line of collimation horizontally, and the tests do not afford enough information to enable this to be done.

In the method described on p. 96, the assumption is made that no pivotal error exists, *i.e.* $\gamma_1 = \gamma_2 = 0$. If this assumption is true, then the error α_1 can be eliminated by making the line of collimation traverse through the tip of the spire and the mean position of (*a*) and (*b*), as explained on p. 97.

The mean position of (*a*) and (*b*) is not necessarily below the tip of

the spire—it is displaced an amount δ—but this fact does not affect the adjustment of the transverse axis.

If the test is repeated (after the axis has been adjusted on (*a*) and (*b*)), with the telescope now F.L., the two points (*c*) and (*d*) should coincide, but at a distance 2δ from the mean position of (*a*) and (*b*). If there is a pivotal error, the displacement is approximately $2\delta \pm H \tan(\gamma_1 + \gamma_2)$ instead of 2δ, and there still remains an error in the transverse axis of $\frac{\gamma_1 + \gamma_2}{2}$. This error is undetected unless it has previously been ascertained that $\delta = 0$, when the displacement is $H \tan(\gamma_1 + \gamma_2)$—an amount which is roughly proportional to the residual error of $\frac{\gamma_1 + \gamma_2}{2}$ in the axis.

Pivotal error cannot be corrected by the Surveyor. He can only estimate its amount. Unless it is sufficiently serious to warrant the return of the instrument to the maker, the following procedure is suggested for a Transit instrument.

(1) After adjusting the parallel plate bubbles (p. 93), adjust the line of collimation laterally, *e.g.* by method (*c*), p. 100; check, if desired, with changed pivots.

(2) Carry out the spire test by method (*c*), p. 97, with pivot D in support A (say), and without changing pivots obtain points (*a*) and (*c*) as described above. If any error is found—assuming that adjusting screws are provided—complete the adjustment of the transverse axis. There is now no error in the transverse axis provided that pivot D is in the support A.

(3) Repeat the spire test, but with pivot D in support B. If any error is found now, it is due to " pivotal error ". Its magnitude can be estimated as explained above.

(4) There are now two alternatives. Either (*a*) the error $\gamma_1 + \gamma_2$ can be distributed so that there always remains an error in the transverse axis of $\pm \frac{(\gamma_1 + \gamma_2)}{2}$, whether pivot D is in support A or in support B; or (*b*) the whole error of $(\gamma_1 + \gamma_2)$ can be allowed to remain in the transverse axis when pivot D is in support B, in which case there will be no error when pivot D is in support A.

In the former case any single observation with the instrument is liable to error. In the latter case only observations taken with pivot D in support B are liable to error from this cause. In either case a pair of F.R. and F.L. observations taken without changing pivot would eliminate the error, but in neither case would the mean of a pair of F.R. and F.L. observations be free from error if pivots were changed between the observations. It is probably preferable then to adopt the latter alternative, as it is a small matter to mark the instrument and arrange that, in normal use, the pivot D shall be held upon the support A.

There appears to be no special advantage resulting from a change of pivots—but if at any time it is desired to repeat a set of observations

under altered conditions, there is, of course, no objection to the pivots being changed, for, provided complete sets of F.R. and F.L. observations are taken, the mean values should agree with corresponding values obtained with the pivots in their normal supports.

In a good instrument, pivotal error should be small, as may be seen from the following values, so that for ordinary purposes the methods of adjustment described in the previous pages are satisfactory. The Surveyor may, however, examine his own instrument as described above, and ascertain whether pivotal error is appreciable.

If it is assumed that $D - d = 3 \times 10^{-4}$ inch; that $2\phi_A = 2\phi_B = 90°$; and that $L = 4$ inches, then $\gamma_1 = \gamma_2 = \frac{\gamma_1 + \gamma_2}{2} = 11$ seconds of arc. If the adjustment is so carried out that the residual error is $\frac{\gamma_1 + \gamma_2}{2}$, this only corresponds to a displacement of 0·025 inch of the mark at the foot of a 40-ft. high spire. The error introduced into horizontal angles due to an error in the transverse axis is discussed on p. 110. It is proportional to the tangent of the angle of elevation.

11. The effect of "obliquity" of the main "vertical" axes has not been discussed in the preceding text. The two axes, theoretically, should coincide, and practically, in the best modern instruments, the error does not exceed a few seconds of arc.

Obviously if the error exists, both axes cannot be vertical simultaneously, and it will be impossible for one of the sensitive bubbles to remain in the centre of its run while the instrument is turned first about one and then about the other axis.

If the most sensitive of the instrument bubbles is made to traverse, for rotation about one axis, the maximum displacement of the bubble when rotation is effected about the other axis will indicate the extent of the error.

Similarly the horizontal transverse axis of the telescope cannot be fixed so as always to be at right angles to both axes, unless these are coincident or parallel. If the adjustment is completed for one relative position of the vernier and scale plates, the adjustment will not be correct for other positions. This may be tested, but the error cannot be remedied by the Surveyor.

The error introduced into horizontal angles owing to obliquity of the main axis, is usually negligibly small, except possibly in mining surveying [1] where very steep sights are taken—the error being proportional to the tangent of the angle of elevation as before mentioned.

12. The graduation marks on the scales of an instrument may not be exactly equidistant, so that to diminish errors due to this cause an angle is sometimes measured several times over different parts of the scale, and all verniers read, when the average is probably more correct than a single reading. The magnitude of the errors due to imperfect graduations may be 10″ or more, though first-class makers

[1] "Methods of Measuring Horizontal Angles that involve Steep or Precipitous Sighting in their Measurement", by L. H. Cooke, *Trans. Inst. Min. and Met.*, March 16, 1922.

can guarantee much greater accuracy than this. *E.g.* in a micrometer theodolite reading directly to 5″ an error of two to three seconds only might be expected.

13. Errors may arise due to "slip" occurring when accurate coincidence is being obtained by the fine motion tangent screws. Reiterations (p. 119) are sometimes taken both clockwise and counter-clockwise to eliminate this as much as possible.

14. In addition to the above-mentioned errors, mistakes may arise due:

(*a*) To the confusion and observation of wrong station-points.
(*b*) To the incorrect reading of the scales, *e.g.* 88°-30′ for 91°-30′, and *vice versa.*
(*c*) To the omission of a half degree when the primary scale is so divided, *e.g.* 181°-15′ instead of 181°-45′.
(*d*) To the reading of the wrong vernier, producing an error of 180° or 120° approximately.
(*e*) To the wrong application of an index error—positive instead of negative, or *vice versa.*
(*f*) To the manipulation of the wrong tangent screw.
Etc., etc.

For extreme accuracy in the measurement of horizontal angles there are two general methods of procedure which may be adopted, viz. Repetition and Reiteration.

Repetition.—To measure an angle BAC by this method—first suggested by Borda—the instrument is set up exactly over the station-point A and accurately levelled.

The vernier is clamped to the zero on the scale, the microscope and the upper fine adjustment screw being used to facilitate this. The lower clamp is loosened and the telescope directed to the left-hand object B; the clamp is re-tightened, and the station signal bisected by means of the lower tangent screw. The leading vernier should now be examined again to see that no slip has taken place, and the readings on other verniers noted. The vernier plate having been unclamped, the telescope is turned towards C; it is then reclamped and C bisected by means of the upper tangent screw. The leading vernier, though not necessarily read at this point, should now indicate the value of the angle BAC $=\alpha$, say.

So far the process described is the ordinary procedure for the measurement of an ângle, but to obtain greater precision the lower plate is now unclamped and the telescope turned about the outer axis, then reclamped, and exact coincidence again obtained with B by means of the lower tangent screw—the vernier still reading α.

The upper clamp is now loosened and the telescope redirected to C, and this station bisected as before by using the upper clamp and tangent screw. The vernier should now read 2α.

This procedure may be repeated any number of times, and the final reading after n repetitions should be $n\alpha$; so that if the actual final reading is observed and 360° added for every complete revolution to

give the value of na, this divided by the number of repetitions n gives the value of the angle BAC $=\alpha$.

The average of all the values given by the different verniers should be adopted.

By this means angles can be read to a finer degree of subdivision than those indicated by the smallest graduations on the vernier.

Any error due to the scales being imperfectly graduated tend to be eliminated, as the angle is measured on several parts of the scale and the result averaged; also errors due to inaccurate bisection of the object, eccentric centering, etc., may be to some extent counterbalanced in the different observations, and any personal error in the reading of the verniers is reduced in the ratio of $n : 1$.

Other errors due to slip, displacement of the station signals B and C, want of verticality of the axis, etc., being cumulative are not eliminated, so that the labour involved in carrying out a *large* number of repetitions is not justified by the increase of precision obtainable, and usually two or three repetitions are quite sufficient.

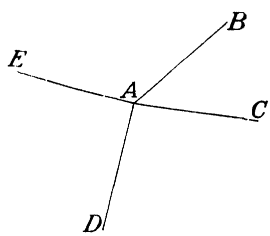


FIG. 4.29
Repetition and Reiteration

Reiteration is a much less tedious method when a large number of angles are to be measured at a station, and is carried out as follows. Let A be the instrument station, above which the theodolite is accurately set up, and let it be required to determine the several angles BAC, CAD, DAE, etc., subtended at this point (Fig. 4.29). The leading vernier is adjusted and clamped at zero by using the upper plate clamp and tangent screw, and the telescope is then directed to some well-defined point, with which exact coincidence is obtained by means of the lower clamp and tangent screw. This point, say B, is known as the " Referring Object ", and may or may not be one of the station-points BCD, etc. After bisection by the cross-hairs the vernier readings are all noted, and the telescope turned about the inner axis towards C, and again clamped and adjusted to C by the upper tangent screw.

The readings of the verniers which are now noted give the value of the angle BAC.

The upper plate is then unclamped, the cross-hairs turned and adjusted to D, and the verniers again read—giving the angle BAD.

Similarly the remaining points E . . . are bisected and finally the telescope is turned to the initial point B, when the leading vernier should read 360°, as the lower clamp and tangent screw have not been touched during the revolution.

There may, however, be an error due to slip, etc.; if small, this may be equally divided among the several angles, but if a large amount, the readings should be disregarded and a new set taken in lieu of them.

If these are satisfactory, the leading vernier is next clamped at some point other than zero, *e.g.* about 60° or 90°, but not necessarily at an even degree, and the angles again observed in a similar manner.

To compensate for slip, the second set may be taken in a counter-clockwise direction; and the telescope must be used face right instead of face left (or *vice versa*), or be inverted in the case of a Y instrument in order to eliminate inaccuracies due to errors 6 and 7 mentioned above.

The cross-wires, too, may be brought into exact coincidence with the object, from a different direction in each reiteration—in order to eliminate the effects of back lash; *i.e.* before applying the fine adjustment, the instrument may be clamped so that the intersection of the cross-hairs falls a little short of the exact position for each reading in one reiteration, and a little beyond the object for each reading in the second reiteration.

Finally, by subtraction, the values of the different angles are deduced, and an average of the various vernier results for the several reiterations may be taken as the correct result. Usually two reiterations will be sufficient, but more may be observed if necessary.

The same remarks concerning the errors which are eliminated, in addition to those specially mentioned, apply as in the method of Repetition.

Accuracy of Angular Measurements.—An ordinary 4-inch vernier theodolite is graduated to read to 1′ or occasionally to 30″.

An ordinary 5-inch vernier theodolite is graduated to read to 1′ or occasionally to 30″.

An ordinary 6-inch vernier theodolite is graduated to read to 20″ or occasionally to 10″.

An ordinary 7-inch or 8-inch vernier theodolite is graduated to read to 10″.

A dial or circumferenter, 6 inches diameter, is graduated to read to 3′.

A prismatic compass, $2\frac{3}{4}$ inches diameter, is graduated to read to 30′.

A box sextant, $2\frac{3}{4}$ inches diameter, is graduated to read to 1′.

A nautical sextant, 7 inches or 8 inches diameter, is graduated to read to 10″.

With micrometer microscopes 5-inch and larger theodolites may read to 10″ on the micrometer drum, and by estimation to 1″. Better class instruments may be still more finely divided (see below).

Considering a 6-inch vernier theodolite, say, graduated to 20″, the maximum error due to reading only would be about 10″, as anything over the 10″ would be recorded as 20″, and anything below 10″ as 0″. Also, as there is an equal likelihood of the value being anything between 0″ and 10″ or between 10″ and 20″, the probable error will be $\pm 5''$ for a single vernier; or if an allowance of 2 or 3 seconds is made for imperfect graduations, $\pm 8''$ say.

For two verniers the p.e. should therefore be about $\pm \frac{8''}{\sqrt{2}}$; and if f.r. and f.l. observations were taken the p.e. of the mean of the four readings due to this source would be $\pm 4''$.

Errors due to centering vary inversely with the length of the lines, as explained on p. 106. In triangulation they are generally negligible, but in traverses where the length of sight is small they may be very serious. Errors in bisection of the object also vary a great deal with the nature of the object sighted and the length of the ray (see p. 107).

Errors due to lack of adjustment of the instrument are obviously very variable, and may amount to several minutes, as explained above.

For a single observation with a 6-inch theodolite, if the two verniers are read, and the p.e. due to reading is $\pm \frac{8''}{\sqrt{2}}$ and due to inaccurate bisection $\pm 6''$ say, then if the instrument is in perfect adjustment, accurately levelled, and the sights are long, the p.e. would be about

$$\pm \sqrt{(5{\cdot}7)^2 + 6^2} = \pm 8''.$$

During the measurement of an angle, however, two points have to be sighted—one at each extremity of the angle—so that the p.e. in the value of the angle would be $\pm 8\sqrt{2} = \pm 11\frac{1}{2}''$, say.

Owing to imperfect adjustment this error might be considerably increased, but with a single f.r. and f.l. set of readings the errors due to lack of adjustment would be largely eliminated, so that the p.e. of the result should not greatly exceed

$$\pm \frac{11{\cdot}5}{\sqrt{2}} = \pm 8'', \text{ say.}$$

This is not the *maximum* error to be expected : the chances are even that it will be either greater or less.

If there are n reiterations the p.e. would be about $\pm 8'' \div \sqrt{n}$.

If there are n repetitions on one face, the p.e. due to inaccurate reading would be $\pm \frac{8\sqrt{2}}{\sqrt{2n}}$ seconds, and that due to inaccurate bisection $\pm \frac{6\sqrt{2n}}{n}$ seconds, so that the total p.e. in the value of the angle would be $\pm \sqrt{\left(\frac{8}{n}\right)^2 + \left(\frac{6\sqrt{2n}}{n}\right)^2}$ seconds.

Thus if $n = 3$, the p.e. would be $\pm \sqrt{(7{\cdot}1 + 24)} = \pm \sqrt{31} = \pm 5{\cdot}6''$.

If there are three repetitions on *each* face, the p.e. of the final result would be $\pm 5{\cdot}6 \div \sqrt{2} = \pm 4''$ about.

In the case of a traverse where the lines are short and the centering error not negligible, the p.e. of a single angle upon one face might easily be $\pm 15''$ to $\pm 20''$, especially as the adjustments cannot readily be made perfect.

A similar rough estimate of the accuracy to be expected may be made in the case of other instruments differently graduated.

A Surveyor may find his own probable error with a particular instrument, by experiment, from observations upon the triangular

and polygonal errors he gets in closed traverses or in triangulation. Thus if his average triangular error is $\pm 30''$ under certain conditions, his average angular error is $\pm \frac{30''}{\sqrt{3}} = \pm 17''$ nearly.

On the Geodetic Survey of the Transvaal and Orange River Colony the angles were measured with a Repsold theodolite, the primary scale of which was graduated to 4′, and on the two micrometer heads each division represented 2″.

The p.e. of a single angle when measured on 8 arcs was found to be $\pm 0{\cdot}30''$, and the p.e. of a single angle when measured on 4 arcs was found to be $\pm 0{\cdot}39''$.

The triangular errors very seldom exceeded $\pm 1{\cdot}5''$, and were generally below $0{\cdot}5''$.

In the measurement of the Geodetic Arc of the 30th Meridian in the Semliki–Ruwenzori region to the west of Lake Victoria,[1] a system of triangulation was built upon an accurately measured base (p. 430). The angles were measured with a 10-inch Repsold instrument, heliostats being employed by day and acetylene lamps by night. Each angle was measured upon eight settings of the horizontal arc, and two measures, f.r. and f.l., taken for each setting. The longest ray was 47 miles.

The average triangular error was $\pm 0''{\cdot}812$ and the p.e. of an observed angle $\pm 0''{\cdot}390$.

The p.e. of a single angle on the Ordnance Survey [2] was about $\pm 1''{\cdot}20$, while the mean value of the p.e. obtained on the Continent was given in 1895 by General Ferrero as $\pm 0''{\cdot}8$.

For modern work under favourable conditions the p.e. may be as low as $\pm 0''{\cdot}25$.

EXAMPLES

1. (U. of L.) In a Transit theodolite the intersection of the webs is displaced 0·05 inch in a horizontal direction from the line drawn through the optical centre of the object-glass parallel to the axis of the draw-tube. Find the angle through which the line of collimation is turned when the focal length is altered from 10 inches to 11 inches.

In measuring the horizontal angle between two points at very different distances, is the error from such a source eliminated by reversing face? Give reasons. How does it affect vertical angles?

2. If the line of collimation of a transit theodolite is adjusted laterally by sighting to points 50 ft. away from the instrument, what will be the error in the line of collimation when sighting to points 200 ft. away, if the optical axis of the telescope is inclined at $(90 - \theta)^\circ$ to the horizontal transverse axis? The principal focal length of the telescope may be taken as 12 in.

If $\theta = 10'$, what will be the approximate distance of p_1' from p_1'' (Fig. 4.18)?

[1] *British Association Report*, 1910.

[2] *Engineering*, January 2, 1914, and January 23, 1914.

CHAPTER V

THEODOLITE SURVEYING AND DIALLING

The Plotting of Angles.—Before dealing with the subject of theodolite surveying, it may be convenient here to summarize the various methods by which the values of angles obtained in the field may be plotted on the plan.

The most important of these methods are :

(1) By means of an ordinary protractor.

(2) By means of a vernier protractor (Fig. 5.1), which is circular in shape, usually about 6 inches in diameter, and fitted with a rotating T frame. On one arm of the T is fixed a clamp and tangent screw arrangement for fine adjustment, while to the other two arms verniers are attached. The outer portions of these two arms are hinged as shown in Fig. 5.1, and fold over the centre of the instrument when not in use, otherwise they open out flat with the points kept slightly above the paper by means of light springs. In the centre is a small glass circle engraved with two cross lines.

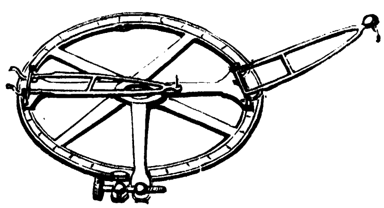


FIG. 5.1.—Vernier Protractor

To plot an angle α at the point C in a line AB, one of the verniers is adjusted to zero by means of the clamp and tangent screw, and the instrument laid upon the paper in such a position that the intersection of the cross-hairs lies exactly above the point C.

If the small pricker on the underside of one of the arms at its extremity is placed on the line AB, the pricker under the opposite arm should also fall on the same straight line through C, unless there is an error in the instrument.

The clamp can then be loosened and the T piece turned until the vernier reads α, exact coincidence being obtained with the tangent screw after reclamping. The arms are then pressed lightly on the paper and the line joining the two small punctured holes should pass through C and make the required angle α with AB. A good instrument of this class may read to single minutes.

(3) By means of a table of tangents. Thus if the tangent of $\alpha = t$, a length CN of n units is measured along AB from the point C (Fig. 5.2), and at the point N a perpendicular is erected, along which a length $NT = n \times t$ is set out.

The point T is then joined to C, and CT makes an angle a with AB at C, because $\frac{\text{NT}}{\text{CN}}=\frac{n\times t}{n}=t=\tan a$.

Usually for convenience n would be some multiple of 10.

Thus in Fig. 5.2 if $a=22°\text{-}42'$, tan $a=\cdot4183$, so that if CN = 10 inches, NT = 4·183 inches.

The method is really one of rectangular co-ordinates, and very accurate results can be obtained if CN and NT are made of considerable length.

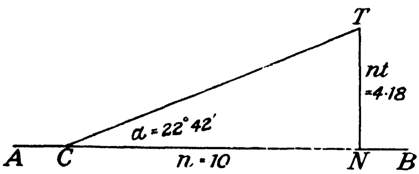


FIG. 5.2

(4) By a table of chords which gives the value of the chord subtended at the circumference by various angles situated at the centre of a circle of unit radius.

Thus, to construct an angle of a at the point C in a line AB, if the chord of $a=c$, an arc of any convenient radius CN $=n$ (preferably a multiple of 10) is described with C as centre to cut the line AB in N, and with N as centre and a radius of $n\times c$, a second arc is described to cut the previous one in E. Then the line joining E to C makes the required angle a with AB at the point C.

For example, if a in Fig. 5.3 $=22°\text{-}42'$, the chord of this angle from published tables is found to be ·3936, so that if CN = 10 inches the radius NE will be 3·936 inches. Or if CN = 20 inches the length NE is 7·872 inches.

It is immaterial what unit of length is employed in setting out CN and NE, provided the same scale is used for each. *E g.* CN might

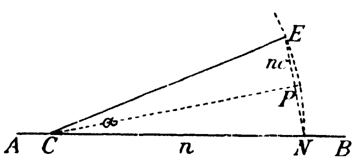


FIG. 5.3

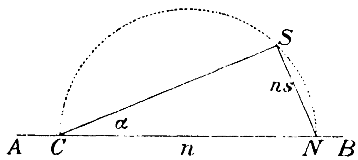


FIG. 5.4

represent 100 ft. to a scale of x ft. to 1 inch, when NE would represent 39·36 ft. to the same scale.

(5) By the aid of a table of sines. Sines are not so convenient for direct use as are tangents or chords, though they can be used as follows (Fig. 5.4). Let the sine of $a=s$.

A length CN is measured along the line CB, equal to n where n is preferably a multiple of 10. With the middle point of CN as centre, a semicircle of radius $\frac{n}{2}$ is described to pass through C and N; and with N as centre an arc of radius ns is described to cut the semicircle in S.

Then SC makes the required angle a with AB at C, because the angle CSN, being in a semicircle, is a right angle, and

$$\frac{NS}{NC}=\frac{ns}{n}=s=\sin a.$$

A table of cosines may be used in a similar manner by describing an arc from C = CS.

For example if $a=22°\text{-}42'$ as before, $\sin a=\cdot 3859$, so that if CN = 10 units, NS = 3·859 units.

Or from a table of cosines, $\cos a=\cdot 9225$, so that CS = 9·225 units when CN = 10.

An alternative method is to deduce the chord of the angle from the table of sines. It can easily be seen from Fig. 5.3 that the chord of an angle is twice the sine of half the angle, *i.e.* $c=2\sin\frac{a}{2}$.

For example, to set out an angle of 22°-42′ by means of a chord when only a table of sines is available, the sine of 11°-21′ is obtained as ·1968, so that the chord is 2 × ·1968 = ·3936 as given in method (4) above.

There are three main types of survey in which the theodolite is employed :

(1) A "Theodolite" survey.
(2) A Trigonometrical survey.
(3) A Traverse survey : (*a*) closed ; (*b*) unclosed.

1. **Theodolite Surveying.**—A simple theodolite survey is very little more than an ordinary chain survey with a few of the longer tie or check lines omitted, the more important angles between the various chain lines being measured in their stead by a theodolite.

A similar survey may be made by using a box sextant, circumferenter, dial, or other angular instrument in lieu of the more accurate but expensive theodolite.

If the whole of the interior angles of a polygon embracing the bulk of the survey are observed, a check may be applied to test the accuracy of the angular measurements, as it is easily proved as a corollary to Euclid I. 32, that the sum of the interior angles of any plane polygon is equal to

$$(2N-4) \text{ right angles},$$

where N = the number of sides of the polygon. In this case the survey becomes a closed traverse (see p. 126).

In addition to these main angles, it is always advisable to observe some others, since little extra labour or loss of time is thereby incurred.

For example, if the instrument is set up at the station B (Fig. 5.6) for the purpose of measuring the angle ABC, it would be advantageous to take both the angles ABE and EBC, as most of the time and labour is usually spent—not in actually measuring the angles—but in carrying

the instrument from one point to another, centering, and levelling up.

The readings may be recorded in several ways :

(1) In the field notes as Fig. 5.5.
(2) By means of a small sketch as in Fig. 5.6, drawn on the side of the field-book page near the commencement of one of the chain lines starting from the instrument station, *e.g.* BC.
(3) By inserting the figures on the small sketch diagram which is

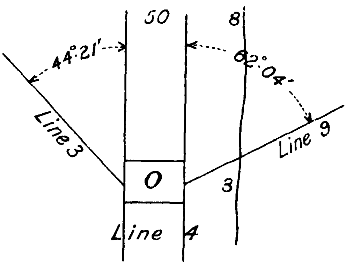


Fig. 5.5

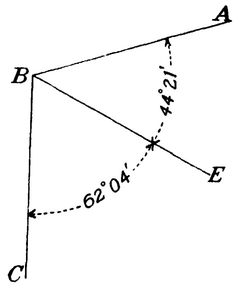


Fig. 5.6

usually drawn to show the arrangement of the chain lines (see p. 8).
(4) By arranging in tabular form, the letters of reference being noted on the sketch plan or in the field notes, *e.g.*

$$\angle ABE = 44°\text{-}21',$$
$$\angle EBC = 66°\text{-}04',$$
$$\angle BCD = 98°\text{-}37',$$
etc.

2. **Trigonometrical Surveying**, in which the highest degree of precision is attained, will be treated in a later chapter.

3. **Traverse Surveying.**—A Traverse Survey consists of a continuous series of lines from which offsets may be taken in the usual manner. The lengths of the lines are determined by chaining or by any other method, and the relative directions of the lines are obtained as described later—the use of triangulation with long tie lines being rendered unnecessary.

Traverses may be classed into two groups :

(*a*) Closed traverses.
(*b*) Unclosed traverses.

(*a*) *A Closed Traverse* is one in which a complete circuit is made, and in which, consequently, the work may be checked and adjusted or " balanced ". It is particularly suitable for the determination of the boundaries of lakes or woods across which tie lines cannot conveniently be measured—or it may be employed to locate the perimeter of any moderately large survey.

The survey may be complete in itself and quite independent of any

other observations, *e.g.* when a continuous series of lines is taken round the boundaries of a wood or lake. In such a case those cumulative errors in the linear dimensions which are directly proportional to the lengths of the lines are undetected, as their effect is merely to increase or decrease the apparent size of the survey without altering the shape or causing any " closing error ".

Thus, in making a traverse round a lake the work may apparently check and " close " quite correctly, though any computed lengths or areas may be inaccurate, as, for instance, if the length of the chain does not agree exactly with the standard.

Errors in the angular measurements, or errors in, but not directly proportional to, the linear dimensions, may be detected, however, as explained later.

When a traverse is not entirely independent, but is " closed " owing to the fact that it connects two points previously located by accurate triangulation, then the above-mentioned cumulative errors in the linear dimensions may also be detected and adjusted.

Primary surveys are occasionally made by means of traverses in lieu of triangulation, but if the country is at all suitable, the latter method is much to be preferred.

(*b*) *An Unclosed Traverse.*—When a traverse does not form a complete circuit it is said to be unclosed, and as a rule cannot be so accurately checked or adjusted.

Such a traverse is useful when the survey of a long narrow strip of country, such as the valley of a river or a coast-line or a long meandering road, is required.

Exceptionally long unclosed traverses may be approximately checked by the determination of the latitude and longitude of each extremity, and possibly of intermediate points, by astronomical observations : but, needless to say, these determinations do not yield results sufficiently accurate to check anything but such considerable errors as might accumulate over very extensive surveys, or in such exploration or geographical traverses as those in which the lengths of the lines are computed from the rate and duration of marches, etc., and the angles from rough compass bearings.

The direction of the traverse may be checked, however, with a considerable degree of accuracy if, periodically, the azimuth of one of the lines is determined astronomically (see Chapter XVII) and the result compared with the traverse data.

Underground surveys are necessarily almost always traverses—either closed or unclosed—and the " dial " is very largely used for these, though the theodolite is also employed.

Methods of Traversing

The chief methods by which the relative directions of the " lines " may be obtained are as follows :

(i) By chain or tape measurements at the junctions of successive lines. This method is illustrated in Fig. 5.7, and needs no

further explanation. It cannot be relied upon to give any great degree of accuracy.

(ii) By an independent observation of the magnetic bearing of each line—*i.e.* by the " Free or Loose Needle Method ".

(iii) By the measurement of the " included angles " between successive lines.

(iv) By the " fast " needle method of determining the bearings from the true or any arbitrarily assumed meridian.

The Free or Loose Needle Method of Traversing.—In determining the magnetic bearing of a line AB with either a dial or a theodolite, the instrument is set up at one extremity, A, of the line, while the vernier is clamped to coincide with the zero of the scale. The line of sight, etc., is then rotated about the vertical axis until the 360° graduation of the scale, or the central division of a trough compass, lies immediately below the N. end of the needle. That is to say, the line of sight is directed along the magnetic meridian, and the vernier reads zero. A tubular compass capable of very accurate setting is shown in Fig. 2.19, the magnified fine end of the edge-bar needle being centred between the two vertical lines ruled on the glass diaphragm.

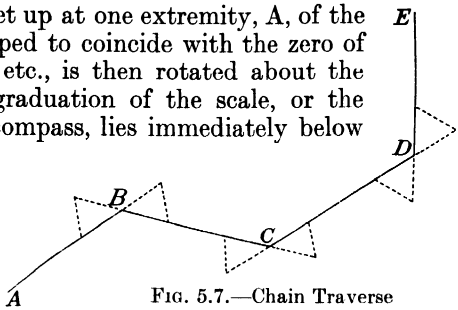


FIG. 5.7.—Chain Traverse

The vernier is now unclamped and the line of sight directed towards B, exact coincidence being obtained with the upper clamp and tangent screw of a theodolite, or by means of the large racking screw of a dial. This motion is recorded by the movement of the vernier over the primary scale of the instrument.

Very accurate results cannot, of course, be expected, as the reading of the needle when in the meridian is not made with a vernier.

When using a dial, or a circular box compass of a theodolite, the vernier may be dispensed with if desired, and *both* readings taken by observing the reading indicated by the end of the needle: it is preferable, however, to use the vernier and read the needle as a check.

The " Free or Loose Needle Method " of traversing especially refers to dialling, and consists of the determination of the bearing of each line independently in this way; the results obtained, however, are not generally so accurate as those found by methods (iii) and (iv).

If time is of much importance, a dial survey may be very quickly executed by setting up the instrument at alternate stations only, and observing the bearings of two lines from each. In this case no check is possible unless the traverse " closes ", and any local attraction of the needle at a station is undetected.

A preferable method is to set up the instrument at *every* station, and to observe the bearings of both the lines meeting there; in this

way the bearing of each line is determined twice, once from either extremity. This procedure enables the local attraction at any particular station to be detected and its amount estimated.

A numerical example will explain this.

Line	Fore Observation	Back Observation	Corrected Bearing	Distance in Links	Remarks
1-2	24°-37′	24°-36′	24°-36½′		
2-3	48°-13′	45°-10′	48°-13′		
3-4	37°-27′	41°-00′	40°-30′		
4-5	18°-04′	17°-32′	17°-33′		

In the above table, column (1) gives the designation of the particular line ; column (2) gives the observed bearing when sighting in a forward direction, while column (3) gives the observed bearing as deduced from the backward reading. Thus in column (2) is given the apparent bearing (24°-37′) of the line 1-2, when sighting forward from station (1) ; and in column (3) is given the apparent bearing (24°-36′) of the same line 1-2 (note, *not* of the line 2-1) when sighting back to station (1) from station 2.

The observed bearing of the line 2-1 from station (2), when using the same hair-line sights as before, would be 204°-36′, and by adding or subtracting 180° from this reading the bearing of 1-2 is deduced as 24°-36′. This calculation may be omitted and the bearing of 1-2 (as distinguished from that of 2-1) observed by applying the eye to the opposite hair-line frame to that used for the forward readings. Thus, for forward readings the eye is applied to that frame which is S., when the vernier is at zero and the line of sight in the meridian, while for back readings the eye is applied to the opposite frame. (With a theodolite the same result would be obtained by reading the opposite instead of the leading vernier ; or by transitting the telescope and still reading the leading vernier.)

For the line 1-2, the discrepancy between the two observed values is small, and the average, *i.e.* 24°-36½′, may be taken as the correct value, and entered in column (4).

If it is known that station (1) is free from any local attraction, evidently station (2) is also free, so that the reading 48°-13′ should be the correct bearing of the line 2-3. The bearing observed from (3) is, however, only 45°-10′, so that the needle is there deflected 3°-03′ from its true position towards the east. Consequently the forward bearing of 3-4 will probably be affected by a like amount, *i.e.* the reading should be 37°-27′ + 3°-03′ = 40°-30′ instead of 37°-27′ as observed.

The back bearing of this line 3-4 is found to be 41°-00′, so that probably at the station (4) local magnetic influences are attracting the needle and causing it to deflect about 30′ towards the west, and there-

fore causing the fore observation 4-5 to have too large a value. The forward bearing of 4-5 then should be 18°-04′ − 30′ = 17°-34′, which agrees very nearly with the observed back bearing of 17°-32′ from (5): the average 17°-33′ is entered in column (4). Station (5) is thus apparently free from any serious local attraction.

Traversing by the Method of Included Angles.—If the relative directions of the lines are to be determined by the measurement of the included angles, the procedure is as follows.

Either the true bearing of one of the lines AB is ascertained by astronomical observations as explained in Chapter XVII, or the magnetic bearing is found as described above, after which the angles between the lines are measured independently by the usual methods, any angular instrument, such as a theodolite or dial or box sextant, being used for the purpose.

Generally the traverse is conducted in a counter-clockwise direction when the inward angles are observed. If a clockwise direction is adopted, the observed angles are the outward angles, as the horizontal scale of an ordinary theodolite or dial is graduated in a clockwise direction.

The method of included angles has the advantages:

(1) That any error in the line of collimation does not seriously affect the results unless the stations are at considerably different levels, because the telescope is not transitted as in the fast needle method described later; and,

(2) The angles can be observed by the process of Repetition to any required degree of accuracy within limits, so that this method would be adopted in a large survey where a traverse is being used in lieu of the more usual triangulation method, for a primary survey.

Fast Needle Method of Traversing.—" Fast needle " is a term generally confined to Dialling, but the principle is the same when applied to a theodolite traverse such as ABCD . . . H (Fig. 5.8).

The theodolite is set up and levelled at A, and the bearing of AB ($=a$) ascertained as already described, the bearing of the last line AH being also noted as a check, if the traverse is " closed ", while the instrument is at this station.

The instrument is then set up and levelled at the next point B, and, with the vernier still clamped at a, the telescope is directed back to A, exact coincidence being obtained with the *lower* clamp and tangent screw. If the theodolite is of the transit form, the telescope may next be rotated through 180° in a vertical plane (*i.e.* transitted), so that the line of sight is directed along the direction of AB produced, *i.e.* along Bb, while the vernier still records a.

If, therefore, the upper clamp is loosened and the telescope rotated in a horizontal plane through an angle a in a counter-clockwise direction, *i.e.* until the vernier reading is zero, the telescope will lie along a meridian Bb_1 parallel to the first meridian Aa_1. Consequently when

the telescope is directed to C, the vernier records the angle b_1BC, which is the bearing (β) of BC.

The process is next repeated by setting up the instrument at C, the vernier still reading β, unfastening the lower clamp, sighting back to B, reclamping and adjusting with the lower tangent screw, and then transitting the telescope so that the line of sight is directed towards c in BC produced, while the vernier still records β. Then on unclamping the upper plates and directing the telescope towards the next point D, the reading of the vernier indicates the bearing (γ) of the line CD: it is unnecessary, of course, actually to adjust the vernier to zero each time, and so to place the telescope in the meridian.

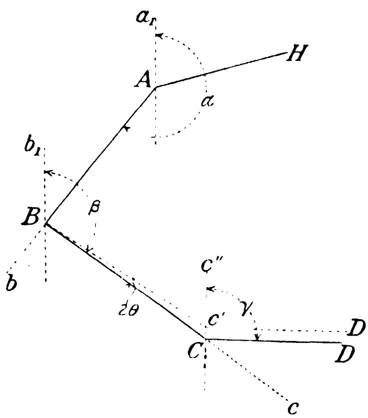


FIG. 5.8

By proceeding round all the points in turn, eventually the bearing of the last line HA is observed from H, and as a check upon the accuracy of the work, this (in the case of a closed traverse) should be equal to the value of the bearing of AH taken from A, plus or minus 180°.

Whether the traverse is closed or unclosed, if a circular box compass is attached to the upper vernier plate, an approximate check upon the angular measurements may be made at any point (*e.g.* C) after the theodolite has been levelled, directed to the preceding point and transitted, by turning the instrument about the inner axis until the vernier records zero reading, when the north and south ends of the compass needle should lie over the N. graduation of the compass scale at alternate stations. For any other position of the telescope the vernier reading should agree with that below the needle on the dial.

If the instrument is provided with a trough compass attached to the lower or scale plate, then the needle when lowered on to its pivot should lie with its north end exactly above the central division of its scale, at alternate stations. This check may be applied at any time after the instrument has been set up and oriented by sighting back to the previous station, and as the position of the scale plate, and therefore that of the trough compass, is not affected by any movement of the vernier relative to the scale plate, it is not necessary to turn the telescope into the meridian and adjust the vernier to zero.

At the remaining stations the N. end of the needle lies in the south end of the trough, and, the state of equilibrium over the centre division of the scale being unstable, no check is afforded.

In the method of procedure described above it will be noted that the telescope is inverted for alternate backward and forward readings, so that, if the line of collimation is not exactly at right angles to the

horizontal transverse axis of the telescope, the " inward angles " are made too large and too small alternately.

Thus, sighting	from	A to B,	the telescope	is in its	normal	position.
,,	,,	B to A,	,,	,,	normal	,,
,,	,,	B to C,	,,	,,	inverted	,,
,,	,,	C to B,	,,	,,	inverted	,,
,,	,,	C to D,	,,	,,	normal	,,
		etc.				

If the line of collimation lies slightly to the left of the longitudinal axis of the telescope when in its normal position, the deviation being say θ, then the observed value of the bearing of C is too small by an amount 2θ. This error tends to be neutralized at the following point, and the correct *bearing* of D is obtained.

The inward	angle at	B	is therefore too	small	by an	amount 2θ.
,,	,,	C	,,	large	,,	,,
,,	,,	D	,,	small	,,	,,

Consequently as the bearings of alternate lines are correct, and those of the remainder are 2θ too small, then, assuming the lines are of approximately equal length, the resulting error in direction over the whole distance will be about θ.

The nature of the total resulting error, *i.e.* the displacement of the last point of the traverse, may be studied by considering the three points BCD.

Thus let ABCD . . . be the true positions of the stations, and ABC′D′ the determined positions, and let the lengths $AB = l_1$, $BC = l_2$, $CD = l_3$, etc.

Then as the angle ABC′ is less than the true value of the angle ABC by an amount 2θ, therefore $\angle CBC' = 2\theta$.

But as $\angle BC'D'$ is too large by an amount 2θ, and C′D′ is parallel to CD, consequently, as C′D′ is assumed to be of the correct length CD, CC′ = DD′.

Similarly, BC′ = BC, so that $CC' = 2BC \sin \frac{1}{2} \angle CBC'$, *i.e.*

$$CC' = DD' = 2l_2 \sin \theta.$$

The additional displacement of the station next but one to D, say F, due to the error of 2θ in the angles at D and E is $2l_4 \sin \theta$, and so on, so that the final displacement of the last point due to this cause may be taken as $2 \sin\theta \, (l_2 + l_4 + l_6 + \ . \ . \ .)$, or, if the courses are approximately equal, this amount is very roughly equal to $L \sin \theta$, where L is the total length of the traverse lines.

Sometimes, as an alternate method,[1] the telescope is transitted at each station immediately before a sight is taken back to the station last vacated; that is to say, the telescope is inverted for each back reading, and is in its normal position for each forward reading.

[1] In this case the trough compass should be correctly oriented at every station instead of at alternate stations only.

Thus, sighting from	A to B,	the telescope is in its	normal	position.
,, ,,	B to A,	,, ,,	inverted	,,
,, ,,	B to C,	,, ,,	normal	,,
,, ,,	C to B,	,, ,,	inverted	,,
,, ,,	C to D,	,, ,,	normal	,,

etc.

Consequently the error at each station tends in the same direction, and in the case assumed the deduced value of each inward angle is 2θ too large, while if the bearing of AB is correct that of BC is 2θ too large, that of CD is 4θ too large, and that of DE is 6θ too large, etc.

The total angular error in a closed traverse of N. sides from this cause is thus $2N\theta$, and the total displacement of the last point of a closed or unclosed traverse is $2 \sin\theta \ (l_2 + l_3 + l_4 + \ . \ . \ . \ )$, or approximately $2L \sin\theta$.

In a closed traverse, however, the total error in the inward angles can be deduced from the formula already stated (p. 125), or by a comparison of the two observed bearings of the closing line AH; so that if the total angular error is divided equally between all the angles, as is usual, the portion of the error due to this source will be correctly distributed and its effect eliminated.

In the first-mentioned method each angle will be incorrect by an amount 2θ, and though the *total* error in the angles may be apparently zero, a portion of the closing error in length will be due to this cause.

For a closed traverse, consequently, the second method is preferable, as the error is capable of accurate adjustment.

For an unclosed traverse conducted by the " Fast Needle " method the first method is preferable, as the distortion, which will probably remain undetected, is only half the amount of that produced by the second method.

In the case of a Y instrument, as the telescope cannot be transitted it could be lifted from its supports and reversed end for end, but this would obviously be undesirable, and preferably the instrument would be rotated through 180° in azimuth, as indicated by the verniers, and readings taken upon the opposite vernier to that used at the previous station.

The advantages of the fast needle method are :

(1) That the bearings are obtained directly and can be roughly checked without any trouble by means of the compass as the work proceeds, so that any large error can be detected almost immediately, whereas with method (iii) no check is afforded until the traverse is closed.

(2) The field work is more expeditious, and the calculations are simplified as the third column of " included angles " (p. 148) is rendered unnecessary because the requisite check is given in a closed traverse by comparing the bearing of the last line HA with that of $AH \pm 180°$.

Underground Traverses.—When traversing underground with a dial—either by the " Loose Needle " method or by the " Fast Needle "

method—it is advantageous to employ three tripod stands, which are usually of very light construction.

The procedure for a Fast Needle Survey is as follows :

One of the spare tripods is set up at A, immediately over the first station-point of the traverse marked by a peg in the ground, or under a fixed mark in the roof of the workings. Upon this tripod a lamp is placed, and the dial is taken ahead as far as possible to the second station B, the position of which is so chosen that the lamp on A is clearly visible.

The third tripod is taken still farther ahead to the third traverse station C, this point being chosen so that a lamp on the dial at B is visible.

There is now one tripod at A, one at C, and one to which the dial is fixed at B.

The vernier is set at the zero of the scale by means of the peg in the bottom of the box, and the head of the instrument is rotated until the N. end of the needle lies exactly over the N. graduation of the card, and it is then clamped.

The line of sight will, in consequence, lie in the magnetic meridian if the station B is free from any local attraction.

Let the vane which is to the south in this position be *s*, and let the one which is north be *n*.

To determine the bearing of the line BA, the needle having been lifted from its pivot to prevent any unnecessary wear, the peg is removed from its socket, and the outer part of the instrument rotated by means of the large racking screw until the sights *sn* are directed to A, the magnitude of the movement being recorded on the vernier.

In order, therefore, to determine the bearing of AB—which differs 180° from that of BA, it is only necessary to sight to A with the vanes reversed, *i.e.* to apply the eye to *n* instead of to *s*.

The bearing (α) of AB having been observed in this way and noted, the line of sight is next directed by means of the racking screw to the lamp at C, the eye being applied to *s* in this case in order to observe the bearing (β) of BC.

The dial is then clamped, detached from the tripod at B, and attached to that already standing at C ; the tripod from A is taken ahead to a new position D, from which a lamp on the dial at C is visible, and a lamp is placed upon this stand at D, and another upon that just vacated at B.

While the reading on the vernier still records β, the dial is oriented at C until the line of sight *ns* is directed to B, with the result that the instrument is placed in the same position relative to the magnetic meridian that it occupied at B.

The lower clamp is then fastened, and, by means of the large racking screw, the line of sight *sn* is rotated until it is directed to D, when the bearing γ of CD is recorded by the vernier.

Similarly, the bearings of the remaining lines are determined, the dial being next transferred to D, and the tripod from B taken ahead to E, and so on.

A rough check upon the accuracy of the angular measurements may be made at any station-point after the dial has been oriented, provided that there is no local attraction.

This is done simply by lowering the needle on to its pivot so that it may swing freely, when the reading under the N. point should agree with that of the vernier.

If it is suspected or known that the point B is influenced by some local magnetic attraction, the traverse may be commenced from any arbitrary meridian, *e.g.* the vernier may be fixed at zero while the vanes *ns* are directed to A.

The bearing of some one or more lines may be observed later by liberating the needle at various intermediate station-points of the traverse.

The lengths of the traverse courses are obtained with a chain or tape in the usual manner.

Expeditionary or Route Surveys are generally compass traverses of a more or less rough nature, as often the main object of the expedition is not primarily the production of a map.

The bearings of the traverse lines may be observed with an ordinary compass, or preferably with a prismatic compass.

The procedure may include :

(1) The observation of the forward and back bearing of each line as in the loose needle method of dialling, or

(2) The forward bearing only of each line, or

(3) General observations as to the mean direction of the route, *e.g.* if the traverse is not being conducted in straight lines, but paths, roads, tracks, or rivers are being followed, possibly through woods or forests. If the bends are not very frequent the method reduces to (2) above, but otherwise a number of bearings may be taken at equal intervals of time, say every 5, 10, 15, or 30 minutes, and a general idea of the route deduced from these records.

In addition, bearings should, if possible, be taken along the route to distinctive objects in the distance, as the information thus obtained materially assists in the final adjustment of the survey.

To determine whether the readings are being affected by variable local magnetic influences, the deviation of the compass should be checked occasionally by comparison with the true meridian determined astronomically ; and both forward and back bearings should be taken of at least a few of the lines and the results compared.

The linear distances are not generally ranged out between station-points, though in a small survey, in order to take forward and back bearings more accurately with the compass, one pole or flag is held at the station last vacated, and another is taken ahead to the extremity of the next line.

Frequently, however, the line of march is governed by the tracks or roads, or straight lines are ensured merely by marching directly towards various prominent natural objects in turn—in which case the bearings are taken without the use of flags or poles, the resulting errors being largely compensating in their nature.

The various methods of determining the distances are :

(1) By pacing—in which case the length of pace should be ascertained by experiment, and checked occasionally, on the level and both up and down hill, as there is a natural tendency to shorten the pace when proceeding up hill, and to lengthen it down hill. After a considerable amount of practice, a person may teach himself to adopt a moderately equal length of pace under various circumstances, and attain results with a probable error of say ± 1 per cent.

This method is, however, very monotonous for long distances, and the liability to error from miscounting is considerable.

The paces of a horse, mule, or other animal may be counted in a similar way, the length of pace being ascertained from a number of observations taken under different conditions (*e.g.* at a gallop, trot, or walk on the level or on inclined ground) over known distances.

(2) By attaching a passometer to the person who is pacing the distances the number of steps is recorded automatically, and this is preferable since, the monotony and strain of continuously counting being obviated, the attention of the Surveyor is not so fully occupied, and he is free to make observations and notes of the surrounding country.

The passometer is generally of watch form, and may be carried in the waistcoat pocket or suspended from one of the buttons.

The pedometer is a similar contrivance, but graduated to read distances directly instead of the number of paces. It can be adjusted to suit different lengths of pace, and is fitted with an arrangement for setting to zero, and a screw stop.

(3) By means of a perambulator—sometimes termed a viameter or wheel pedometer. This contrivance is similar to a single bicycle wheel provided with forks and a handle ; it is wheeled along the route, and the number of revolutions recorded automatically. The distance traversed is equal to the number of revolutions multiplied by the circumference of the wheel, and this is registered on a counter or dial. On rough or hilly country the registered reading is obviously too high, and an allowance must be made.

(4) By means of a trocheameter or odometer, attached to the wheel of a carriage, cart, wagon, bicycle, etc., the number of revolutions is recorded, and from this, knowing the diameter of the wheel, the distance traversed may be deduced. There are several other contrivances of a similar nature.

(5) By means of a telemeter as explained in Chapter VIII.

(6) The distances may be less accurately determined by time intervals, *i.e.* the rate of progress is estimated, and the time interval between the various instrument observations noted. From this, after deducting for halts, etc., the lengths may be roughly computed.

In addition to land marches, this method is also applicable to journeys by boat along a river or stream.

The observations are supplemented by sketches and notes of features which it is desirable to record.

If the survey is long, approximate checks may be obtained by

means of astronomical observations of latitude and azimuth, and the survey "balanced" as explained later: similarly if any points *en route* have been otherwise located, a correction may be applied.

Generally, observations for latitude are not sufficiently refined to afford much direct check on a traverse, but observations for azimuth are very useful.

Reduction of Field Notes by the Method of Co-ordinates

To plot a traverse, the lines may be scaled directly from the field notes, and the direction of each fixed by means of a protractor, but a preferable procedure is to employ the method of co-ordinates, after "balancing" or distributing any resultant error over the whole of the work.

A convenient form into which the necessary data abstracted from the field book may be entered is given in Tables I and II, p. 148.

The angles, if preferred, may be entered directly into these columns at the time of observation, instead of into an ordinary field book.

As previously mentioned, when the fast needle method of traversing is adopted, it is not necessary to reduce the values for the third and fourth columns of inward angles, as in this case the bearing of the line AF would be observed from A, and the amount of any inaccuracy in the sum of the angles would be apparent from the comparison of this bearing with the bearing of FA observed from F $\pm 180°$.

When the method of included angles is adopted for a closed traverse the sum of the angles in column (3) should be equal to $(2N - 4)$ right angles where N is the number of "sides" of the polygon. If the total of these proves incorrect, the angles are adjusted by distributing the discrepancy as equally as possible between them, because there is no reason why a large angle should be either more or less inaccurate than a small one.

This adjustment would not be carried out very rigorously—*e.g.* if the original readings had been taken to the nearest minute, the corrections would not usually be applied in any smaller units than one minute, but the error would otherwise be distributed as evenly as possible.

The adjusted inward angles are entered as shown in column (4). It is then unnecessary to work out the values of column (5)—but the reduced bearings of column (6) are calculated directly.

A reduced bearing is the smaller of the two angles which the chain line makes with the meridian, so that it cannot be greater than 90°. It is always stated as the angle measured from the meridian, *e.g.* N. $a°$ E., or N. $a°$ W., or S. $a°$ E., or S. $a°$ W., and not as an angle from the E. and W. line.

For example, a whole circle bearing of 268°-30′ is expressed as a reduced bearing of S. 88°-30′ W., and not as W. 1°-30′ S.

The method of calculating the values of the reduced bearings given in column (6), Table I, from the included angles in column (4),

is preferably by the aid of small diagrammatic sketches rather than by the application of formulae.

Thus in Table I, if the inward angle at C is 64°-57′, and that at D is 211°-30′, and the reduced bearing of CD is S. 24°-16′ E., as in Fig. 5.9, then the reduced bearing of DC is N. 24°-16′ W., and the reduced bearing of DE is (211°-30′ - 24°-16′) - 180°-0′ = S. 7°-14′ W.

Similarly if the whole circle bearings are observed in the field instead of the included angles, the reduced bearings can be easily deduced, and the included angles calculated by the aid of a sketch such as Fig. 5.9.

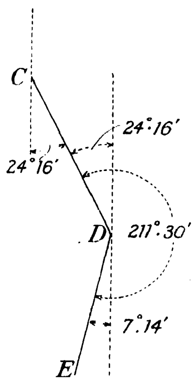


Fig. 5.9

The bearings tabulated in columns (5) and (6) may be (1) actual bearings computed from the observed azimuth [1] of one of the traverse lines, or (2) magnetic bearings computed from the observed magnetic bearing of one of the lines as described previously, or (3) any arbitrary line, *e.g.* one of the long chain lines, may be referred to as a datum.

Occasionally the magnetic south or the true south is taken as the zero, instead of the more usual convention which assumes the northerly direction to be zero.

Consecutive Co-ordinates.—In the first instance, the co-ordinates of each point are calculated with reference to the point immediately preceding it; *i.e.* the resolved components parallel and at right angles to the meridian are computed for each line taken in order round the polygon. These distances, or consecutive co-ordinates, are termed Northings, Southings, Eastings, or Westings, though sometimes the terms Latitudes and Departures are employed,

A Northing being a + Latitude,
A Southing being a − Latitude,
An Easting being a + Departure,
A Westing being a − Departure.

This notation is to be preferred when a " meridian " other than the true or the magnetic meridian is assumed as a datum.

It is easily seen, then, that the magnitude of any particular latitude is obtained by multiplying the length of the traverse line to which it refers, by the cosine of its reduced bearing, while the departure is similarly calculated from the sine of the angle.

Example.—The line CD in Table I has a length of 296·4 ft., and a reduced bearing of S. 24°-16′ E.

The Southing is 296·4 × cos 24°-16′ = 270·2 (by logarithms).

The Easting is 296·4 × sin 24°-16′ = 121·8 (by logarithms).

In the alternative notation the Latitude of D from C is − 270·2, and the Departure is + 121·8.

Error of Closure.—It is now apparent that when the co-ordinates of each point have been determined with reference to the preceding

[1] See Chapter XVII.

point, and a complete circuit has been made, if the work is correct the sum of the Northings will equal those of the Southings, and the sum of the Eastings will equal those of the Westings.

If not, the discrepancy gives what is known as the " closing error " or " error of closure ".

In the particular example in Table I the Southings exceed the Northings by 1·3, and the Westings exceed the Eastings by 1·2, so that the resultant error is $\sqrt{1{\cdot}3^2 + 1{\cdot}2^2} = 1{\cdot}77$ ft., as shown in Fig. 5.10, where A is the initial point and A_1 the point at which the circuit finishes.

The bearing of the closing error AA_1 is S. $\tan^{-1} \dfrac{1{\cdot}2}{1{\cdot}3}$ W., *i.e.* S. 42°-42½′ W.

This error is usually expressed as the fraction

$$\frac{\text{Displacement}}{\text{Perimeter of traverse}} = \frac{1{\cdot}77}{3464{\cdot}6} = \frac{1}{2000} \text{ nearly.}$$

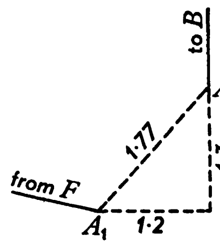


FIG. 5.10

The maximum allowable value of this fraction of course depends upon the instruments available, and upon the scale to which the survey is to be plotted.

It may be noted that any small inaccuracy in the length of the chain itself will practically have no effect upon the closing error, as cumulative errors of this kind merely enlarge or diminish the figure as a whole in the proportion that the actual length of the chain is less or greater than the true length.

If, however, the traverse is being conducted between two points previously determined by triangulation or other means, then cumulative as well as compensating errors have effect, and the allowable error should be larger than in the former case.

Balancing the Survey.—It is generally desirable that the closing error—which though within the allowable limit, may be an appreciable amount—shall be so distributed throughout the whole of the survey that its effects shall be as little apparent upon the plan as possible. This operation is known as " balancing " and is usually accomplished by one of the methods described below.

1. *Rational Methods.*—The closing error is the cumulative effect of errors in the measurement of the angles and errors in the linear measurements. The relative importance of these two sources of error varies according to the instruments used, the character of the country, etc. A logical method of balancing should make allowance for these factors. One such method is suggested by Higgins,[1] and others by Crandall,[2] Rappleye,[3] etc. These methods, though more rational than those usually employed, involve more laborious calculations, and consequently they have not been widely adopted by practical surveyors. The general view is that, provided the closing error is distributed throughout the survey, the exact procedure is not very important,

[1] " The Adjustment of Transverse Surveys ", *Engineering*, vol. **148**, p. **218**.
[2] *Text Book on Geodesy and Least Squares.*
[3] *Proc. Am. Soc. C.E.* vol. 55.

since the differences in the effects on the plan are almost indistinguishable for the different methods.

This argument is not perhaps very satisfactory, and there is much to be said for adopting more rational methods. It must be remembered, however, that even the rational methods only estimate the most likely corrections : the real errors may be either more or less in any particular case.

2. *Bowditch's Method.*—Bowditch's rule for distributing the "error of closure" has a theoretical basis as is explained later, though the fundamental assumptions may not be altogether justified. It is assumed that errors in the linear dimensions are proportional to $\sqrt{l}$,[1] where l is the length of a line, and that errors in angular measurements are inversely proportional to $\sqrt{l}$, which is questionable. On these bases it is deduced (p. 144) that, under conditions which render it probable that the angles and lines are equally liable to error, the most probable distribution of the error is obtained by making the corrections for latitudes and departures each proportional to the lengths of the *traverse* lines (see below), *e.g.* if Y be the total closing error in latitude and l_1, l_2, . . . the lengths of the respective lines, the corrections to be applied to the calculated latitudes are

$$\frac{l_1}{\Sigma l} \cdot Y, \quad \frac{l_2}{\Sigma l} \cdot Y, \quad \ldots,$$

the sign in each case being such as to diminish the error.

Similarly, the total error in departures is distributed

$$\frac{l_1}{\Sigma l} \cdot X, \quad \frac{l_2}{\Sigma l} \cdot X, \quad \ldots,$$

among the lines l_1, l_2, etc.

It is usual to balance the angles before calculating the co-ordinates if the data permit of this being done, but this point is discussed more fully later.

Example.—In the traverse, details of which are given in Table I, the error of 3 minutes in the sum of the included angles was distributed arbitrarily, 1 minute being added to each of the angles B, E, and F. It would have been equally satisfactory to have distributed the error evenly by adding 30 seconds to each of the six angles—particularly if convenient tables were available. If it were felt that certain angles were more liable to error than others, a larger share of the total error could be assessed arbitrarily to those angles.

The co-ordinates having been calculated, the sum of the Southings is found to exceed the sum of the Northings by 1·3 feet. The correction to be applied to AB is therefore $(1080 \div 3464{\cdot}6) \times 1{\cdot}3 = 0{\cdot}4$ nearly. This is entered in column 2, Table II, and is added to the Northing 1080, giving a corrected consecutive co-ordinate of 1080·4 (column 4).

Similarly the correction for DE is $(718{\cdot}8 \div 3464{\cdot}6) \times 1{\cdot}3 = 0{\cdot}3$ nearly. This is entered in column 2, and is deducted from the Southing 713·1, giving a corrected consecutive co-ordinate of 712·8 (column 5).

As shown in Table I, the sum of the Westings exceeds the sum of the Eastings by 1·2 feet. The corrections to the departures are given in column 3 and the corrected co-ordinates in columns 6 and 7, Table II.

All such corrections may be calculated quickly and conveniently on a slide-rule to a degree of accuracy amply sufficient for the purpose.

[1] See p. 42 and Chapter XIX.

3. *Graphical Method.*—For a rough survey such as that made with a prismatic compass, the above method of correction may be applied graphically, without any previous adjustment of the angles. Thus let ABCDA (Fig. 5.11) be the uncorrected plan of a closed traverse, plotted directly from the field notes, and let the closing error be AA_1.

To any convenient scale—not necessarily that of the traverse itself—measure off along a straight line A′A″ consecutive distances A′B′, B′C′, C′D′, D′A″, to represent the sides of the traverse.

From A″ draw A″a' parallel and equal to the closing error AA_1 (Fig. 5.11). Join A′a', and through B′C′D′ . . . draw a series of lines parallel to A″a', cutting A′a' in $b'c'd'$ . . .

The intercepts B′b', C′c', D′d' A″a' are then the corrections in magnitude and direction to be applied at the points $BCDA_1$.

Through BCD draw short rays parallel to AA_1, and measure off along them distances Bb, Cc, Dd, A_1A equal to B′b', C′c', D′d', A″a' respectively giving AbcdA as the adjusted plan of the traverse.

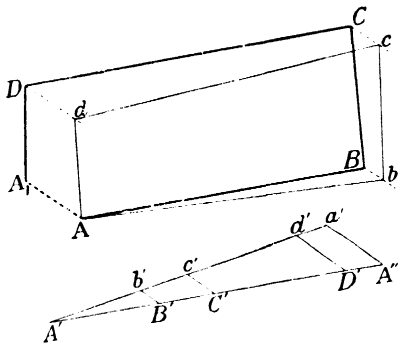


FIG. 5.11

4. *Transit Rule.*—Generally, the angles of a theodolite traverse are measured to a much greater degree of accuracy than the sides, and a well-known empirical rule for such a case states that the latitude (or departure) correction for any line should be proportional to the *latitude* (or *departure*) instead of to the length of the line itself. It is claimed that the angles are less affected by these corrections than by those derived by Bowditch's method. This claim is probably true in most cases, but it does not follow that the angular corrections do not then exceed the probable errors in their measurement (see p. 150).

If the closing error in latitude is Y and the latitudes of the various lines are $y_1, y_2, y_3, \ldots$, while the sum of the latitudes is Σy, the corrections to be applied to the various latitudes are

$$\frac{y_1}{\Sigma y} \,.\, Y, \quad \frac{y_2}{\Sigma y} \,.\, Y, \quad \ldots,$$

and the corresponding departure corrections are

$$\frac{x_1}{\Sigma x} \,.\, X, \quad \frac{x_2}{\Sigma x} \,.\, X, \quad \ldots$$

Example.—The error in latitude (Table I) is 1·3, while the sum of the latitudes $= 1090{\cdot}8 + 1089{\cdot}5 = 2180{\cdot}3$.

The correction to be applied to the line FA is therefore

$$\frac{100{\cdot}7}{2180{\cdot}3} \times 1{\cdot}3 = 0{\cdot}1$$

approximately, which makes the corrected Southing 100·6 (Table III).

Similarly, in Table IV the total error in departures = 1·7, while the total of the Eastings and Westings is 2314·1, so that the correction for AB is

$$\frac{764 \cdot 3}{2314 \cdot 1} \times 1 \cdot 7 = 0 \cdot 6,$$

making the Easting for AB = 764·9 (Table VI).

5. "*Axis*" *Method.*—The "Axis" method[1] preserves the general shape of the figure by applying all the corrections to the sides of the traverse and leaving the angles unaltered. The distribution of the errors is obviously an unlikely one, but the displacements on the plan may be so small that this is not important.

Thus, let $ABCDEFA_1$ (Fig. 5.12) be the uncorrected plan of a closed traverse, and let the closing error be AA_1.

Join AA_1. If this line when produced does not cut the figure into

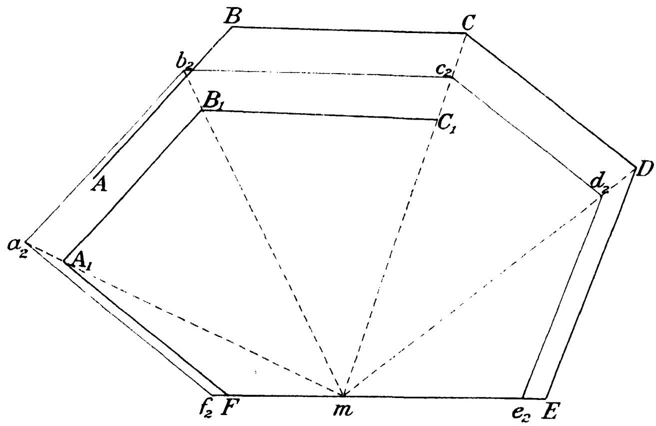


FIG. 5.12.—Axis Method of Balancing Traverse

two nearly equal parts, the error is transferred to some other point, say C_1, by drawing A_1B_1 parallel and equal to AB, and B_1C_1 parallel and equal to BC, etc.

The figure $CDEFA_1B_1C_1$ is now the uncorrected traverse and CC_1 the closing error, and the line CC_1 produced to m divides the polygon into two moderately equal portions.

The line CC_1 is divided at c_2 in the ratio of the perimeter of one portion to the perimeter of the remainder (exclusive of Cm), *i.e.*

$$\frac{Cc_2}{c_2C_1} = \frac{CD + DE + Em}{mF + FA_1 + A_1B_1 + B_1C_1}.$$

The point c_2 is determined from this equation and its position plotted on CC_1, and lines are then lightly drawn from m to D, A_1, B_1 . . .

From c_2, c_2d_2 is drawn parallel to CD to cut mD in d_2.
and ,, d_2, d_2e_2 ,, ,, DE ,, mE in e_2.

[1] See *A Treatise on Surveying*, Middleton and Chadwick, vol. i

Similarly, from c_2, c_2b_2 is drawn parallel to C_1B_1 to cut mB_1 in b_2.
and from b_2, b_2a_2 ,, ,, B_1A_1 ,, mA_1 in a_2.
and from a_2, a_2f_2 ,, ,, A_1F ,, mF in f_2.

The figure $a_2b_2c_2d_2 \ldots f_2a_2$ is then the adjusted traverse, the one portion being slightly enlarged in the ratio $\frac{mc_2}{mC_1}$, and the other reduced in the ratio $\frac{mc_2}{mC}$, from the derived field observations.

If required, the lengths CD, DE, Em can be mathematically adjusted in the ratio $\frac{mc_2}{mC}$, and the lines mF, FA, AB, BC in the ratio $\frac{mc_2}{mC_1}$, the new values being substituted in the traverse table and the co-ordinates calculated as already explained.

Independent Co-ordinates.—After the survey has been balanced so that the sum of the Northings is equal to that of the Southings, and the sum of the Eastings is equal to that of the Westings, *i.e.* after the closing error, instead of being concentrated at (say) A, is distributed throughout the whole of the traverse, the next step is to calculate the " Independent Co-ordinates ".

As already stated, the consecutive co-ordinates of any point are the distances measured from the preceding point in directions parallel and at right angles to the assumed meridian or datum line.

Thus in Table III the point B is 1080·6 ft. N. and ·9 ft. E. of A ; while C is 9·5 ft. N. and 694·6 ft. W. of B ; C is 270·0 ft. S. and 121·9 ft. E. of B ; and so on.

If the survey is plotted from these figures, the accuracy of the position of each point upon the plan evidently depends upon the accuracy with which the previous points are plotted ; so that even although no large mistake is made at any point, the final points may still be displaced to a considerable extent from their true positions, on account of the accumulation of small and possibly unavoidable errors in scaling, etc.

In order to obviate this, the co-ordinates of each point are calculated with reference to one common origin, *e.g.* A, and each point may then be plotted quite independently of any other station.

Example.—In Table III, as B is 1080·6 ft. N. of A and C is 9·5 ft. N. of B, C is 1080·6 + 9·5 = 1090·1 ft. N. of A.

Then as D is 270·0 ft. S. of C, its independent latitude is 1090·1 − 270·0, *i.e.* 820·1 ft. N. of A.

Similarly with the departures, as B is ·9 ft. E. of A and C is 694·6 ft. W. of B, C is 694·6 − ·9 = 693·7 ft. W. of A, and is booked as shown. The independent departure of D is in the same way 693·7 − 121·9 = 571·8 ft. W. of A.

As the last point of a closed traverse is identical with the first point, its independent co-ordinates should be zero, and this fact consequently affords a check upon the correctness of the numerical calculations involved in the preparation of the independent from the consecutive co-ordinates.

By the aid of the table of independent co-ordinates prepared and

checked in this manner each station-point may be plotted from two rectangular axes which pass through the initial point A, and the accuracy of the drawing may be checked by noting whether the scaled distances between consecutive points agree substantially with the chainage values given in the field book and tabulated in column (2), Table I.

A little discrepancy is unavoidable, if the survey has been adjusted, on account of the distribution of the closing error, but any large mistake in the plotting is easily detected.

The offsets, taken and booked in the manner described in Chapter I, are plotted from these chain lines in the usual way.

When the plan of a traverse extends over several sheets of paper, the positions of various points on other sheets than that which contains the origin may be calculated mathematically and plotted by means of co-ordinates referred to the margin lines as axes. Possible errors which might occur in measuring across the intervening sheets are thus eliminated, and any one sheet may be completed quite independently of any other.

Derivation of Bowditch's Rule.

Let $l_1, l_2 \ldots$ represent the measured lengths of the lines,
$\Sigma l \ldots$,, the perimeter $= L$,
$x_1, x_2 \ldots$,, the calculated departures,
$y_1, y_2 \ldots$,, ,, latitudes,
$a_1, a_2 \ldots$,, ,, reduced bearings,
$e_1, e_2 \ldots$,, the probable errors in departures,
$c_1, c_2 \ldots$,, ,, ,, latitudes,
X . . . ,, the total error in departure,
Y . . . ,, ,, ,, latitude.

It is assumed that the probable error δl in the measurement of a line of length l is proportional to $\sqrt{l}$ and this is reasonable for compensating errors. It would not be true for cumulative errors, but, as already explained, these do not affect the closing error of a completely closed self-contained traverse. If we are weighting the lines, therefore, and considering only this error, the weights, which should be inversely proportional to the *squares* of the probable errors (see p. 533), are in the ratios

$$w_1 : w_2 : w_3 : \ \ldots \ = 1/l_1 : 1/l_2 : 1/l_3 : \ \ldots$$

It is further assumed that if the total displacement δd of one end B of a line relative to the other end A is made up of two parts—a displacement δl along the line and a displacement δr at right angles to the line—then the probable values of these two displacements are equal. Thus if $\delta r = \delta a \,.\, l = \delta l = k\sqrt{l}$, then δa varies as $1/\sqrt{l}$, *i.e.* again the weights of the lines are

$$w_1 : w_2 : w_3 : \ \ldots \ = 1/l_1 : 1/l_2 : 1/l_3 : \ldots$$

It is usually considered that the accuracy of measurement of an

angle does not depend very much upon the lengths of the lines, except that for very short sights the error is likely to be more than with long sights. The assumption that $\delta r = \delta l$ cannot be true for all types of traverse; and where the angles are measured with a theodolite, and the lines by direct chaining, δr and hence δa is overestimated. Bowditch's rule is considered most suitable for compass traverses, for which it was originally formulated.

The probable total displacement of one end of a line relative to the other is obtained by combining the probable errors δl and δr, *i.e.* $\delta d = \sqrt{\delta l^2 + \delta r^2}$.

It can also be expressed as $\sqrt{c^2 + e^2}$, where c is the probable error in latitude and e that in departure, so that the weights of the lines when considering either form of δd are in the ratios of the reciprocals of the lengths of the lines.

Now by the theory of least squares the sum of the weighted squares of the errors must be a minimum, *i.e.*

$$\Sigma w(\delta d)^2 = \text{minimum},$$

i.e.

$$\Sigma \frac{c^2 + e^2}{l} = \text{minimum}. \qquad (1)$$

We also have

$$\Sigma c = \text{Y} \quad \text{and} \quad \Sigma e = \text{X}. \qquad (2)$$

Differentiating (1) and (2) we have

$$\left.\begin{aligned} \Sigma \frac{c\delta c + e\delta e}{l} &= 0 \\ \Sigma \delta c &= 0 \\ \Sigma \delta e &= 0 \end{aligned}\right\} \qquad (3)$$

As explained in Chapter XIX, these three equations can be written as

$$\left.\begin{aligned} c_1\delta c_1/l_1 + c_2\delta c_2/l_2 + c_3\delta c_3/l_3 + \ldots \\ + e_1\delta e_1/l_1 + e_2\delta e_2/l_2 + e_3\delta e_3/l_3 + \ldots \end{aligned}\right\} = 0, \qquad (4)$$

$$\lambda_1\delta c_1 + \lambda_1\delta c_2 + \lambda_1\delta c_3 + \ldots = 0, \qquad (5)$$

$$\lambda_2\delta e_1 + \lambda_2\delta e_2 + \lambda_2\delta e_3 + \ldots = 0, \qquad (6)$$

where λ_1 and λ_2 are undetermined coefficients known as correlates or correlatives.

Now $\delta c_1, \delta c_2, \ldots \delta e_1, \delta e_2 \ldots$ are not definitely related, so that the coefficients of each must vanish separately when the three equations (4), (5), and (6) are combined.

Adding, we have—

$$\left.\begin{array}{lll} \text{for } \delta c_1, & c_1/l_1 + \lambda_1 = 0, \text{ and hence } & \lambda_1 = -c_1/l_1 \\ \text{for } \delta c_2, & c_2/l_2 + \lambda_1 = 0, \quad ,, \quad ,, & \lambda_1 = -c_2/l_2, \text{ etc.} \\ \text{and for } \delta e_1, & e_1/l_1 + \lambda_2 = 0, \text{ and hence } & \lambda_2 = -e_1/l_1 \\ \text{and this is equal to} & & -e_2/l_2, \quad -e_3/l_3, \text{ etc.} \end{array}\right\} \qquad (7)$$

Substituting for $c_1 + c_2 + \ldots + c_n = Y$ in equation (2),

$$-\lambda_1(l_1 + l_2 + \ldots l_n) = Y,$$

i.e.
$$\lambda_1 = -Y/\Sigma l = -c_1/l_1 \text{ from (7),}$$

and
$$c_1 = \frac{l_1}{\Sigma l} \cdot Y,$$

$$c_2 = \frac{l_2}{\Sigma l} \cdot Y.$$

etc.

Similarly,
$$e_1 = \frac{l_1}{\Sigma l} \cdot X,$$

$$e_2 = \frac{l_2}{\Sigma l} \cdot X,$$

etc.

i.e. the errors, and hence the corrections, are proportional to the lengths of the sides.

Comparison between Bowditch's Method and the Transit Method.

(*a*) Firstly, consider a traverse, the lines of which lie along and at right angles to the axes of co-ordinates.

The correction of an error in latitude by Bowditch's method affects all the lines and angles in the traverse.

The correction of the same error by the Transit rule only affects lines in the direction of the error (*i.e.* N. and S.)—because the lines at right angles to these have no latitudes ; the angles are not affected.

Similarly, by the correction of an error in departure, the angles are affected if Bowditch's method is applied, and unaltered if the Transit rule is applied.

It follows, then, that as the actual closing error is the resultant of the two co-ordinate errors, the balancing by Bowditch's method is apt to distort the angles from the observed values, while the balancing by the Transit rule produces no angular distortion—provided the lines of the traverse lie exactly along and at right angles to the axes of co-ordinates.

(*b*) Secondly, consider a similar traverse turned so that the lines lie at 45° to the axes of co-ordinates.

The corrections for an error in latitude, or for an error in departure, and therefore for any error, are now identical whichever of the two methods of balancing is employed, because each latitude and each departure bears the constant ratio of $\frac{1}{\sqrt{2}}$ to the line to which it refers, and the sum of the latitudes or the sum of the departures bears the same ratio $\frac{1}{\sqrt{2}}$ to the sum of the lines.

Hence the angular distortion produced is evidently the same amount for each method.

In the one limiting case (*a*) the Transit rule produces no angular distortion, and in the other limiting case (*b*) it produces exactly the same angular distortion as Bowditch's method.

In any general case, then, the amount of angular distortion must lie somewhere between these extreme limits, and the Transit rule will usually give less angular distortion than Bowditch's method—particularly if the axes of co-ordinates are so chosen as to be approximately along and at right angles to the main lines of the traverse.

Bowditch's method is particularly applicable to the case of a compass traverse, when the angles are liable to considerable error. The Transit rule is particularly applicable to the case of a theodolite traverse, when very little error might be expected in the angles in comparison with that in the sides.

Example.—Tables I to VI show the calculations of a traverse based on an actual survey. The data obtained in the field are shown in italics in Tables I and IV.

In the former case the axes of co-ordinates are taken approximately parallel and at right angles to the main lines of the traverse: in the latter case they are taken inclined at about 45° to the main lines of the traverse.

For an ordinary compass survey only Tables I and II or IV and V would be used, while for an ordinary transit traverse only Tables I and III or IV and VI would be necessary. In any of these cases the two tables used could be combined into one large table.

Example.—To examine the amount of angular distortion resulting from the application of the Bowditch and Transit rules, the traverse described in Table I was balanced under several different conditions.

(I) AB reduced bearing N. 00°-00′ E,
(II) AB ,, ,, N. 45°-00′ E;

and

(*a*) Angles not corrected before balancing,
(*b*) Angular error distributed evenly among the six angles.

The corrections to latitudes and departures were computed by the Bowditch and Transit rules. Then the alterations in the reduced bearings of the various lines were calculated and the change in value of the inward angles.

To calculate the alterations in bearings we have

$$x = l \sin a,$$
$$y = l \cos a,$$

employing the same notation as on p. 144.

Then
$$\delta x = \delta l \sin a + l \cos a \, \delta a,$$
$$\delta y = \delta l \cos a - l \sin a \, \delta a.$$

Eliminating δl from these two equations, we get

$$\delta a = (\delta x \cos a - \delta y \sin a) \div l.$$

or, if δa is expressed in seconds,

$$\delta a \text{ (seconds)} = (\delta x \cos a - \delta y \sin a) \div 48l \times 10^{-7}.$$

The results are summarised in Table VII.

[TABLES

Table I

When the axes of co-ordinates are approximately parallel and at right angles to the main traverse lines.

Line	Length	Inward Angle	Corrected Inward Angle	Bear-ing	Reduced Bearing	Consecutive Co-ordinates			
						N.	S.	E.	W.
AB	*1080*	*76°-03′*	76°-03′	*0°-03′*	N. 0°-03′ E.	1080	..	0·9	..
BC	*695·2*	*90°-43′*	90°-44′	..	N. 89°-13′ W.	9·5	..	..	695·1
CD	*296·4*	*64°-57′*	64°-57′	..	S. 24°-16′ E.	..	270·2	121·8	..
DE	*718·8*	*211°-29′*	211°-30′	..	S. 7°-14′ W.	..	713·1	..	90·5
EF	*258*	*84°-17′*	84°-17′	..	S. 88°-29′ E.	..	6·8	257·9	..
FA	*416·2*	*192°-28′*	192°-29′	..	S. 76°-00′ E.	..	100·7	403·7	..
Sum .	3464·6	719°-57′	720°-00′	..	..	1089·5	1090·8	784·4	785·6
					Sum . .	2180·3		1570·0	
					Diff. . .	1·3		1·2	

Table II

Corrections by Bowditch's Method

When the axes of co-ordinates are approximately parallel and at right angles to the main traverse lines.

Line	Corrections		Corrected Consecutive Co-ordinates				Independent Co-ordinates			
	N. & S.	E. & W.	N.	S.	E.	W.	N.	S.	E.	W.
AB	·4	·4	1080·4	..	1·3	..	1080·4	..	1·3	..
BC	·3	·2	9·8	..	..	694·9	1090·2	..	..	693·6
CD	·1	·1	..	270·1	121·9	..	820·1	..	..	571·7
DE	·3	·3	..	712·8	..	90·2	107·3	..	..	661·9
EF	·1	·1	..	6·7	258·0	..	100·6	..	..	403·9
FA	·1	·1	..	100·6	403·9	..	0·0	..	..	0·0
Sum .	1·3	1·2	1090·2	1090·2	785·1	785·1	..	..	..	..

Table III

Corrections by Transit Method

When the axes of co-ordinates are approximately parallel and at right angles to the main traverse lines.

Line	Corrections		Corrected Consecutive Co-ordinates				Independent Co-ordinates			
	N. & S.	E. & W.	N.	S.	E.	W.	N.	S.	E.	W.
AB	·6	..	1080·6	..	·9	..	1080·6	..	·9	..
BC	..	·5	9·5	..	..	694·6	1090·1	..	..	693·7
CD	·2	·1	..	270·0	121·9	..	820·1	..	..	571·8
DE	·4	·1	..	712·7	..	90·4	107·4	..	..	662·2
EF	..	·2	..	6·8	258·1	..	100·6	..	..	404·1
FA	·1	·3	..	100·6	404·1	..	0·0	..	..	0·0
Sum .	1·3	1·2	1090·1	1090·1	785·0	785·0	..	..	..	..

Table IV

When the axes of co-ordinates are approximately at 45° to the main traverse lines.

Line	Length	Inward Angle	Corrected Inward Angle	Bear-ing	Reduced Bearing	Consecutive Co-ordinates			
						N.	S.	E.	W.
AB	*1080*	*76°-03′*	76°-03′	*45°-03′*	N. 45°-03′ E.	763·0	..	764·3	..
BC	*695·2*	*90°-43′*	90°-44′	..	N. 44°-13′ W.	498·2	..	..	484·8
CD	*296·4*	*64°-57′*	64°-57′	..	S. 20°-44′ W.	..	277·2	..	104·9
DE	*718·8*	*211°-29′*	211°-30′	..	S. 52°-14′ W.	..	440·2	..	568·2
EF	*258*	*84°-17′*	84°-17′	..	S. 43°-29′ E.	..	187·2	177·5	..
FA	*416·2*	*192°-28′*	192°-29′	..	S. 31°-00′ E.	..	356·8	214·4	..
Sum .	3464·6	719°-57′	720°-00′	..	..	1261·2	1261·4	1156·2	1157·9
					Sum . .	2522·6		2314·1	
					Diff. . .	0·2		1·7	

Table V

Corrections by Bowditch's Method

When the axes of co-ordinates are approximately at 45° to the main traverse lines.

Line	Corrections		Corrected Consecutive Co-ordinates				Independent Co-ordinates			
	N. & S.	E. & W.	N.	S.	E.	W.	N.	S.	E.	W.
AB	·1	·5	763·1	..	764·8	..	763·1	..	764·8	..
BC	·1	·3	498·3	..	..	484·5	1261·4	..	280·3	..
CD	..	·2	..	277·2	..	104·7	984·2	..	175·6	..
DE	..	·4	..	440·2	..	567·8	544·0	..	..	392·2
EF	..	·1	..	187·2	177·6	..	356·8	..	..	214·6
FA	..	·2	..	356·8	214·6	..	0·0	..	..	0·0
Sum .	·2	1·7	1261·4	1261·4	1157·0	1157·0	..	..	..	..

Table VI

Corrections by Transit Method

When the axes of co-ordinates are approximately at 45° to the main traverse lines.

Line	Corrections		Corrected Consecutive Co-ordinates				Independent Co-ordinates			
	N. & S.	E. & W.	N.	S.	E.	W.	N.	S.	E.	W.
AB	·1	·6	763·1	..	764·9	..	763·1	..	764·9	..
BC	..	·3	498·2	..	..	484·5	1261·3	..	280·4	..
CD	..	·1	..	277·2	..	104·8	984·1	..	175·6	..
DE	·1	·4	..	440·1	..	567·8	544·0	..	..	392·2
EF	..	·1	..	187·2	177·6	..	356·8	..	..	214·6
FA	..	·2	..	356·8	214·6	..	0·0	..	..	0·0
Sum .	·2	1·7	1261·3	1261·3	1157·1	1157·1	..	..	..	..

TABLE VII

	AB = N. 00°-00′ E.				AB = N. 45°-00′ E.			
	No Pre-correction to Angles (= 180″ to A)		Pre-correction 30″ to each Angle		No Pre-correction to Angles (= 180″ to A)		Pre-correction 30″ to each Angle	
b = Bowditch *t* = Transit	*b*	*t*	*b*	*t*	*b*	*t*	*b*	*t*
	ft.	ft.	ft.	ft.	ft.	ft.	ft.	ft.
Error of closure . .	1·16	1·16	1·70	1·70	1·16	1·16	1·70	1·70
Sum of corrections to the six included angles:	″	″	″	″	″	″	″	″
(*a*) Excluding initial correction	320	217	430	328	320	254	430	342
(*b*) Including initial correction	500	397	470	376	500	434	470	378
Maximum correction to any angle:								
(*a*) Excluding initial correction	−117 (C)	+67 (D)	−172 (C)	−103 (C)	−117 (C)	+93 (A)	−172 (C)	+141 (A)
(*b*) Including initial correction	+281 (A)	+220 (A)	+185 (A)	+92 (A) +129 (D)	+281 (A)	+273 (A)	+185 (A)	+171 (A)

It will be noted that—

(I) When nominally no pre-correction is made to the angles, actually the procedure is equivalent to applying the whole correction to the angle at A. When the bearing of AB is N. 00°-00′ E. that of FA reduces to S. 76°-06′ E., from which the angle at A is 76°-06′ instead of 76°-03′ as measured.

(II) The " error of closure " is 1·70 feet when the angles are pre-corrected by adding 30″ to each. This is about the same as that in Table I (*i.e.* 1·77 feet), when the distribution of the errors was different and the corrections were not calculated to so many significant figures. It is greater than the " closing error ", 1·16 ft., for the case where the whole correction is applied to A (see (I) above). These figures show that the size of the " error of closure " remaining to be distributed may be increased by the pre-correction of angles, though this is not likely to be so in all cases.

(III) Whether Bowditch's method or the Transit rule is adopted, the angles are affected—and the correction applied to them may exceed the probable error in those angles. The p.e. of one angle measured with a theodolite reading to 1 minute is hardly likely to be as much as 3 or 4 minutes as are the figures in the last line of Table VII. The assumption that the angles are little affected when the transit rule is applied, is evidently not valid.

(IV) The effect on the angles tends to be less with the Transit rule than with Bowditch's rule. This is seen by comparing the average corrections, or the figures for the sum of the corrections given in Table VII, which are, of course, six times the averages.

(V) The maximum correction to any particular angle is also less when the Transit rule is applied, as the last two lines in Table VII

show. The difference, as was deduced on p. 146, is more marked when the main traverse lines are roughly parallel to the axes of co-ordinates. When the main traverse lines are roughly at 45° to the axes of co-ordinates, the distortion of the angles is nearly as great with the Transit rule as with Bowditch's.

Calculation of Enclosed Area.—The area enclosed by the traverse lines may be calculated by the application of formula (14), p. 39, Chapter I, *i.e.*

$$\text{Area} = \Sigma\frac{(d_2 + d_1)}{2}(l_2 - l_1).$$

Example.—From the independent and corrected consecutive co-ordinates in Table II

$-(692{\cdot}3 \times 9{\cdot}8) =$	-6784	
$1265{\cdot}3 \times 270{\cdot}1 =$		341,758
$1233{\cdot}6 \times 712{\cdot}8 =$		879,310
$1065{\cdot}8 \times 6{\cdot}7 =$		7,140
$403{\cdot}9 \times 100{\cdot}6 =$		40,632
$1{\cdot}3 \times 1080{\cdot}4 =$		1,404
	-6784	1,270,244
		6,784
		2)1,263,460
		631,730 square links
		$=63{\cdot}17$ square chains
		$=6{\cdot}317$ acres.

Similarly, Tables III, V and VI yield practically identical results, differing by less than one unit in the third decimal place of an acre.

Observations of Unequal Reliability.—Occasionally it may be judged by the surveyor that, owing to particular difficulties encountered in the field, errors were more likely to have occurred in certain measurements than in others. In such cases arbitrary allowances can be made when distributing the errors. Thus it may be considered that the angles B and D are more likely to be in error than the others. The surveyor may then judge that it would be advisable to attribute say k times as much error to these as to the other angles A, C, E, F, . . . N. The total correction, E, may then be distributed (*e.g.* in the pre-adjustment of the angles) $E/[n+2(k-1)]$ to $n-2$ of the angles, and $kE/[(n+2(k-1)]$ to B and D.

Example.—The error in a traverse ABCD is 3′, and the errors in B and D are assumed to be twice as large as those in A and C.

The corrections for A and C are therefore $\frac{1}{6} \times 3' = 30''$, and for B and D are $\frac{2}{6} \times 3 = 1'$, making a total of 3′.

Similarly, the distribution of the total error in latitudes and departures may be modified. If two lines are considered to require correction k_2 and k_5 times the usual amount, the latitude correction

$$c_1 = y_1 Y/(y_1 + k_2 y_2 + y_3 + y_4 + k_5 y_5 + \ . \ . \ .),$$
$$c_2 = k_2 y_2 Y/(y_1 + k_2 y_2 + y_3 + y_4 + k_5 y_5 + \ . \ . \ .),$$
$$c_3 = y_3 Y/(\quad\text{,,}\quad\text{,,}\quad),$$
$$c_5 = k_5 y_5 Y/(\quad\text{,,}\quad\text{,,}\quad),$$

etc.

Example.—If in Tables I and II it is considered that DE is three times as liable to error as the remaining courses, the equivalent sum of the Northings =1089·5 and of the Southings 1090·8 + 2 × 713·1 = 2517·0, making a total for the latitudes of 3606·5.

The correction for AB is then		$\frac{1080}{3606 \cdot 5} \times 1 \cdot 3 = \cdot 4$
,, ,,	CD ,,	$\frac{270 \cdot 2}{3606 \cdot 5} \times 1 \cdot 3 = \cdot 1$
,, ,,	DE ,,	$\frac{3 \times 713 \cdot 1}{3606 \cdot 5} \times 1 \cdot 3 = \cdot 8$
,, ,,	FA ,,	$\frac{100 \cdot 7}{3606 \cdot 5} \times 1 \cdot 3 = \cdot 0$
	Total . . .	1·3

the corrections for BC and EF both being negligible.

Alternatively, the different quantities may be given "weights" proportional to the squares of the probable errors, and the errors distributed as explained in the notes on Adjustment of Observations, p. 534 *et seq.* (Cf. also the equations on p. 145.)

Additional Methods of checking Unclosed Traverse Surveys.—The co-ordinates of an unclosed traverse survey cannot be balanced in the same way as those of a closed traverse—as the closing error is not known. It is desirable, therefore, that additional information should be obtained in the field to provide some check upon the work. The methods, of course, can be equally well applied to closed traverses.

(1) The bearings of certain well-defined objects—*e.g.* steeples, distinctive trees, ranging rods, etc., though not necessarily required for the survey proper, are observed from a few of the station-points.

The directions of these observed bearings are plotted upon the plan by means of a protractor, and, as a check, all those rays (at least three in number) taken towards the same object, should be concurrent.

This graphical method is not always very satisfactory, so that at times it may be advantageous to check the work by calculation.[1]

Thus let ABC be three traverse points from which observations are taken to O, and let the co-ordinates of the points be aa_1, bb_1, cc_1, and XY respectively, and let the bearings of O from A, B, and C be α, β, and γ respectively.

Then if the angles α and β are in the quadrants indicated in Fig. 5.13,

$$X = a + AO \sin \alpha = b - BO \sin \beta, \qquad (1)$$

$$Y = a_1 + AO \cos \alpha = b_1 + BO \cos \beta. \qquad (2)$$

$$\therefore AO = \frac{b - a - BO \sin \beta}{\sin \alpha} = \frac{b_1 - a_1 + BO \cos \beta}{\cos \alpha},$$

from which
$$BO = \frac{(b-a)\cos \alpha - (b_1 - a_1)\sin \alpha}{\sin(\alpha + \beta)}. \qquad (3)$$

Similarly,
$$AO = \frac{(b-a)\cos \beta + (b_1 - a_1)\sin \beta}{\sin(\alpha + \beta)}. \qquad (4)$$

[1] *Proc. Inst. C.E.* vol. clviii. Crosthwaite on Traverse Surveying.

X and Y may be found from equations (1) and (2).

Similarly, by using the co-ordinates aa_1 and cc_1 and bearings α and γ, AO and CO can be calculated and co-ordinates X_1 and Y_1 computed. These should, of course, be identical with the values X and Y obtained from A and B—or at any rate the discrepancy should be small.

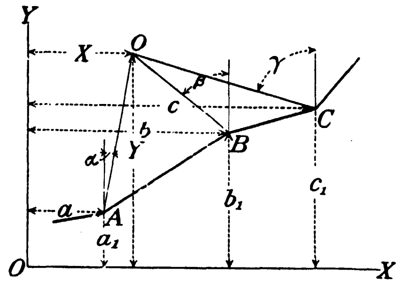


FIG. 5.13

(2) A more complete check is afforded if observations are also taken from the observed point, to the selected traverse station-points. In this case a number of triangles or small polygons are formed, and as all the included angles may be calculated from the observations, the sum may be checked. It is only necessary here to observe any particular object from *two* traverse points, instead of from three as above.

(3) When the point O observed from the traverse points is not suitable for an instrument station—*e.g.* when a spire or a flag-pole is bisected—a satellite station S may be employed. The selected traverse stations are observed from S, and the observations reduced to O as follows.

Let B and E (Fig. 5.14) be two points on the traverse whose co-ordinates are b, b', and e, e' respectively, so that the calculated length of BE

$$=\sqrt{(b-e)^2+(b'-e')^2}=l_1 \text{ say,}$$

the bearing of E from B

$$=180^\circ+\tan^{-1}\frac{b-e}{b'-e'}=\beta \text{ say,}$$

and the bearing of B from E

$$=\tan^{-1}\frac{b-e}{b'-e'}=\alpha \text{ say.}$$

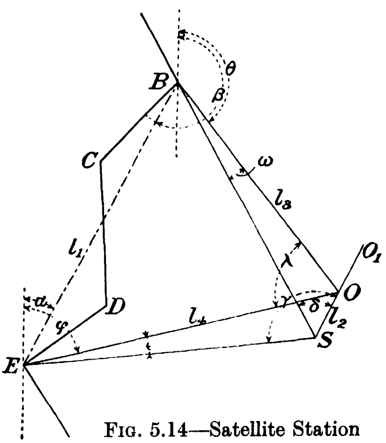


FIG. 5.14—Satellite Station

Let the bearing of O from B $=\theta$ and from E $=\phi$.

And as O is assumed unsuitable for an instrument station, let S be the "satellite station"—*i.e.* any suitable point near O at which the instrument is set up.

The distance SO is measured with a tape $=l_2$ say, and the angles OSE $=\gamma$ and OSB $=\delta$ determined. In the same way, if more than two traverse points are to be observed from S, the various angles may be observed as bearings from SO (say), and, if necessary, subtracted from 360°.

In the triangle EBO the length EB $= l_1$ and the two angles EBO $= \beta - \theta$ and BEO $= \phi - a$ are known, so that, if the work is correct

$$\angle \text{EOB} = 180 - (\beta - \theta) - (\phi - a) = \lambda' \text{ say,} \qquad (1)$$

and $$\text{BO} = \frac{l_1 \sin (\phi - a)}{\sin \lambda'} = l_3 \text{ say,}$$

and $$\text{EO} = \frac{l_1 \sin (\beta - \theta)}{\sin \lambda'} = l_4 \text{ say.}$$

Now in the triangle BSO, SO $= l_2$, BO $= l_3$, and $\angle$BSO $= \delta$,

$$\therefore \text{ the angle SBO} = \sin^{-1} \frac{l_2}{l_3} \sin \delta = \omega \text{ say,}$$

and similarly, $$\angle \text{SEO} = \sin^{-1} \frac{l_2}{l_4} . \sin \gamma = \epsilon \text{ say.}$$

Then from the figure O_1OE = sum of interior and opposite angles $= \epsilon + \gamma$ and O_1OB = sum of interior and opposite angles $= \omega + \delta$, so that by subtraction

$$O_1\text{OE} - O_1\text{OB} = \lambda = \epsilon + \gamma - \omega - \delta. \qquad (2)$$

If this value of λ agrees with that λ' obtained from equation (1), the inference is that all the work on the traverse between B and E is correct, both as regards angular and linear measurements. If not, the presence of an error is indicated but not definitely located or measured : it may be present in any of the lengths BC, CD, DE, or in any of the bearings observed from B, C, D, or E ; or in the length SO, or in the angles δ and γ ; or in the numerical calculations (see Examples, pp. 395, 396, 408).

Accuracy.—The effect of a small error in the line of collimation has been mentioned in the previous pages.

An attempt will now be made to discuss briefly the effects of small errors in the linear and angular measurements.

Whether the angles of the traverse are measured directly by the method of included angles, or by the fast needle method, they are liable to small errors due to inaccuracies of centering, bisecting, reading, etc.

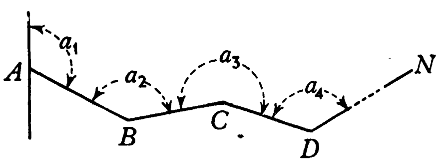


FIG. 5.15

Let the p.e. of each angle be $\pm \theta$ say.

Then if the bearing a_1 of the first line AB (Fig. 5.15) is determined from the meridian, an error of $\pm \theta$ will displace B an amount $\pm$ AB . θ. All the remaining lines will be twisted through this same angle, and the displacement of C $= \pm$ AC . θ, of D $= \pm$ AD . θ, and of the last point N of the traverse, the displacement $= \pm$ AN . θ.

Thus, if the traverse is closed A and N coincide and there is no displacement of N due to this error in a_1. An error of $\pm \theta$ in a_2 will similarly twist all the remaining points relatively to AB, and the last point

N will be displaced an amount $\pm\theta\,.\,\mathrm{BN}$. Similarly an error in a_3 at C displaces N an amount $= \pm\mathrm{CN}\,.\,\theta$.

The total displacement of N that might be expected would therefore be

$$\pm\theta\sqrt{\mathrm{AN}^2+\mathrm{BN}^2+\mathrm{CN}^2+\ldots}$$

If the traverse is unclosed, and in nearly a straight line, the displacement of the last point would be

$$\pm\theta\sqrt{l_n^2+(l_n+l_{n-1})^2+\ldots+(l_n+\ldots l_1)^2}.$$

Thus, if the lengths are approximately equal the probable displacement

$$=\pm\theta\,l\sqrt{\frac{n\,.\,\overline{n+1}\,.\,\overline{2n+1}}{6}}.$$

Again, if X and Y are the co-ordinates of N measured from the various stations A,B, . . . and β is the inclination or reduced bearing of a line such as AN, BN, etc., then, for instance, $\mathrm{X}=\mathrm{AN}\sin\beta$,

and $$\delta\mathrm{X}=\mathrm{AN}\cos\beta\,\delta\beta=\mathrm{Y}\delta\beta,$$

$\therefore$ the probable error in departure of N relative to A

$$\delta\mathrm{X}_1\ (\text{say})=\pm\theta\sqrt{\mathrm{Y}_1^2+\mathrm{Y}_2^2+\ldots}$$

and similarly the p.e. in latitude of N relative to A

$$\delta\mathrm{Y}_1\ (\text{say})=\pm\theta\sqrt{\mathrm{X}_1^2+\mathrm{X}_2^2+\ldots}$$

The error in the linear measurements is composed of two portions, as explained in Chapter I—a cumulative error, and a compensating error.

The cumulative error will be proportional to l, *i.e.* of the form c_2l, and in any particular traverse will be always + or always −.

The error in latitude due to this cause will therefore be

$$\delta\mathrm{Y}_2\ (\text{say})=c_2\,(y_1+y_2+\ldots y_n),$$

and the error in departure

$$\delta\mathrm{X}_2\ (\text{say})=c_2\,(x_1+x_2+\ldots x_n),$$

where c_2 may be either $+^{ve}$ or $-^{ve}$, and $x_1\ldots$ and $y_1\ldots$ are consecutive co-ordinates.

If the traverse is " closed ", $y_1+y_2+\ldots=0$, and $x_1+x_2+\ldots=0$, so that the error of closure due to this source would be nil.

In an unclosed traverse the displacement of the final point would be

$$\sqrt{(\delta\mathrm{Y}_2)^2+(\delta\mathrm{X}_2)^2}.$$

The compensating error may be assumed proportional to $\sqrt{l}$, *i.e.* $c_1\sqrt{l}$ say.

The p.e. in latitude in any given line will thus be $\pm c_1\,.\,\frac{y}{l}\sqrt{l}$,

and the p.e. in departure in any given line will thus be $\pm c_1 \frac{x}{l} \sqrt{l}$.

The total p.e. in latitude,

$$\delta Y_3 = \pm c_1 \sqrt{\frac{y_1^2}{l_1} + \frac{y_2^2}{l_2} + \dots}$$

and the total p.e. in departure

$$\delta X_3 = \pm c_1 \sqrt{\frac{x_1^2}{l_1} + \frac{x_2^2}{l_2} + \dots}$$

and the total p.e. in the position of the last point of the traverse due to this source will be

$$\pm \sqrt{(\delta Y_3)^2 + (\delta X_3)^2} = \pm c_1 \sqrt{l_1 + l_2 + l_3 \dots}$$

The total error in latitude at N will therefore be

$$\delta Y = \delta Y_2 \pm \sqrt{(\delta Y_1)^2 + (\delta Y_3)^2},$$

and the total error in departure will be

$$\delta X = \delta X_2 \pm \sqrt{(\delta X_1)^2 + (\delta X_3)^2},$$

and the total displacement will be

$$\pm \sqrt{\delta X^2 + \delta Y^2}.$$

If the linear errors are assumed all to be compensating and proportional to l, *i.e.* of the p.e. $= \pm c_3 l$, as is very often assumed when stating the limits of accuracy, then the p.e. in latitude due to this source

$$\delta Y_4 = \pm c_3 \sqrt{y_1^2 + y_2^2 + \dots}$$

and similarly $\quad \delta X_4 = \pm c_3 \sqrt{x_1^2 + x_2^2 + \dots}$

The total displacement of the last point will then be

$$\pm \sqrt{(\delta Y_1)^2 + (\delta Y_4)^2 + (\delta X_1)^2 + (\delta X_4)^2}.$$

In the case of a loose needle survey the error in the bearing of a line only effects the same displacement at the end of the traverse as at the end of the particular line to which it refers.

I.e. the p.e. in the position of N due to a small error $\pm \phi$ in each bearing is

$$\delta X_1 = \pm \phi \sqrt{x_1^2 + x_2^2 + \dots x_n^2},$$

and $\quad \delta Y_1 = \pm \phi \sqrt{y_1^2 + y_2^2 + \dots y_n^2},$

and on account of errors in linear measurements the same exprcssions tor δY_2, δY_3, δY_4, δX_2, δX_3 and δX_4 are applicable as in a theodolite fraverse.

A few examples of the accuracy obtained in practice are given below.

(1) On the Ohio River Survey,[1] commenced in August 1911, for

[1] Ohio River Survey. G. G. Graetcr. *Engin. News*, vol. lxxi. No. 12.

maps to a scale of 500 ft. to 1-inch, a control base line was run in duplicate along each bank. The instrument work was executed with a 10-inch transit, and all angles repeated 10 times, 5 times to the right and 5 times to the left, both verniers being read and recorded.

The linear dimensions were measured with a steel tape 200 ft. long, corrected for temperature, roughly levelled, and a uniform pull of 18 lb. exerted.

At intervals of 10 miles the two traverses were closed, the error of closure being not more than 1 in 20,000.

(2) The topography was then ascertained by transit and stadia traverses closing on the main traverse.

The allowable errors were

10 ft. to 1 mile in distance, *i.e.* ($\frac{1}{528}$),
2′ in azimuth,
0·4 ft. in elevation,

though the actual closures averaged much less.

(3) On the traverses connecting the triangulation points of the Topographical Survey of Cincinatti,[1] using a 150 ft. steel tape, and a transit reading to single minutes, the average closing error was 1 in 3000.

(4) The traverse stations were plotted on the sheet, and a blue print of all the traverse work on that sheet supplied separately.[1] Each topographer worked in a circuit, and for the plane-table traverses, some of which were between points as much as 3000 ft. apart, the limit of error was 5 ft., or 1 division of the scale.

(5) On a Reconnaissance Traverse in Guatemala[2] on muleback, with compass, aneroids, and watch, the distances were computed by timing the mule, and a closing error of only $\frac{3}{8}$th of a mile was obtained in a circuit of 81 miles (*i.e.* 1 in 216 about).

The rate of the mule on ground up to 10° slope averaged about $\frac{1}{20}$th mile per minute.

(6) For an ordinary compass and chain traverse a closing error of 1 in 400 to 1 in 500 might be expected, while for an ordinary transit and chain traverse an error of 1 in 1000 might easily result.

(7) For compass traverses in tropical forest countries,[3] conducted with a tarred manilla rope 310 ft. long (three lengths being considered as 900 ft., the length of leg, to allow for twist in paths, etc.), and compass bearings, taken on to the forward man by the sound of a " coo-ee " when necessary, an error of the order of 1 in 40 may be anticipated.

EXAMPLES

1. (U. of L.) The extracts from the field book given below relate to a complete traverse survey of a pond :

(*a*) Draw up a complete traverse sheet, and apply the usual checks.

[1] Topographical Survey of Cincinnati. Mitchell. *Engin. News*, vol. lxix. No. 14.
[2] *Proc. Inst. C.E.* vol. clxxxvi. p. 457.
[3] *Topography in the Tropical Forest Belt.* Ordnance Survey Training Series, No. 2.

(*b*) Plot the survey to a scale of 1 inch = 50 links.

(*c*) Compute in any way you please the area of the pond in acres and decimals of an acre.

Station	Exterior Angle	Bearing	Length of Side Links
A	288°-57′	33°-20′ E. of N.	643
B	276°-05′	..	354
C	213°-47′	..	168
D	233°-45′	..	342
E	247°-26′	..	485

Line 1		Line 2		Line 3		Line 4		Line 5	
643	..	..	..	..	..	..	..	..	..
610	47	..	..	..	..	..	..	..	..
540	36	..	..	..	..	..	..	..	..
510	23	..	..	..	..	..	..	..	..
436	20	..	..	..	..	..	..	..	..
370	14	..	..	..	..	..	..	..	..
305	8	..	..	..	..	342	..	485	..
240	10	..	..	..	..	320	40	460	30
190	12	354	16	..	..	255	35	340	16
152	15	330	8	..	..	220	16	310	12
110	24	210	5	168	..	110	12	190	9
75	38	120	14	120	12	85	15	140	14
40	50	60	20	50	4	20	25	80	20
0	0	0	0	0	0	0	0	0	0

2. (I.C.E.) The following bearings were taken in running a compass traverse:

Station 1 to station 2	351°	Station 3 to station 4	90°		
,, 2 ,, 1	$170\frac{3}{4}$°	,, 4 ,, 3	273°		
,, 2 ,, 3	356°	,, 4 ,, 5	176°		
,, 3 ,, 2	173°	,, 5 ,, 4	$355\frac{3}{4}$°		

At what station do you suspect local attraction, what is probable amount and in what direction?

(3) (U. of B.) Given in a closed traverse:

Side	Length	Azimuth
AB	391 feet	42°-38′
BC	..	..
CD	407 ,,	250°-18′
DE	791 ,,	280°-53′
EA	782 ,,	61°-05′

Find length and azimuth of BC.

4. (I.C.E.) A theodolite traverse has been run round a figure with six sides. The internal angles have been measured and their sum found to be 720°-1′-20″.

The consecutive co-ordinates of each point were calculated and found to be

	N.	S.	E.	W.
A	..	2257·9	..	51·2
B	..	1325·5	..	15·5
C	..	259·4	913·0	..
D	1500·3	..	..	138·8
E	2191·1	..	..	478·8
F	..	47·0	..	210·3

Can you say if the traverse closes, and if not, whether the error is likely to be in the measurement of an angle or a side ? Also can you say from inspection which pair of co-ordinates are probably wrong ?

5. (U. of L.) A four-sided traverse ABCD has the following lengths and bearings :

Side	Length in Ft.	Bearing
AB	500	Roughly East
BC	245	178°
CD	Not obtained	270°
DA	216	1°

Find the exact bearing of the side AB.

6. (U. of L.) Two points A and D are connected by a traverse survey ABCD and the following records are obtained :

AB = 876 ft.
BC = 682 ft.
CD = 983 ft.
Angle ABC = 118°-15′
Angle BCD = 108°-40′

Assuming that AB is in the meridian, determine :

(1) The latitude and departure of D relatively to A.
(2) The length AD.
(3) The angle BAD.

7. From Tables III, V, and VI (pp. 148 and 149) find the area enclosed by the traverse AB . . . FA, in each case, assuming a Gunter's chain to have been employed.

8. (U. of L.) A straight tunnel is to be run between two points A and B, whose co-ordinates are given in the annexed table :

Point	Co-ordinates N.	Co-ordinates E.
A	0	0
B	3014	256
C	1764	1398

It is desired to sink a shaft at D, the middle point of AB, but it is impossible to measure along AB directly, so D is to be fixed from C, a third known point. Calculate

(*a*) The co-ordinates of D.
(*b*) The length and bearing of CD.
(*c*) The angle ACD given that the bearing of AC is 38°-24′ E. of N.

9. (U. of L.) Describe the principle on which the arc of a sextant is graduated. A closed survey ABCD is made with sextant and chain. The bearing of AB is

found by means of a compass to be N. 30° E. and the angles ABC and BCD are 90° and 120° respectively as measured by the sextant. The lengths of the lines and heights of the station points are as follows :

AB	600 ft. long		A	300 ft. above datum
BC	200 ,,		B	330 ,, ,,
CD	400 ,,		C	320 ,, ,,
			D	310 ,, ,,

Obtain the correct bearings of all the lines and the length of AD.

CHAPTER VI

THE LEVEL AND LEVELLING

THE LEVEL

THE level is an instrument for determining the relative heights of different points on the earth's surface.

There are various patterns, including the "Dumpy", and the "Y" instruments; patent reversible types such as Cushing's and Cooke's, and later models with special features, some of which will be described below.

The telescope is provided with a diaphragm usually carrying cross-hairs of the type shown in Fig. 2.15 *b*, though often there are three horizontal webs as in Fig. 2.15 *e* or *f*, or metal points as in Fig. 2.15 *d* or *b*, the two outer being used for tacheometric purposes. The telescope is generally similar to that of a theodolite, and has already been described in Chapter II. Some further details of "internal" focusing arrangements are given later.

The telescope is provided with a bubble tube, and the supporting frame is furnished with "parallel plates" and either three or four levelling screws. Sometimes a compass box is enclosed in the space between the supports (Fig. 6.1).

A small cross-bubble is to be found on some "Dumpy" instruments, but this is generally not much used, and the instrument is levelled by the longer and more sensitive main bubble, by means of the parallel plate screws, exactly in the same manner as has already been described for a theodolite.

Sometimes the vertical axis is provided with a clamp and slow-motion tangent screw, especially when the diaphragm is fitted with metal stadia points (Figs. 6.2 and 6.3).

The Dumpy Level.—The telescope of the Dumpy level (Figs. 6.1 and 6.2) is firmly fixed to the frame by the makers, and there are usually only two permanent adjustments.

(1) To make the bubble "traverse"—*i.e.* to ensure that the longitudinal axis of the bubble is at right angles to the vertical axis of the instrument. The instrument is first levelled approximately, and the telescope turned until it is parallel to the two opposite parallel plate screws *b* and *c* (Figs. 4.12 and 4.13), and by their aid the bubble is brought to the centre of its run. On turning through 180° in azimuth so that the telescope is again parallel to *b* and *c*, if the bubble is no longer in the central position, its displacement is halved by means of the screws *b* and *c*, and the remaining half compensated

for by means of the capstan-headed screws attaching the bubble tube to the telescope.

(2) To place the line of collimation parallel to the axis of the bubble, when it should then ideally coincide with both the optical axis

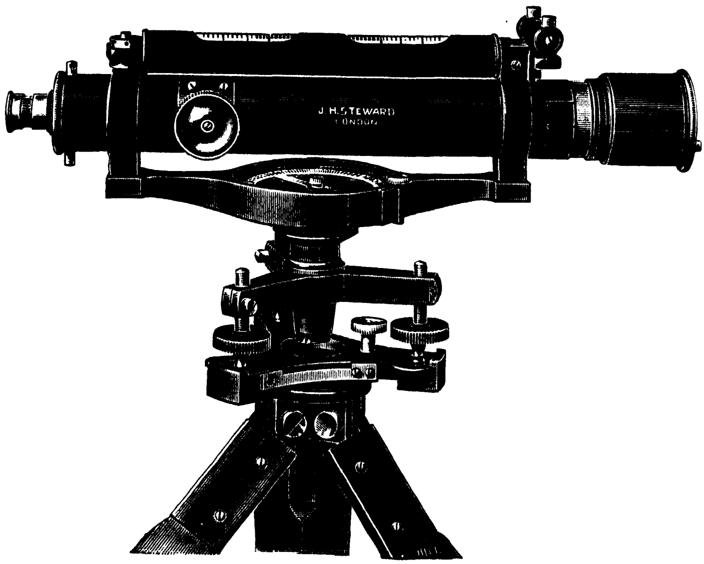


FIG. 6.1.—Dumpy Level, with Compass

and the geometrical axis of the telescope. The procedure is the same as in the two-peg test described on p. 100 for the theodolite telescope. The true difference in level between two pegs A and B, 200 feet or so apart, is ascertained by setting up the instrument equidistant from

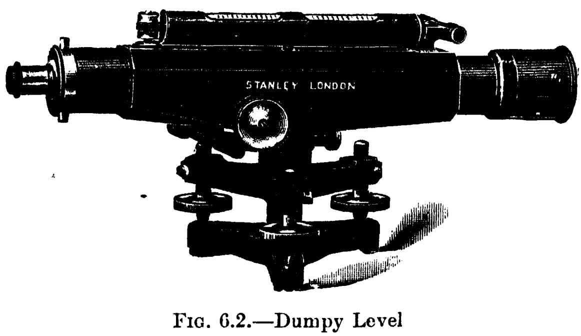


FIG. 6.2.—Dumpy Level

them, and taking staff readings on each peg, the bubble being in the centre of its run for each reading (Figs. 4.19 and 4.20). The level is then set up at D, a point near one of the pegs A, and the bubble being in the central position, readings are again taken on the pegs A and B.

If the difference of level so obtained is correct, the line of collimation is parallel to the axis of the bubble. If not, the adjustment is made in this case, without any alteration of the bubble tube, by altering the diaphragm screws until the correct difference is obtained, *e.g.* the diaphragm must be raised/lowered if the peg B is apparently too low/high.

The Y Level.—The Y level differs from the Dumpy in that by unpinning the two clips shown in Fig. 6.3 the telescope may be lifted bodily from the Y supports in which it is secured. The adjustments are :

(1) Vertical adjustment of the line of collimation of the telescope.
(2) Adjustment of the bubble axis parallel to the line of collimation.
(3) Adjustment of the line of collimation and the bubble axis at right angles to the vertical axis.

Adjustment (1) may be tested in the manner that was described

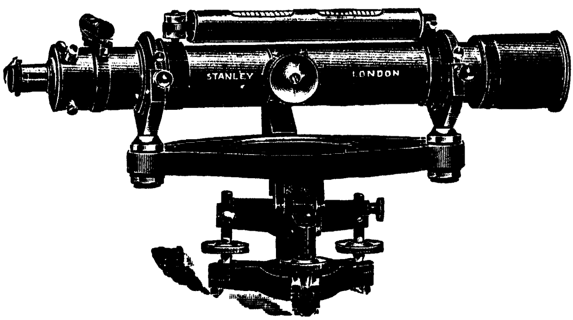


FIG. 6.3.—Y Level

on p. 97 for a Y theodolite. The telescope supports being uncovered, the cross-hairs are focused on to a levelling staff held at a convenient distance away and the reading taken. The telescope is carefully lifted from its bearing, replaced in an inverted position, and a second reading taken. If the two readings agree, the line of collimation is in the axis of the telescope : if not, the difference is halved by adjusting the two diaphragm screws.

Adjustment (2) may be made by the method described on p. 104 for a Y theodolite. Alternatively, the two-peg method as described above for the Dumpy level may be used—the correction being made by adjusting the capstan-headed screws at the end of the bubble tube, the diaphragm screws, of course, remaining undisturbed after adjustment (1).

Adjustment (3)—to make the bubble "traverse" correctly—is carried out as described above for the Dumpy level, except that any correction required is made by manipulating the capstan-headed screws provided for this purpose at the foot of the Y supports—where they are attached to the frame of the instrument. The bubble tube

capstan screws are, of course, not altered after the completion of adjustment (2).

When a Dumpy level has two vertical supports beneath the telescope, as in Fig. 6.1, very occasionally adjusting screws are provided at the feet, like those of the Y level. In such circumstances, three adjustments may be made: (1) the diaphragm hair-lines are placed centrally in the telescope, by making the axial reading on a distant staff halve the extreme readings within the field of view (*cf.* p. 98); (2) the bubble axis is adjusted to be parallel to the line of collimation, by manipulating the bubble-tube capstan screws, in completing the two-peg test; and (3) the bubble is made to traverse, as in the Y level, by manipulating the adjusting screws at the feet of the supports.

The Cushing's Level.—In Cushing's level the object-glass, and the eye-piece carrying with it the diaphragm, are interchangeable, and each can be rotated in the socket in which it fits. The screw (Fig. 6.4)

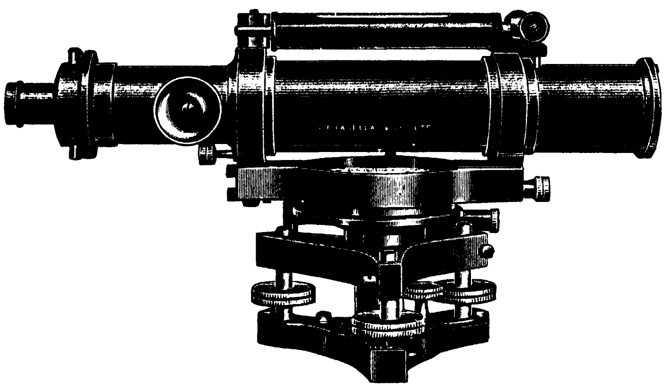


FIG. 6.4.—Cushing's Level

ensures that the diaphragm is in its correct vertical position when a reading is to be taken.

The adjustments, which are the same as those for a Dumpy level, may be tested by the general methods there described, but they may be more easily executed by the special methods and in the order stated below:

(1) **To adjust the Line of Collimation.**—The instrument is levelled in the usual manner, and, the bubble being preferably though not necessarily in its central position, a reading is taken on a levelling staff held a convenient distance away; or, if preferred, a point p_1 may be made on a wall as in adjustment (4), p. 98, for a Y theodolite (Fig. 4.16).

The eye-piece and diaphragm are inverted, *i.e.* are rotated through 180° in their socket, and the levelling staff is again read. If the two readings do not coincide, half the difference is compensated for by means of the diaphragm screws, with the result that the new reading should be then identical for either position of the eye-piece.

(2) **To adjust the Line of Collimation at Right Angles to the Vertical Axis** the instrument is levelled, and, as before, a reading taken on a levelling staff (or a point p_1 is marked to coincide with the cross-hairs). The instrument is rotated through 180° in azimuth, and the eye-piece and object-glass interchanged but not inverted, so that the line of sight is directed to the same object as before. The reading on the staff is again noted, and half the deviation, if any, is taken up by means of the screws provided for this purpose beneath the telescope supports. The line of collimation should now be at right angles to the vertical axis, and no deviation should be apparent on repeating the test.

It should be noted that, as in adjustment (1), it is not *necessary* that the bubble should be in the centre of its run, nor that the vertical axis be truly vertical. For if the line of collimation is inclined to the central axis at an angle $90° - \alpha$, then on rotating the telescope through 180° about this axis, the axis of course remains in the same position, but the centre line of the telescope makes an angle of 2α with its original position—as explained in Fig. 4.14 ; so that half this deviation (*i.e.* $\frac{2\alpha}{2} = \alpha$, the actual error) being taken up by the screws above mentioned, the adjustment is completed.

If the bubble *is* brought to the centre of its run for the first observation and is found to leave that position when the telescope is rotated through 180°, it must *not* be brought back to the normal position, as any movement of the parallel plate screws to effect this will also alter the position of the central axis.

(3) **To ensure that the Bubble shall " traverse " correctly**, after setting up the instrument in an approximately accurate position, the telescope is rotated into a line parallel to two base screws, *b* and *c* (Figs. 4.12 and 4.13), and by them the bubble is brought to the centre of its run. With a tribrach arrangement it may be somewhat more simple to orient the telescope over a single screw at right angles to *bc*.

If the bubble does traverse correctly, no displacement should occur on rotating the telescope through 180° in azimuth ; but if there is a deviation of, say $2n$ divisions, half this amount is corrected by means of the screws *b* and *c*, and the remaining *n* divisions by means of the capstan-headed screws which attach the bubble tube to the telescope.

After manipulating the remaining parallel plate screw or screws to " level " the instrument in a plane at right angles to *b*, *c*, the bubble should remain in its central position for a complete revolution of the telescope.

Cooke's Level.—This is shown diagrammatically in Fig. 6.5. The telescope is furnished with two equal cylindrical flanges which fit into two corresponding collars, S and S′, on the instrument frame, and from which the telescope may be withdrawn in an endwise direction. The collars are connected by a rigid cylindrical tube SSS′.

When replacing the telescope, the object-glass end is inserted into either socket and the telescope pushed forward until a rim on the rear flange prevents further motion.

The small screw Sc then secures the telescope in the socket in such a position that the cross-hairs are correctly placed.

The permanent adjustments and the special methods of testing and correcting these are as follows :

(1) **To place the Line of Collimation in the Axis of the Telescope**—or in the axis of the collars. The instrument is set up and approximately levelled in the usual manner and the cross-hairs directed to any well-defined mark (p. 98), *e.g.* a pencil-mark on a piece of white paper or a graduation on the levelling staff. Without disturbing the instrument, the telescope is then rotated in the sockets and about its longitudinal axis, through 180° (*i.e.* it is inverted), when the cross-hairs should again cut the same mark p_1. If not, half the deviation is corrected

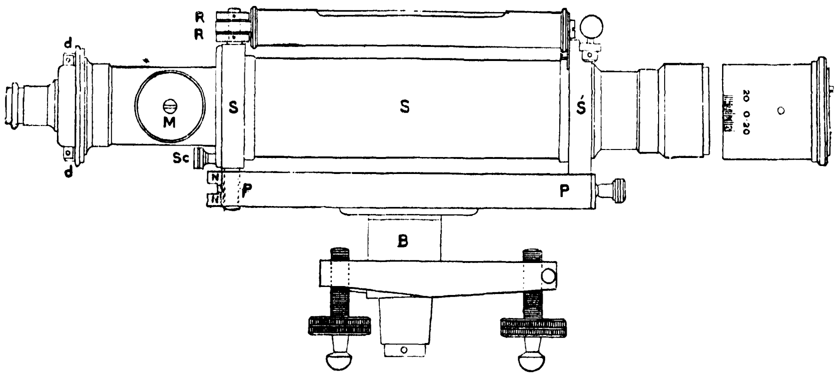


FIG. 6.5.—Cooke's Reversible Level

by means of the diaphragm, screws *d*, *d*, and the operation repeated until the adjustment is perfect.

(2) **To place the Line of Collimation at Right Angles to the Vertical Axis.**—The instrument is set up and approximately levelled, and the telescope having been brought over a pair of parallel plate screws *b*, *c*, or over a single screw in a tribrach arrangement, is directed to a well-defined mark (p. 98). The telescope is withdrawn from its sockets and replaced from the opposite end, so that the telescope flanges are now in the opposite sockets to their former positions. The whole instrument is then rotated through 180° in azimuth, to enable the telescope to be directed again towards the mark p_1.

If the cross-hairs do not now bisect p_1, half the deviation is corrected by the parallel plate screws and half by the screws NN_1, which permit the socket immediately above them to be raised or lowered as desired. The procedure is repeated until the adjustment is correct (compare adjustment (2), p. 165).

(3) **To make the Bubble traverse**, exactly as in the case of Cushing's level, half the deviation of the bubble from its central position, on reversal, is corrected by means of the bubble capstan-headed screws RR and half by the parallel plate screws.

A precise levelling instrument may be either of the Dumpy or the Y [1] type, but the telescope usually has a greater magnifying power (40 to 50 diameters) than the ordinary instrument. It is provided with three parallel plate screws and a very sensitive bubble, which is brought to the centre of its run for each reading by means of a fine adjustment screw under the eye-piece of the instrument, thus slightly altering the angle between the line of collimation and the vertical axis each time. In an ordinary level the bubble is made to " traverse ", and is brought to the centre of its run by means of the parallel plate screws ; but with such a sensitive bubble as that of a precise instrument accuracy can be obtained more expeditiously with the fine adjustment screw by centering the bubble independently *for each reading.*

A mirror is provided over the bubble tube so that the observer can see from the reflection that the bubble is in the correct position while the staff is being read.

In the U.S. Coast Survey Level a small auxiliary telescope was provided for viewing the bubble with the left eye, while the staff was being read through the main telescope with the right eye.

Occasionally the bubble is not accurately centered, but the deviation is noted and a correction applied to the results, as shown on p. 204.

Certain adjustments of the instrument need to be tested daily—before and after the day's operations—but, unless the error is greater than some specific amount, the instrument itself is not adjusted, but a calculated correction is applied to the *results* obtained during the day.

For example, to test the line of collimation the staff is set up at a measured distance, say d, from the instrument, and a number of readings taken with the telescope in its normal position, and also in its inverted position, *i.e.* when rotated through 180° about its longitudinal axis.

If the mean of the observations in these two positions be s_1 and s_2 respectively, then the error of collimation at the distance d is $\frac{s_1 - s_2}{2}$ (*cf.* Fig. 4.17), so that the correction to be applied at any distance x is therefore

$$x \times \frac{s_1 - s_2}{2d}.$$

In the Mississippi River Survey the instrument was not necessarily adjusted unless the collimation error was 1·25 mm. or more at a distance of 50 metres.

Zeiss Level.—The Zeiss level differs in several respects from the instruments described above.

Nos. II and III are of the form shown in Fig. 6.6, the larger size being suitable for precise levelling operations.

[1] The " Kern " level, of Swiss make, is of the Y type and has been largely used in U.S.A.

No. I is a smaller instrument of the same general design as Nos. II and III, but it differs in a few of the details.

The telescope of this, which is only 8 inches long, magnifies 20 diameters, and though the whole instrument weighs but $4\frac{1}{2}$ lbs. in its case, it is considered equivalent to an ordinary 12″ or 14″ Dumpy.

No. IV is a cheaper form and differs in many respects from Nos. I, II, and III.

The type shown in Fig. 6.6 will be described here.

The normal object-glass A consists of an achromatic plano-convex

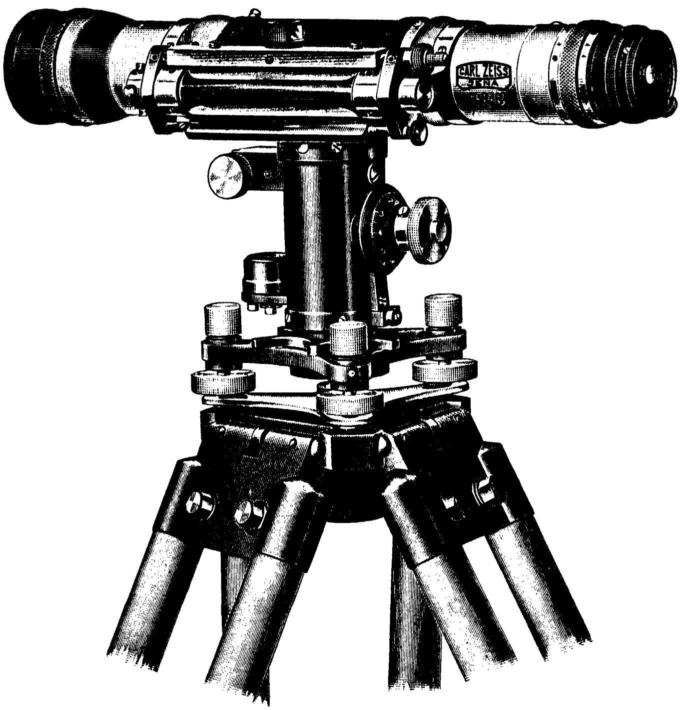


FIG. 6.6.—Zeiss Level

lens permanently fixed in the tube by the makers, and there is a similar lens B fixed at the opposite end of the tube as shown in Fig. 6.7.

Cross-hairs are engraved on the interior surface of each of these lenses, consequently they cannot be adjusted and diaphragm capstan screws are unnecessary.

The image formed by the object-glass A is focused on to the opposite lens B by means of a negative lens C. This lens can be made to slide in the tube by the manipulation of the milled-headed screw on the outside of the tube, to which is connected a rack and pinion arrangement.

The compound eye-piece, which magnifies the image, can be ad-

justed for parallax by a rotary motion—a much more convenient method than the usual sliding motion—and the edge of the eye-piece is graduated in diopters (Fig. 6.6).

The principle of the internal focusing telescope is explained on p. 57.

Before taking any readings, the instrument is approximately levelled by means of the 3-foot-screws, the bubble of the circular level on the left of the standard in Fig. 6.6 being brought by their aid to the centre of its run.

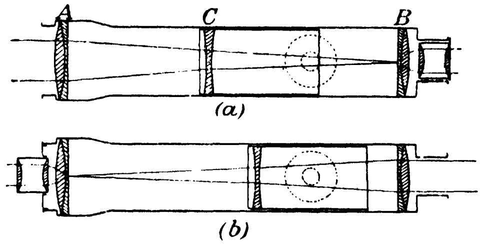


FIG. 6.7

In addition to this circular bubble, however, there is a long sensitive bubble, enclosed in a glass tube to protect it somewhat from extreme changes in temperature and carried on brackets from the side of the telescope. As a final operation the angle between the longitudinal axis of the telescope and the main vertical spindle of the instrument is altered by means of the fine adjustment screw seen on the right of the standard; and in this way the bubble is accurately centered. The result of this motion is that the telescope axis is not necessarily at right angles to the main axis, which is only approximately vertical; consequently the bubble will not traverse, but must be readjusted whenever the telescope is moved in azimuth, as explained on p. 94 and in Fig. 4.14.

The need for such readjustment of the bubble is not as serious a drawback as might be anticipated, because, as the bubble can be seen from the eye-piece end of the telescope by reflection in the small prism above the bubble tube, it is not necessary for the operator to change his position to the side of the instrument, and the fine adjustment levelling screw is quickly and easily manipulated.

One half of each extremity of the bubble—illuminated by means of the adjustable mirror placed immediately below it—is reflected by the prisms in the long rectangular casing immediately above the bubble tube into the small prism box. The arrangement is so effected that when the bubble is "central" the reflections of the two halves make one curve, as in Fig. 6.8, *a*, while when slightly out of truth the appearance of the reflections is as in Fig. 6.8, *b*.

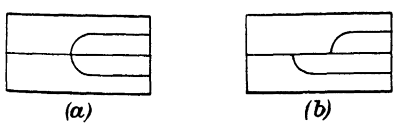


FIG. 6.8

There is also a clamp and tangent screw to effect accurate adjustment in azimuth. This is seen above the spherical bubble on the left of the standard in Fig. 6.6.

Adjustment.—When the bubble is seen as Fig. 6.8, *a*, the line of

collimation of the telescope should be horizontal, and the truth of this adjustment is examined as follows:

A staff is held vertically upon a peg or other firm object at a distance of about 150 ft. from the instrument, and readings are taken under the four following conditions, the bubble being accurately centered for each by means of the fine adjustment levelling screw:

(*a*) With the telescope in its normal position, the eye-piece at the opposite end of the tube to the usual object-glass, and with the bubble tube on the left of the telescope as in Figs. 6.6 and 6.7, *a*.

(*b*) With the clamp, which is seen projecting over the top of the telescope in Fig. 6.6, loosened, and the telescope rotated through 180° on its longitudinal axis so that the bubble tube is on the right of the telescope, and the prism box below the bubble.

(*c*) With the eye-piece removed from its normal position and screwed into the cap in front of the normal object-glass, the telescope and bubble remaining as in (*b*) (Fig. 6.7, *b*).

(*d*) With the eye-piece removed from its normal position and screwed into the cap in front of the normal object-glass, but with the telescope and bubble as in (*a*).

In positions (*c*) and (*d*) the small prism may be turned through 180° in azimuth, so that the bubble can be seen from the end of the tube at which the eye-piece is now fixed.

The arrangement of the lenses in the cases (*a*) (*b*) and (*c*) (*d*) respectively is seen in Fig. 6.7, *a* and *b*.

If the instrument is in perfect adjustment the four readings will be coincident.

For ordinary work the instrument is used throughout as in method (*a*), hence it is necessary that when the eye-piece is normal and the bubble on the left of the telescope, the line of collimation shall coincide with the true horizontal line so determined.

Consequently, if the adjustment is not perfect, the instrument is again arranged as in (*a*) and the telescope tilted by means of the fine adjustment levelling screw until the staff reading is made to agree with the mean value of the four results.

If the eccentricity of the bubble is now considerable, the bubble tube is tilted by means of the antagonizing capstan-headed screws at its end until the bubble is *approximately* in the centre of the tube.

When this has been done, or if the divergence of the bubble is such a slight amount that it is unnecessary to alter the tube itself, the final adjustment is completed by releasing the clamp at the end of the long rectangular prism box, and adjusting this by means of the screw until the ends of the bubble appear in the small prism box, as in Fig. 6.8, *a*.

That is to say, the line of collimation is set truly horizontal and, at the same time, the reflections of the two ends of the bubble are made to coincide. This can be accomplished even though the bubble is not in the exact centre of its tube, provided that it is approximately so.

For precise work it is sometimes customary to adopt as correct the mean of two readings, for one of which the telescope is in the normal position (Fig. 6.6), and for the other the telescope is inverted as

in (*b*). For such work it is essential that the *mean* of these two readings shall agree with the mean of the four values (*a*), (*b*), (*c*), (*d*) above.

Thus if the four readings on the staff are r_a, r_b, r_c, r_d respectively, then, if the instrument is in adjustment,

$$\frac{r_a+r_b}{2} \text{ will be equal to } \frac{r_a+r_b+r_c+r_d}{4}=\text{R, say.}$$

If this is not the case, but

$$\frac{r_a+r_b}{2}=\frac{r_a+r_b+r_c+r_d}{4}\pm m, \quad i.e. \ \text{R}\pm m,$$

then the telescope must be tilted by means of the screw A until the reading r_a' in the normal position (*a*) of the instrument is equal to $r_a \mp m$, as any movement of the telescope affects r_a and r_b equally.

The long prism casing is then adjusted by the screw at its extremity —and if necessary the bubble tube also by means of the capstan screws—until the reflection in the small prism appears as in Fig. 6.8, *a*.

It now remains to adjust the bubble tube so that its *horizontal* projection is parallel to the axis of the telescope, and to do this the telescope is rotated through 90°, *i.e.* until the bubble is immediately above the telescope. The correct reading R is then bisected on the staff, and the bubble brought to its central position, as seen in the small prism, by means of those capstan-headed adjusting screws at the end of the bubble tube, which are horizontal in the normal positions (*a*) or (*b*).

An alternative and more accurate method of performing this adjustment is to place the tripod so that one leg is directed towards the staff. The bubble is then placed above the telescope and brought to the centre of its run while a reading on the staff is noted.

The foot screws are now in the positions shown in Fig. 6.9, so that if the screw 2 is turned through, say, 180 or more degrees a tilt is given to the instrument in a direction at right angles to the line joining the screws 1 and 3. The component of this movement along the direction 1S will slightly alter the reading on the staff S, *i.e.* in Fig. 6.9, if 2 is raised, the reading will be increased. Similarly, if the screw 3 be turned, the instrument will be tilted in a direction at right angles to 1-2.

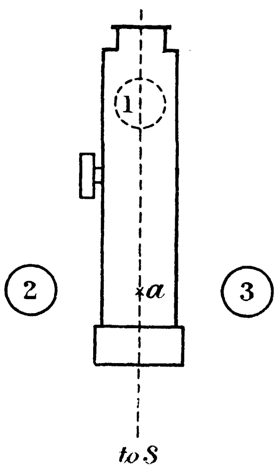


FIG. 6.9

It is evident, then, that if 3 is screwed in the opposite direction to 2, the component along 1S of the latter movement can be made to neutralize that caused by 2.

When the original reading has been obtained on the staff by rotating 3, it signifies that the point *a*, midway between 2 and 3, is raised to its original level; and as 1 is unaltered the resultant effect on the instrument is a tilt round 1*a*S as an axis.

Consequently, if the axis of the bubble is parallel to 1*a*S, the position of the bubble as seen in the small prism box will be unaffected by this motion, *i.e.* it will revert to its central position when the original staff reading is re-obtained by the turning of screw 3. But if the horizontal projection of the bubble tube axis is *not* parallel to 1*a*S, the tilting movement at right angles to 1*a*S will have a component along the tube and cause a displacement of the bubble from its central position.

The adjustment is completed if necessary by means of the horizontal screws provided for the purpose at the end of the bubble tube.

Other Modern Levels.—A good example of a modern level made by Messrs. Cooke, Troughton and Simms, is shown in Fig. 6.10.[1]

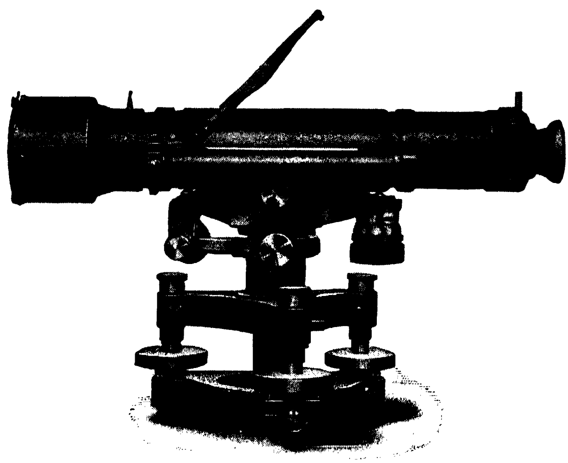


FIG. 6.10.—Cooke's Level

A noteworthy feature in the design of this instrument is the employment of a spirit bubble of the reversion type.[2] The telescope, which is capable of rotation about its longitudinal axis, has attached to it a fitting which carries a sensitive spirit-level. The bubble tube is graduated upon the two opposite sides of the vial, and if the bubble is central on one scale, X, then when the bubble tube is rotated through 180° about its longitudinal axis, the bubble should be central on the opposite scale, Y. The makers claim that the graduations are made to register with each other within 2 seconds of arc.

The line of collimation of the telescope is fixed during manufacture to coincide with the mechanical axis of rotation, and no means of altering this adjustment is provided. Its truth can be tested as in a Y or as in a Cooke's Reversible Level.

In the normal working position, the bubble tube is on the left of the telescope, and the position of the bubble can be read from the eye-piece

[1] See also *Engineering*, June 3, 1921, pp. 680-682.

[2] "New Types of Levelling Instruments using Reversible Bubbles", by T. F. Connolly, *Trans. Opt. Soc.* 25 (1923–24).

end of the instrument. If the telescope be rotated about its longitudinal axis, the bubble tube is brought over to the right of the telescope. If the telescope axis and the bubble axis are parallel, the bubble reading on the scale Y will agree with its former reading upon the opposite scale X. If the readings differ, the two axes are not parallel, and adjustment must be made by means of the capstan-headed screws at the end of the bubble tube.

This is the only permanent adjustment that can be made by the Surveyor, and its truth can be tested very quickly at any time merely by rotating the telescope through 180° about its longitudinal axis and noting the readings of the bubble before and after this movement.

A circular spirit-level is mounted within the tribrach for rough levelling. This enables the main axis to be adjusted into an approximately vertical position very quickly. The line of collimation of the instrument is then brought into an accurately horizontal position by means of the differential setting screw, which is shown under the eyepiece end of the telescope. This screw does not affect the " vertical " axis, but tilts the telescope about a transverse axis. The angle between the line of collimation and the vertical axis is thus not necessarily 90°, as it is altered each time the differential setting screw is manipulated.

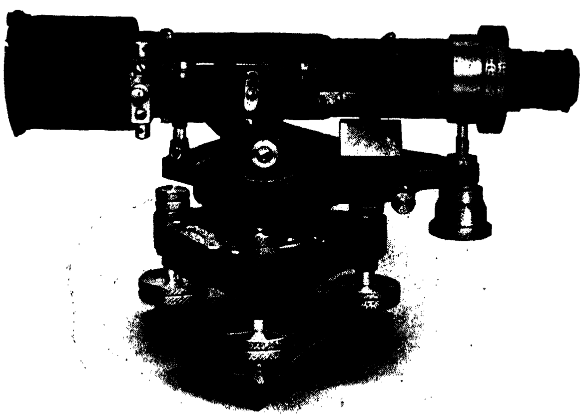


FIG. 6.11.—Watts " Self-adjusting " Level

The Surveyor can view the bubble without moving his position, as the end portions of the bubble are reflected in the speculum mirror shown in the illustration. This mirror is of special design and does not reflect the whole length of the bubble—the image is foreshortened so that the central portion is eliminated, and the magnified end portions are brought closer to each other.

The head of the differential screw is furnished with a graduated drum, which enables the degree of tilt to be measured when the telescope is tilted from its horizontal position. The instrument can thus be employed as a gradienter.

The telescope has internal focusing, and the diaphragm is fitted with an interchangeable cell which can be removed and replaced without altering the collimation adjustment of the instrument.

Reference may also be made to a self-adjusting Precision Level [1] (Fig. 6.11) made by Messrs. E. R. Watts & Son. Though differing in detail from the instrument described above, the main features are very similar. In this example, however, the bubble has been designed so as to remain of constant length under changing conditions. Each bubble is tested, and its constancy of length under a wide range of temperatures is certified by the N.P.L. One end only of the bubble is viewed, either when in the normal or in the reversed position, a prism reflector being provided to enable the Surveyor to see this in each case from the eye-piece end of the telescope.

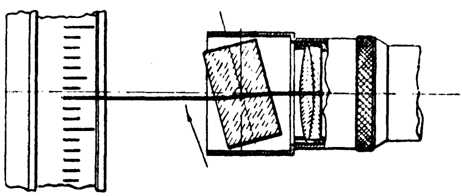


FIG. 6.12.—Parallel Plate Micrometer

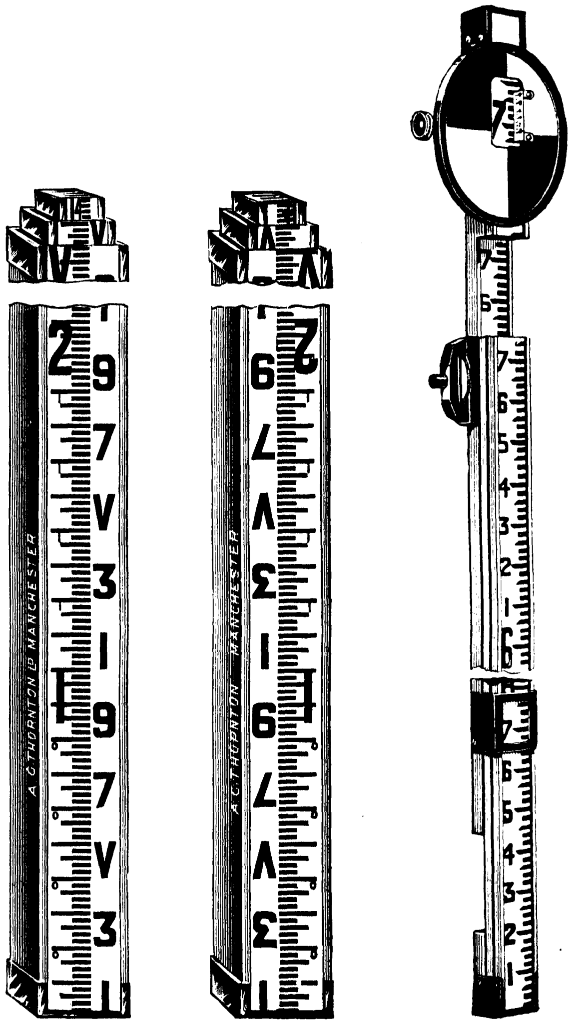
Parallel Glass Plate Micrcmeter.—This simple contrivance, which is fitted by Messrs. E. R. Watts & Son to their precise levels, consists of an optically worked parallel disc of glass, horizontally pivoted in front of the object glass of the level telescope. The line of sight is not affected when the disc is vertical, but the interval between the hair-line and the nearest staff graduation mark below it may be measured, instead of estimated by eye, by tilting the disc as in Fig. 6.12, until coincidence with the graduation mark is obtained. The displacement of the line of sight is parallel to the original line, and independent of the staff distance; and as 0·02 ft. displacement corresponds to an angular movement of the disc through 20 divisions on a calibrated scale, it is possible to read the staff to the 3rd decimal place of a foot, and estimate closely the 4th place at short ranges.

The **levelling staff** is usually graduated in feet, decimally divided into tenths and hundredths of a foot. The length, cross-section, and method of graduation, etc., vary considerably in different patterns, but the most usual form is known as Sopwith's and is shown in Fig. 6.13, *a*.

This staff is of telescopic design and of rectangular section, the total length when extended being usually 14 ft., though other lengths are also obtainable. Sometimes it is in one length of, say, 10 ft., and at other times it is hinged to fold in two lengths. It is made of timber such as mahogany, well seasoned, and bound, shod, and capped with brass. The graduations are painted direct on the face, or a varnished scale is pasted on. In the usual pattern the figures indicating the whole numbers of feet are painted in red, some—such as V and X—being for distinction in Roman figures, while the remainder are of other forms. The tops of these red figures denote the level of the foot which they specify. Sometimes the 11, 12, and 13 are replaced by one, two, or three square dots respectively, as the top length of the staff is necessarily

[1] *Engineering*, June 12, 1925.

rather narrow. The staff when viewed through the majority of levelling and stadia instruments appears inverted, so that as confusion is liable to arise between the 6 and the 9 (note the 6 does not appear among the



(*a*) Upright (*b*) Reverse (*c*) Target Rod

FIG. 6.13.—Level Staves

black figures), the 9 is often made blind as 9, while the eye of the 6 is clear as 6.

On some staves the figures are inverted so that they appear erect when viewed through the telescope (Fig. 6.13, *b*). The subdivisions of a

foot are marked by the black numerals **1**, **3**, **V**, **7**, and **9**, each of these being exactly ·1 of a foot in height.

Thus the top	of the black	**1**	denotes	·1
,, bottom	,,	**3**	,,	·2
,, top	,,	**3**	,,	·3
,, bottom	,,	**V**	,,	·4
,, top	,,	**V**	,,	·5 and so on.

Each tenth of a foot is subdivided into ten parts, and the alternate spaces filled in as shown in Fig. 6.13. The top of a white space thus indicates an odd number of hundredths, while the top of a black space indicates an even number. Usually the further subdivision of these spaces is not estimated, the second decimal place being taken as odd if the hair-line intersects upon a white space, and even if on a black space. Others forms of graduation are shown in Fig. 8.1.

To obviate mistakes which may arise in those cases where a large red figure does not happen to fall within the field of the telescope, sometimes one or two smaller supplementary red figures are painted in intermediate positions in each foot length.

Otherwise the staff man may be instructed to raise the whole staff slowly until a red figure comes into view—when this figure will be the correct number of whole feet to be booked ; or the staff man may run his finger along the staff until this is seen through the telescope—when the correct number of whole feet to be added to the observed decimal readings may be noted directly.

If it is intended that the staff shall be read by the instrument man, through the telescope, the staff is known as self-reading (Fig. 6.13, *a* and *b*), but for longer distances a "target" staff is employed (Fig. 6.13, *c*). This is provided with a movable vane which is adjusted by the staff man until the instrument man signals that the central mark coincides with the diaphragm web. The staff man then observes and books the vernier reading. For important work several readings would be taken and a mean adopted (see Reciprocal Levelling).

The Philadelphia rod, by Gurley, shown in Fig. 6.13, *c*, is 13 ft. long, and is graduated in feet, tenths and hundredths, while the vernier reads to thousandths.

Staves used for precise survey [1] work are generally not extensible, but, like those used on the Ordnance Survey of Great Britain and the Indian Survey, are in one length. These were 10 ft. long, and self-reading, graduated on both faces from different zero positions, and could be read either in an upright normal position or reversed, so that four different readings could be taken at any particular staff station.

American types are generally of + or T section, and are graduated in metres, subdivided to centimetre or even 2-millimetre divisions. A spirit level, a plumb-line or pendulum device is provided to ensure verticality, and sometimes a thermometer is attached for determining temperature.

[1] See also p. 216.

LEVELLING

A level surface may be defined as a surface, the tangent plane at every point of which is perpendicular to the direction of gravity at that point, and it coincides with the shape adopted by the surface of a free liquid.

The surface of the earth at " mean sea-level ", *i.e.* the mean level of the great oceans, is known as the Geoid, and this is by no means a regular figure, because the direction which a plumb-bob assumes under the action of gravity depends upon many factors, such as the difference in density of different portions of the earth's crust, and the attraction in a lateral direction due to large mountain masses.

Near mountain ranges, though there are many exceptions to the rule, the surface of the geoid tends to be raised higher than would otherwise be the case, while proximity to the great oceans tends to produce depressions in the geoid.

For many purposes it has been assumed that the geoid is approximately spherical in form, but for more precise work it is taken as an oblate spheroid, *i.e.* the surface of revolution of an ellipse about its minor axis.

Other forms—such as the Ellipsoid, which has three unequal principal axes—have been suggested as agreeing more closely with the actual surface of the earth, but need not be further considered here.

The size of the particular sphere, spheroid, ellipsoid, or other shape adopted is such that, as far as the known data allow, the differences between it and the geoid, as determined by the method of Least Squares, are a minimum.

If it were truly spherical, any plane section would be circular ; the mean radius of the sphere which most nearly coincides with the geoid may be taken as 20,890,000 ft.

If considered as an oblate spheroid, any section parallel to the equatorial plane would be circular, but any meridional section (*i.e.* through the poles) would be elliptical. For the dimensions of the oblate spheroid the reader is referred to Chapter XVI.

These measurements have been computed in a variety of ways, *e.g.* by the actual measurement of arcs at different latitudes, by pendulum observations for the intensity of the force of gravity, by calculations based on astronomical observations, etc.

The use of the terms " Level " and " Horizontal " is somewhat confused in every-day parlance, but here it is assumed that a Level line is a line lying wholly in the geoidal or corresponding spheroidal surface, while a horizontal line is that lying in a plane tangential to this. A level line is therefore curved, while a horizontal line is straight.

If the effect of refraction upon it is neglected, the line of collimation of a properly adjusted level is a horizontal line, but as the distances at which a levelling staff can be read with an ordinary instrument are comparatively small, the divergence of this from a level line is inappreciable, though in the case of exceptionally long sights an allowance for " Curvature and Refraction " is made as explained later.

Considering, then, for the present that the horizontal line of collimation is a level line, the ordinary method of levelling is shown in the following example.

Fig. 6.14 shows the plan of a road, of which a longitudinal section is required for sewering purposes. The existing sewer finishes at the manhole A, where the depth to the invert is 10 ft. 6 ins.

The road is bounded by thick hedges which cannot easily be seen through; and there is a bench mark ⊼ on the house shown, the reduced level of which is 80·60 ft. above Ordnance datum. These "bench marks" have been left at various points all over the country by the Ordnance Surveyors, and are usually V-shaped incisions cut into the stonework or brickwork of churches, bridges, or other permanent structures. Upon the maps, the positions of these are given, together with the height of the centre of the top horizontal bar of each above the "Ordnance Datum Level". It was intended that the datum, as marked on the entrance to the Mersey Docks at Liverpool, should coincide with the "mean sea-level", and though doubts have existed since, yet the latest data indicate that any difference is small, and that this old datum is practically the same as the mean sea-level round the coast, and 0·13 foot above the mean sea-level at Newlyn. Apparently, however, the Old Ordnance Survey datum surface for the whole country was not a level plane through the datum at Liverpool.

As shown in this example, Ordnance bench marks form a very convenient method of checking long lines of levels.

The first position at which the instrument is set up and levelled is in the main road opposite the entrance to Parkgate House. From this position both the B.M. and the initial point A can be seen. The levelling staff is held vertically against the wall of the house and with the bottom of its base plate level with the centre of the horizontal bar of the bench mark. The telescope is directed towards this, the hair lines having previously been focused quite clearly by moving the eye-piece inwards or outwards as required. The staff is then focused by the use of the large milled-head on the side of the telescope, with the result that the image should be distinct and parallax absent, *i.e.* there should be no apparent movement of the image relative to the hair-lines when the eye is moved. (See p. 56.)

The reading at which the horizontal web of the diaphragm appears to cut the staff is now noted (say 1·04) and entered in the first column of the level book as a Back Sight, while in the "Remarks" column a note is made ⊼ "on Parkgate House". The reduced level 80·60 is filled in either now or later from the Ordnance maps.

It may be stated here that the first reading at any setting of the instrument is booked as a "Back Sight", while the last reading for any setting is a "Fore Sight".

The two forms of level book in general use are shown on pp. 180 and 181, and the columns to be entered up in the field are identical in each. The difference between the two forms lies in the method of "reduction" afterwards in the office.

The first column on the page is sometimes "Chainage".

The staff is next placed on the manhole cover at A and a reading (4·58) taken, and entered in the "Intermediate" column. The chainage commencing from this point, 0 is entered in the "Distance" column and a note is made as shown in the "Remarks" column. A

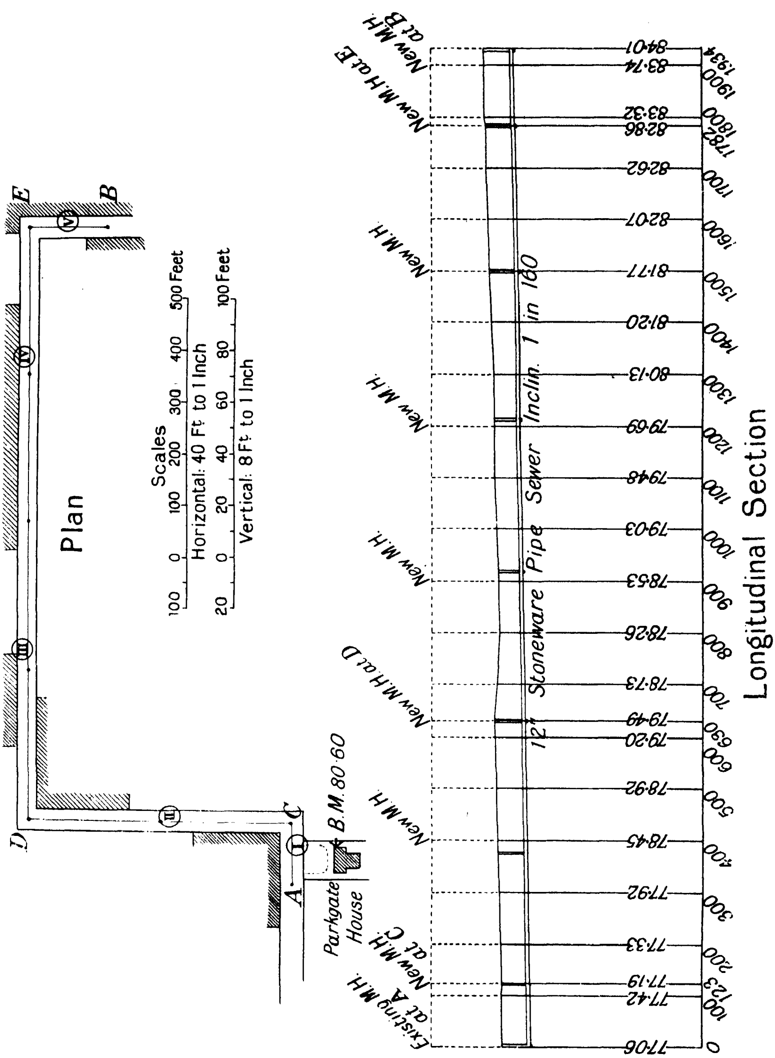


FIG. 6.14

further reading may be taken upon the invert of the sewer if this can be conveniently obtained.

The chain is stretched along the road from A, and the staff held vertically in the centre of the road at the end of every chain (or half chain) length when the slope is fairly regular, or at every change of

slope if sufficiently well defined; and the reading at 100 is booked (4·22) in the " Intermediate " column.

The chainage to the centre of the road at the bend C is noted as 123 in the " Distance " column, and a reading 4·45 is taken on the staff at this point, and entered in the " Fore Sight " column, as it is

LEVELS FOR CONTINUATION OF SEWER IN WELLER ROAD

(Collimation Method.) *March* 5, 19—.

Back Sight	Inter-mediate	Fore Sight	Reduced Level		Distance	Remarks
			Instr. Axis	Staff Station		
1·04	..	..	81·64	80·60	..	⊼ Parkgate House.
..	4·58	..	..	77·06	0	Manhole cover at A. Depth to invert 10·50 ft.
..	4·22	..	..	77·42	100	Centre of Road.
5·28	..	4·45	82·47	77·19	123	Bend of road.
..	5·14	..	..	77·33	200	Centre of road.
..	4·55	..	..	77·92	300	,, ,,
..	4·02	..	..	78·45	400	,, ,,
..	3·55	..	..	78·92	500	,, ,,
..	3·27	..	..	79·20	600	,, ,,
4·74	..	2·98	84·23	79·49	630	Bend of road.
..	5·50	..	..	78·73	700	Centre of road.
..	5·97	..	..	78·26	800	,, ,,
..	5·70	..	..	78·53	900	,, ,,
..	5·20	..	..	79·03	1000	,, ,,
..	4·75	..	..	79·48	1100	,, ,,
6·38	..	4·54	86·07	79·69	1200	,, ,,
..	5·94	..	..	80·13	1300	,, ,,
..	4·87	..	..	81·20	1400	,, ,,
..	4·30	..	..	81·77	1500	,, ,,
..	4·00	..	..	82·07	1600	,, ,,
..	3·45	..	..	82·62	1700	,, ,,
5·13	..	3·21	87·99	82·86	1782	Bend of road.
..	4·67	..	..	83·32	1800	Centre of road.
..	4·25	..	..	83·74	1900	,, ,,
5·29	..	3·98	89·30	84·01	1934	Position of new manhole B.
8·22	..	6·76	90·76	82·54	..	..
..	..	1·34	..	89·42	..	⊼ Station Bridge 89·40.
36·08	..	27·26	..	89·42	..	..
27·26	..	..	..	80·60	..	..
8·82	..	..	..	8·82		

apparent that the next staff station will not be seen from the present position of the instrument.

A new position for the instrument must now be chosen, and the points to be borne in mind are: (1) the instrument must be sufficiently high to enable the line of collimation to cut the staff held at the old fore sight station, when used as the proposed back sight station; (2)

the distances from the instrument to the points at which back and fore sight readings are to be taken should be approximately equal, or exactly so for very accurate work; and (3) these distances should not be so great as to cause the readings to be very indistinct.

It is, of course, not necessary—nor even advisable—that the instrument should be placed upon the section line, any convenient

LEVELS FOR CONTINUATION OF SEWER IN WELLER ROAD

(Rise and Fall Method.) *March* 5, 19—.

Back Sight	Inter-mediate	Fore Sight	Rise	Fall	Reduced Level	Distance in Feet	Remarks
1·04	..	..	..	..	80·60	..	⊼ Parkgate House.
..	4·58	..	..	3·54	77·06	0	Manhole cover at A. Depth to invert, 10·50 ft.
..	4·22	..	·36	..	77·42	100	Centre of road.
5·28	..	4·45	..	·23	77·19	123	Bend in road.
..	5·14	..	·14	..	77·33	200	Centre of road.
..	4·55	..	·59	..	77·92	300	,, ,,
..	4·02	..	·53	..	78·45	400	,, ,,
..	3·55	..	·47	..	78·92	500	,, ,,
..	3·27	..	·28	..	79·20	600	,, ,,
4·74	..	2·98	·29	..	79·49	630	Bend in road.
..	5·50	..	..	·76	78·73	700	Centre of road.
..	5·97	..	..	·47	78·26	800	,, ,,
..	5·70	..	·27	..	78·53	900	,, ,,
..	5·20	..	·50	..	79·03	1000	,, ,,
..	4·75	..	·45	..	79·48	1100	,, ,,
6·38	..	4·54	·21	..	79·69	1200	,, ,,
..	5·94	..	·44	..	80·13	1300	,, ,,
..	4·87	..	1·07	..	81·20	1400	,, ,,
..	4·30	..	·57	..	81·77	1500	,, ,,
..	4·00	..	·30	..	82·07	1600	,, ,,
..	3·45	..	·55	..	82·62	1700	,, ,,
5·13	..	3·21	·24	..	82·86	1782	Bend in road.
..	4·67	..	·46	..	83·32	1800	Centre of road.
..	4·25	..	·42	..	83·74	1900	,, ,,
5·29	..	3·98	·27	..	84·01	1934	Position of new manhole B.
8·22	..	6·76	..	1·47	82·54	..	
	..	1·34	6·88	..	89·42	..	⊼ Station Bridge (89·40).
36·08 27·26 8·82	.. .. ..	27·26 ..	15·29 6·47 8·82	6·47 ..	89·42 80·60 8·82	.. ..	.. ..

and firm position from which all the points are visible being suitable.

If the point D be chosen as the instrument station, the distance to a back sight station at C is rather excessive, so that a point (II) approximately midway between C and D would be more suitable.

The levelling staff is accordingly again held at C, and the reading (5·28) entered in the " Back Sight " column and on the same line as the previous fore sight reading of 4·45. The readings at 200, 300, 400,

500, and 600 ft. are booked as intermediate sights, and that at the corner D (2·98) as a fore sight.

The instrument is again moved, and the next length DE being about 1100 ft., and accordingly too great to be taken in one setting, the third instrument station (III) chosen is 250 ft. or so from D. Readings are booked as shown, and the (IV) and (V) instrument stations chosen are indicated on the plan.

The exact position of points (known as "Change" points or "Turning" points) upon which back and fore sight readings are taken is immaterial, and unless it chances to be convenient to use points upon the section, *any* other firm points approximately equidistant from the instrument may be employed.

In order to check the accuracy of the field work, the levelling may be continued from B either to some other bench mark F whose reduced level is known, or back to the initial bench mark on Parkgate House, and for this work no chaining is required—simply a series of back and fore sights—with no intermediates—taken at *any* convenient points along the route. The point B may be an intermediate sight if more convenient.

The "reduction" of the levels to some assumed datum plane will now be explained. The two methods are known as the "Collimation Method" (p. 180) and the "Rise and Fall Method" (p. 181).

Collimation Method.—Let *ab* (Fig. 6.15) represent the plane in which the line of collimation lies when the instrument is in its first position by the entrance to Parkgate House. Then the reading on the staff when held on the bench mark being 1·04, the line of collimation is 1·04 ft. higher than the bottom of the staff, *i.e.* than the bench mark. But the "reduced level" (*i.e.* the altitude above Ordnance datum) of this is 80·60, therefore the reduced level of the line of collimation, *i.e.* of the instrument axis, is $80{\cdot}60 + 1{\cdot}04 = 81{\cdot}64$ ft.

This is entered in column (4), while 80·60 is entered in column (5), because the B.M. is the first "staff" station.

Again, the reading on the staff when held at A is 4·58 ft., *i.e.* the manhole cover is 4·58 ft. below the level of the instrument axis, and hence is $81{\cdot}64 - 4{\cdot}58 = 77{\cdot}06$ ft. above datum. This value is entered in column (5). Similarly, the reduced level of the 100 chainage point is $81{\cdot}64 - 4{\cdot}22 = 77{\cdot}42$; and that of the 123 point $81{\cdot}64 - 4{\cdot}45 = 77{\cdot}19$ (the fore sight reading is taken here, as both values, 81·64 and 4·45, refer to the first setting of the instrument).

When the instrument is moved to the second position, the new line of collimation is almost sure to be at a different level from that in the first case, but the two levels are correlated by means of the back and fore sight readings. Thus let *cd* (Fig. 6.15) represent the new line of collimation. The level of the staff station at 123 chainage is already reduced as 77·19, and as the back sight reading here is 5·28, the reduced level of *cd* is $77{\cdot}19 + 5{\cdot}28 = 82{\cdot}47$, and this is entered in column (4).

The reduced level of the staff station at 200 is now $82{\cdot}47 - 5{\cdot}14 = 77{\cdot}33$, and, similarly, the other reduced levels are calculated as in the table.

Rise and Fall Method.—In this method the reduced level of each point is deduced from that of the one immediately preceding it. It may be seen from the above explanation of the collimation method, and by reference to Fig. 6.15, that the bench mark is 1·04 ft. below the line of collimation, while the point A is 4·58 ft. below the same line, so that the latter point must be (4·58 − 1·04) = 3·54 ft. below the bench mark.

That is to say, there is a " fall " of 3·54 ft. from the B.M. to A, and this is entered in the " Fall " column as shown. The reduced level of A is therefore 80·60 − 3·54 = 77·06, a value which is entered in column (6) (p. 181).

Similarly, the 100 ft. point is 4·22 ft. below the line of collimation, while A is 4·58 ft. below, so that there is a " rise " of (4·58 − 4·22) =

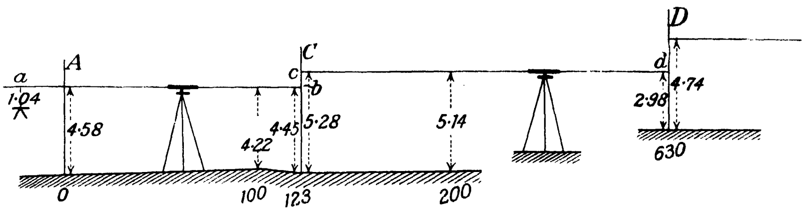


FIG. 6.15

·36 ft. from A to 100. This is entered in the " Rise " column and the reduced level calculated as 77·06 + ·36 = 77·42 ft.

(Note, it is the " Rise " or " Fall " *to* the second *from* the first of each pair of values which is calculated—not conversely.)

The next position (123) is 4·45 ft. below the first line of collimation, while the 100 is 4·22 ft. below the same line. Hence there is a " fall " of ·23 ft. from the 100 to the 123 point, and the reduced level is 77·42 − ·23 = 77·19 ft.

The position 200 is 5·14 ft. below the *second* line of collimation, while the 123 point is 5·28 ft. below *this* line: there is accordingly a rise of ·14 ft. from 123 to 200, and the reduced level of the 200 position is 77·19 + ·14 = 77·33.

It must be noted that the 4·45 is *not* subtracted from the 5·14 in computing the Reduced Level of the 200 position, as the two values refer to different settings of the instrument, and hence to different lines of collimation. For the same reason 4·22 is not subtracted from 5·28 to obtain the reduced level of the 123 point (see Fig. 6.15).

The remainder of the computations, which are made in a similar manner, may be followed from the level book (p. 181).

Checking.—The first value (1·04) in the back sight column gives the position of the first point below the first line of collimation.

The difference between the first fore sight (4·45) and the corresponding back sight (5·28) gives the difference in level between the first and second lines of collimation. Similarly, the difference between 4·74 and 2·98 gives the difference in level between the second and third lines of collimation, and so on. Finally, 1·34 gives the position of the last

point below the last line of collimation. Consequently, if these calculations are all done together, by subtracting the summation of all the fore-sight values from the summation of all the back-sight values, or *vice versa*, the result will be the difference in level between the first and last points, *e.g.* in this case 36·08 − 27·26 = 8·82 ft. rise. This should agree with the difference between the first and last reduced levels, 89·42 − 80·60 = 8·82 ft. rise.

Also in the "Rise and Fall" method (p. 181) the difference between each pair of consecutive points is entered either in the fourth or fifth column, so that the difference between the summation of the values in these two columns should again give the difference in level between the first and last points, *i.e.* 15·29 − 6·47 = 8·82 ft. rise.

Each page may be checked independently in this way, the last reading on each page being entered, while in the field, in the fore-sight column, and the same value again entered at the head of the next page in the back-sight column, although in the usual course it would be booked simply as an intermediate sight.

This may be considered as equivalent to a new setting of the instrument, with the line of collimation at exactly the original level. When the last booking on a page chances to be a true fore sight, and the instrument is actually moved, the back-sight reading is not entered on the same (bottom) line, but alone on the top line of the next page—in the back-sight column. When one page has been "reduced", the last value in the "Reduced Level" column is transferred to the top line of the next page, and the calculations proceeded with as before.

In the "Rise and Fall" method, the accuracy of each reduced level depends upon that of the preceding one; *e.g.* if the value 77·92 had been miscalculated as 77·82 either through an error in the "Rise" column or an error in computing the reduced level itself, then all the following values would be 0·10 too small, and the final value would be 89·32 instead of 89·42, so that the error could be detected.

If the difference between the "Rise" and "Fall" totals was not 8·82, the mistake would be in the calculations of the rises and falls from the staff readings, and these would need re-working until the mistake was located.

If the difference between these columns (4) and (5) checked correctly with (1) and (3) to 8·82, the mistake would be in the successive additions or subtractions to compute the values in the "Reduced Level" column, and these would require re-working.

It is therefore advisable that on each page the rise and fall calculations shall be completed and checked by comparing with the difference of the back and fore sight column summations, before the reduced-level calculations are commenced.

In the "Collimation" method, any mistake in the calculations which refer to a back or fore sight station will be carried through all the computations which follow, but any mistake in connection with an intermediate point will not affect other stations, *e.g.* if the underlined value 77·92 had been miscalculated as 77·82, the following values would be unaltered, and the last value would still be obtained as 89·42.

The mistake might thus be undetected—which is a disadvantage.

To conclude: the method of checking in the "Rise and Fall" system checks *all* the office calculations—intermediates and back and fore sights, unless by chance two equal and opposite mistakes occur, while in the collimation system only the back and fore sight calculations are checked.

The "Rise and Fall" method is therefore the more popular with Surveyors and Engineers, although the office work in connection with the "Collimation" method is slightly less laborious.

The field work may be checked, as already mentioned, by continuing the levels back to the original bench mark, possibly by another route, or by proceeding to some other bench mark, as in this example, where it is assumed the B.M. on Station Bridge has a reduced level of 89·40. The error in this case is thus ·02 of a foot, an amount which does not exceed the maximum allowable (see p. 219).

When the levels are continued back to the original B.M., the sum of the back sights should equal those of the fore sights, and the sum of the Rises those of the Falls. It is, of course, not always essential or possible that an *Ordnance* bench mark shall be used as a starting-point from which to commence levelling, but it is always advisable to commence from *some* permanent and easily located position, such as the end of a particular doorstep, the top of a mile post, etc. The datum to which the levels are all reduced may then be assumed at any convenient and arbitrary distance, *e.g.* say 20 ft., below this. One advantage of the use of Ordnance bench marks is that the labour in checking back to the original bench mark is often obviated, while another great advantage is that sets of levels taken at considerable distances apart can be quickly and easily correlated. Fig. 6.14 shows the section plotted to the same horizontal scale as the plan, viz. 40 ft. to an inch, and to a vertical scale of 8 ft. to an inch—the datum being Ordnance datum. This increased vertical scale enables small differences of level to be made easily distinguishable without at the same time making the plan and section so long as to be unwieldy. When the horizontal and vertical scales are similar, the section is said to be plotted to "Natural Scale". The new work upon such a section would preferably be executed in coloured lines, say carmine, crimson lake, vermilion, blue, burnt sienna, burnt umber, etc. etc.

Cross-Sections.—In levelling for works, the exact line of which has been already determined, a longitudinal section is often all that is required, particularly when the works are of narrow widths, as in the laying of sewerage or water mains; but at other times, in order to decide upon the most suitable line, or to enable quantities to be computed for cuttings, embankments, etc., then cross-sections become necessary. Cross-sections also form a convenient method of locating contour lines (see p. 189).

The method of levelling these depends very largely upon the nature of the ground and upon the amount of assistance available. A main longitudinal line is first of all set out upon the ground, and the whole or a portion of this is chained, pegs being left at all points at

which a cross-section is required. The distances apart for cross-sections depend, of course, upon the nature of the works for which they are required, and upon the conformation of the ground. For quantities—if the ground is sufficiently regular to admit of it—they are preferably equidistant, and perpendicular to the main longitudinal line. This may be arranged by setting out each right angle independently with the chain and tape, or with a box sextant or optical square; or by ranging in an auxiliary longitudinal line, pegging in a similar manner to the main line, and ranging the cross-sections through corresponding points on the two lines.

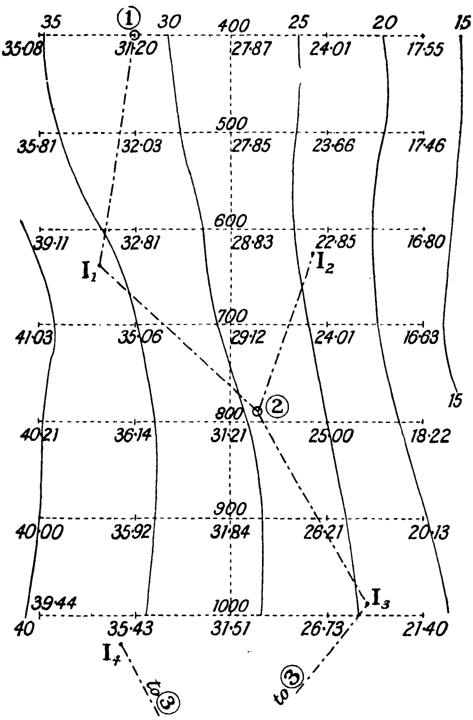


Fig. 6.16

If the differences in altitude are sufficiently small to enable one or more whole cross-sections to be "levelled" from one instrument station, the work can be very quickly done when four operators are available. Two are employed with the chain in setting out the lines, while one proceeds along the cross-section lines and holds the levelling staff at each point where occurs a change in the inclination of the ground. He either notes and books the chainages himself or calls out the information to the fourth man stationed at the instrument. If only three operators are available, the staff man might also act as follower and assist with the chaining.

If the ground is fairly uniform in slope, "levels" may be read at even distances, *e.g.* 25, 50, 75, 100 ft.

In the case of ground the slope of which will not allow of a complete cross-section being taken from any one instrument station, there are two methods available. Each section may be completed independently before another is commenced by moving the instrument to as many other stations in turn as are required, and taking back and fore sight readings as already explained: this simplifies the work and renders the booking very straightforward, but unless the sections are very long, and a considerable distance apart, it entails a great deal of unnecessary labour and loss of time.

An alternative method consists in setting out the cross-section lines

with the chain, marking points at which levels are required by means of " whites " or twigs, and booking the readings, then levelling the accessible portions of several sections from each instrument station. If the readings are to be taken at even distances, *e.g.* every 25 ft. along the cross-sections, and the " whites " are collected by the staff man from each point at which he holds the staff, the work can be quickly and easily done without confusion. Two men, in this case, can proceed with the chaining and setting out, while the instrument and staff men follow for the levelling operations; or, if only two men are available, the setting out can be done first of all by the instrument and staff men before they commence the actual levelling. If the staff stations are not at even distances the work is more confusing, but can be carried out if the chainage distances are previously booked on a slip of paper, and entered in the level book as the readings are taken. The instrument man can generally judge from his own position if the correct points are being taken.

Cross-section

Back Sight	Inter-mediate	Fore Sight	Rise	Fall	Reduced Level	Distances			Remarks
						Left	Centre	Right	
10·67	..	..	..	..	31·20	..	400	100	..
..	6·06	..	4·61	..	35·81	..	500	200	..
..	9·84	..	..	3·78	32·03	..	500	100	..
..	2·76	..	7·08	..	39·11	..	600	200	..
..	9·06	..	..	6·30	32·81	..	600	100	..
..	13·04	..	..	3·98	28·83	..	600	..	..
	⋮								
..	1·66	..	..	..	..	..	..	..	..
..	5·73	..	..	4·07	36·14	..	800	100	..
..	10·66	..	..	4·93	31·21	..	800	..	..
0·41	..	12·54	..	1·88	29·33	..	..	..	Peg (2)
..	1·89	..	..	1·48	27·85	..	500	..	..
..	6·08	..	..	4·19	23·66	100	,,	..	..
..	12·28	..	..	6·20	17·46	200	,,	..	..
..	6·89	..	5·39	..	22·85	100	600	..	..
	⋮								
..	8·21	..	6·37	..	25·00	100	800	..	..
..	1·43	..	..	6·78	18·22	200	,,	..	..
0·28	..	12·54	11·11	..	29·33	..	..	..	Peg (2)
..	3·40	..	..	3·12	26·21	100	900	..	..
..	9·48	..	..	6·08	20·13	200	,,	..	..
..	2·88	..	6·60	..	26·73	100	1000	..	..
..	8·21	..	..	5·33	21·40	200	,,	..	..
	Etc.								

Example.—The table shows the application of this method to the cross-sections shown on the plan in Fig. 6.16. The assumed instrument stations are indicated as I_1 I_2 I_3, but of course many other positions are possible and suitable. Change points are marked (1).

It is assumed that the 400 cross-section has already been levelled, and that a fore sight has been taken on the point 100 ft. to the " right " of the longitudinal

line, giving its reduced level as 31·20 ft. above some arbitrary datum. The 500, 600, 700, and 800 cross-sections are taken together from stations I_1 and I_2 as shown in the level book.

As the length of an ordinary levelling staff is 14 ft., the height of the instrument axis, when the level is set up at I_1 must not be more than this interval above the back-sight position (1), otherwise the required fore-sight reading will not be obtainable; but, on the other hand, if it is possible, the axis should be sufficiently high to enable readings to be taken on the highest points of the "right" sections.

From the contours it is seen that the reduced level of the chosen position of I_1 is a little over 37·00, and therefore that of the axis is approximately 42·00 (assumed 41·87 in the table); the position will consequently allow of a back sight being taken on (1), and will permit of the highest point 41·03 on the 700 cross-section being levelled. On the ground, of course, these facts have to be judged by the eye. Another precaution to be observed is that readings (particularly back and fore sights) should not be taken at greater distances than, say, 250 to 300 ft., depending upon the power of the telescope.

After levelling, and booking in the manner indicated in the table, all stations within range, *e.g.* in this case the right sections of the 500, 600, 700, and 800 points, it becomes necessary to choose a "change" point. Often one of the main line pegs, such as the 800 point, would be suitable for this purpose and would form a permanent bench mark until the job was completed; but as the reading of the 800 point is only 10·66, probably in this case it would be advisable to choose another point at a lower level, *e.g.* a peg at (2) where the reading is, say, 12·54.

Another instrument station I_2 is chosen, which will enable a back sight to be read from the peg (2), and at the same time include, if possible, the lowest staff station on the 500-800 left sections. These lines having been levelled, a check reading might again be taken on (2) as a fore sight, and the instrument set up at I_3 to enable the 900-1200 left sections to be levelled after taking a back sight from here on to (2).

A new change point (3) is then selected, as high as possible up the bank near the 1200 line, and the 900-1200 and 1300-1600 right sections taken, using this point (3) as a back sight point for the instrument stations I_4 and I_5 in turn.

It may not always be possible to take the cross-sections in two settings each, as in this case, but if three or even more instrument stations are required, the booking is done in a similar manner, so that the data for plotting can be easily abstracted in the office from the level book.

A longitudinal section may be plotted from the figures which refer to the main line distances, but it is not generally necessary that it should be plotted to the same horizontal scale as the cross-sections; the vertical scales, however, usually would be the same for each.

For the method of calculating quantities from the sections, the reader is referred to Chapter XI.

Cross-sections may also be taken with a theodolite, Abney level, or clinometer, as explained in methods 8 and 9, pp. 193 and 194 (Contouring), or, if under water, by means of soundings.

Contours.—A contour may be defined as a line of equal altitude upon the earth's surface, and is usually shown on a map by a dotted, dot and dash, or other distinctive line, or by colours, hatchures, or shadings. Thus the boundaries or traces of the sections in which an imaginary level plane of 100-ft. altitude cuts the various features of a country represent the 100-ft. contour lines. Similarly, if the mean level of the sea were raised 100 ft., its boundary would coincide with the 100-ft. contour —just as its present mean boundary may be considered to represent a contour of zero altitude. Consequently, all contour lines—though not necessarily upon any particular plan or map—must eventually form complete circuits. The levels of the contours may, however, be referred to any arbitrary datum, *e.g.* the old Ordnance datum of Great Britain is not *exactly* at mean sea-level.

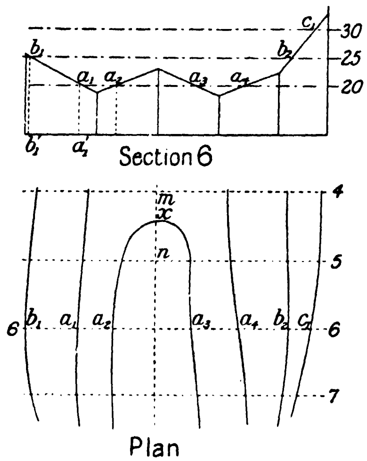


FIG. 6.17

Contour lines are not of necessity above sea-level, those under water being frequently determined by means of soundings. The heights of these may be stated in negative values and referred to mean sea-level, or in positive values, with reference to some arbitrary datum of a lower altitude.

The vertical distances between the consecutive contour lines shown on a plan are known as " Contour Intervals ", and the magnitude of these depends upon several factors, such as the expense to be incurred in their determination ; the scale to which the map is to be produced ; the nature of the country, *e.g.* whether mountainous or otherwise ; and the purpose for which they are required.

An area may be contoured, *i.e.* the position of the various contour lines determined, by several different methods, the chief of which are as follows:

1. A main line is chained and pegged out on the ground, and its position ascertained with reference to a sufficient number of fixed points to enable it to be correctly plotted on the plan. Cross-sections are taken in the usual way, at various points along its length, a Dumpy or other level being used for the purpose. The section lines are lightly plotted in pencil upon the plan, and the complete cross-sections upon other paper. Horizontal lines at the required intervals are then drawn on the sections, and the points of intersection with the ground level noted, *e.g.* *a*, *b*, *c*, in Fig. 6.17, after which the horizontal distances of these intersections from the main line are plotted on the section lines on plan (Fig. 6.17).

As an alternative, each section may be plotted lightly upon its plan line as a base, and the points of intersection projected down to this, to give the required points on the plan: thus in Fig. 6.17 the section 6 would be plotted on the line 6, 6 in plan, and the points a_1, b_1, etc., projected down to this as a_1a_1', b_1b_1', etc. Afterwards the lines not required are erased, and the contour lines completed by joining up all the corresponding points so obtained.

The point x might be got from the longitudinal section, or by interpolation between the known values at m and n.

Instead of *plotting* the sections, the reduced levels of the various points may be written on the plan, directly from the level book, and the contours interpolated as in method 2.

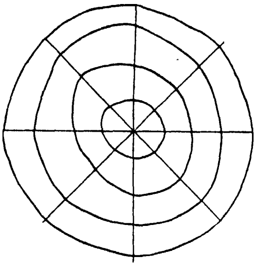


FIG. 6.18

2. A hill is sometimes contoured by levelling along a series of radiating sections, set out at 30°, 45°, or other suitable intervals from a point on the crest (Fig. 6.18).

The positions of the various staff stations are then marked on the plan, and the reduced levels of each abstracted from the level book and written against them. Between these values the contour lines may be interpolated—either by judging the intersection points on the radiating lines with the eye, or by means of the device shown in Fig. 6.19.

If desired, the cross-sections may be plotted, and the required points determined as explained in method 1.

The diagram [1] shown in Fig. 6.19 affords a very convenient means of interpolating contours from spot levels.

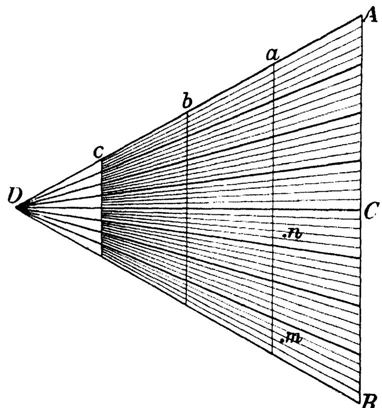


FIG. 6.19
Interpolation Diagram for Contours

A line AB is drawn upon a sheet of tracing cloth or tracing paper, and along this a number of equal parts are set out.

From the centre C a perpendicular CD is erected of any convenient length; and rays are then drawn from D to the various points along AB, each fifth line being rather thicker than the remainder to avoid confusion.

Parallel to AB a number of lines such as a, b, c are drawn to cut the radiating lines as shown, and it is evident that these are now equally divided also.

To interpolate the 5-ft. contours between two points m and n having reduced levels of say 127·4 and 141·3, the lowest thick radiating

[1] Kennedy, *Surveying with the Tacheometer.*

line may be considered as 125, and the other thick lines as 130, 135, 140, 145, etc., respectively.

The tracing paper is placed over the plan so that the point *m* lies between the 127 and 128 rays, and the diagram is then moved over the paper until the point *n* at the same time lies between the 141 and 142 rays as shown in the figure. The line *mn* must be parallel to the line AB (or to *a*, *b*, or *c*) as oblique lines across the rays are not equally divided. The thick lines which represent 130, 135, and 140 are then pricked through on to the plan to give the 130, 135, and 140 contour points.

In this way, when the spot levels have been plotted, the contour lines may be very quickly and accurately interpolated.

If desired, each thick line may represent a 1-ft., 10-ft., 50-ft., or other interval as the case may require.

Another method of interpolating the contours between points of known altitude on the plan is as follows.

Let the two points *m* and *n* in Fig. 6.20 have reduced levels of 127·4 and 148·3 ft. respectively.

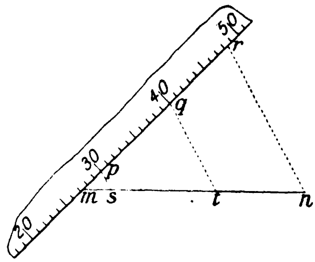


FIG. 6.20

Any convenient plotting scale is laid on the paper with the point *m* coinciding with 27·4 graduation, and in such a direction *mr* as will give good intersection in the following construction. The 30, 40, and 48·3 graduations are marked on the paper at *p*, *q*, and *r*, a set square is placed along the line joining *r* to *n*, and parallels are drawn through *p* and *q* to intersect the line *mn* in *s* and *t*, when *s* and *t* lie on the 130 and 140 contours.

The lines shown dotted in the figure need not be drawn, as the points *s* and *t* may be marked with dots while the edge of the set square lies along *ps* and *qt* respectively.

3. The "cutting" points of the several contours with each section line in either method 1 or 2 may be found by trial in the field and the chainages booked direct, thus obviating the necessity of plotting the cross-sections in the office.

Thus, if the back-sight reading on to a bench mark or a change point —the reduced level of which is, say, 29·76—is 3·58, the height of the instrument axis is 29·76 + 3·58 = 33·34 ft. above datum. Consequently, if the levelling staff is moved along the section line until, by trial, the reading is found to be about 3·34, its foot marks a point on the 30 contour, and the chainage can be observed and booked.

Similarly, when the staff is held at a point on the 25 contour the reading is 8·34, and on the 20 contour, 13·34.

4. Points at intervals along each contour may be found by this same method of trial, and pegged out on the ground with "whites" or other pieces of stick, after which a chain line may be laid down on the ground and the contours surveyed by means of offsets in the usual manner. To enable the work to be plotted, the chain line must, of course, be fixed relatively to some known points on the plan.

5. If the reduced levels of the bench marks are not known at the moment, or to dispense with the calculations in the field, and to save time in finding points exactly on the various contour lines, a number of " spot " levels may be taken at any arbitrary points, such as changes of slope, ridges, valleys, etc. These points are " pegged ", surveyed, and plotted, and their reduced levels afterwards written in pencil upon the plan to enable the required contours to be interpolated as explained in method 2.

6. For small-scale topographical maps, if the reduced levels of a few prominent points are determined, the contours may be sketched on the map with sufficient accuracy, judging their position and shape by the eye.

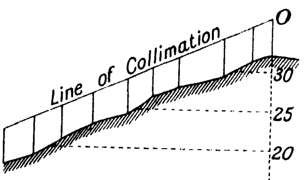


FIG. 6.21

7. A theodolite may be used instead of an ordinary level, in which case the line of collimation need not be horizontal, but may be inclined at any convenient angle approximately parallel to the slope of the ground. Thus, in Fig. 6.21 the instrument could be set up at the point of intersection O, and readings taken on the staff at various points, with the line of collimation inclined downwards at an angle (a) say.

The reduced levels of the staff stations may be calculated in the usual way after adding an amount $d \tan a$—the values of which can very readily be deduced on an ordinary slide rule—to each, where d is the horizontal distance from the instrument to the staff station, or $d' \sin a$, where d' is the slope distance. Or preferably the cross-sections may be plotted very quickly by measuring the actual readings downwards from the plotted inclined line of collimation as in the figure. Similarly the chainage distances, if measured along the slope, may be plotted directly along the inclined datum line, or the horizontal distances may be deduced from the slope distances as $d' \cos a$.

In order to draw the inclined line of collimation upon the plan, or in order to calculate the reduced levels of the staff stations, it is necessary that the reduced level of the instrument axis shall be known.

This may be determined for each position of the theodolite :

(i) By taking a sight on to a levelling staff held upon some permanent or temporary bench mark of known altitude, preferably with the line of collimation horizontal. Thus if a reading of 7·46 is obtained on the staff placed on the top of a peg 80·60 ft. above datum, the reduced level of the instrument axis—if the line of collimation is horizontal—is 88·06 ft.

If the telescope is inclined at an angle a, and the horizontal distance from the instrument to the staff is d'', the reduced level of the axis is $80{\cdot}60 + 7{\cdot}46 \pm d'' \tan a$, according as a is an angle of depression or elevation.

(ii) By ascertaining the reduced levels of the various instrument stations by direct levelling before setting up the theodolite : then

measuring directly the height of the axis above the peg marking the particular instrument station.

Thus, if the theodolite is to be set up at the intersection of successive cross-sections with the longitudinal section line, these points may be pegged, and their levels ascertained in the usual manner with a dumpy or other level, before commencing the cross-sections: unless the pegs are driven flush with the surface, two levels are required at each point, viz. that of the ground, and that of the top of the peg; or if a reading is taken on the top of the peg only, the amount of projection above the ground should also be booked to enable a correct longitudinal section to be plotted.

Example.—If one peg, say M, has a reduced level of 83·60, and the height of the instrument above M is 4·76, the reduced level of the axis is 83·60 + 4·76 = 88·36 ft. above datum.

8. The longitudinal section levels having been taken in the usual manner with a "Dumpy" or other instrument, the contouring may be executed with a simple hand level, Abney level, or clinometer as follows:

Let the reduced level of the longitudinal section peg A be, say, 81·43—this figure being entered in the contour book from the figures furnished by the levelling party before the contouring work commences.

A ranging or other rod is held vertically upon the peg A, and the hand level or other instrument is steadied by resting it against a nail driven into the rod at say 5 ft. above its lower extremity. The reduced level of the instrument axis is consequently 81·43 + 5·00 = 86·43 ft.

To determine the 80 contour, the contour staff must be taken down the slope, along the cross-section line as set out with an optical square, until a reading of 6·43 ft. is obtained by sighting the staff through the aperture of the instrument, while at the same time the bubble is seen by reflection to be in the centre of its run.

Similarly, up the slope, the 85 contour is determined by moving the staff until a reading of 1·43 ft. is obtained.

The distance from the main line to the contour is measured with a tape and booked as below.

To proceed to determine the 75 contour, the instrument man moves to and occupies the position located as the 80 contour, while the staff man moves the staff down the slope until a reading of 10·00 ft. is obtained.

Similarly to determine the 90 contour the instrument man proceeds up the slope until he finds by trial a position from which the reading on the staff held upon the 85 contour is 10 ft.

An alternate method may be employed to determine the second and following contours, *e.g.* the 80 contour having been located, the staff man remains at that point, while the instrument man proceeds down the slope until he finds, by trial, a position from which he sights exactly to the foot of the staff, when the instrument is level and 5 ft. above the ground. His ranging rod then fixes the 75 contour, and the distance from the 80 contour is taped and booked.

The 70 contour is found from the 75 position in the same way. Similarly when proceeding up bank, the instrument man moves to the 85 contour and takes the position occupied by the staff man, who now moves up the slope until he finds by trial another position at which the foot of his staff is just on the same level as the eye of the instrument man.

Sometimes the staff is used to determine only the first contour on each side of the main line: for the remaining points the foot of the staff man is sighted instead of the foot of the staff.

Such methods may be sufficiently accurate for small-scale plans, but it is preferable to adopt the method of sighting to the 10 ft. mark on the staff for more accurate work—especially if there is long grass which may tend to hide the lower portions of the staff or the staff man's foot.

The horizontal dimensions that are booked may be either the distances from one contour to the next, or they may be the total distances of each contour from the main line.

The following shows the method of booking the contours which are delineated on the plan in Fig. 6.22; the distances being entered as they are measured in the field.

180	170	160	150	140	130	122·64	120	110	100	90	80
184	155	123	92	58	24	702 + 00	10	48	83	124	171
180	170	160	150	140	130	120·22	120	110	100	90	
210	179	148	111	75	38	700 + 00	3	40	80	127	

In the central column are shown the horizontal distances along the main longitudinal section line, with the reduced levels of the various points abstracted from the level book. To the right and left are shown the values of the contours with the respective horizontal distances from the main line.

The staff may be an ordinary Sopwith levelling staff—or a more simple pattern with clearer graduations reading to $\frac{1}{10}$th of a foot only.

When the reduced levels of the longitudinal section pegs are not known, ordinary cross-sections may be run with the hand level, the readings on the slope at definite distances from the main line being noted and reduced to the proper datum later.

9. Cross-sections may be determined with the clinometer or Abney level, by steadying the instrument against the 5 ft. mark of a ranging rod as in method 8, and by sighting to a vane fixed at a similar height above the ground upon another ranging rod. The average inclination of the ground is then recorded upon the graduated arc of the instrument (Fig. 3.7 or 3.9).

If the slope of the ground is not uniform, a number of sights may be

taken over consecutive lengths, and the data booked in the following form—the inclination being placed above the line and the distance over which the slope is constant below.

In the central column are placed the chainages along the main line of the survey, and the reduced levels as previously determined by the levelling party.

$20\frac{1}{2}°$	17°	122·64	−15°	−12°
70	165	702·00	120	50
18°	15°	120·22	−14°	−10°
75	150	700·00	110	45

The cross-sections may be plotted directly from these figures with a protractor, provided that the vertical and horizontal scales are equal: and the contours may then be interpolated as in method 2. If the cross-sections are only required for the location of the contour lines, squared paper may be used with advantage.

An alternate method of interpolating the contours is to tabulate the results in a table such as (I), when l represents the slope distance, or (II), when l represents the horizontal projection.

Thus if the measured distances are slope distances, the horizontal projections are represented by $l \cos \alpha$, and the vertical differences by $l \sin \alpha$.

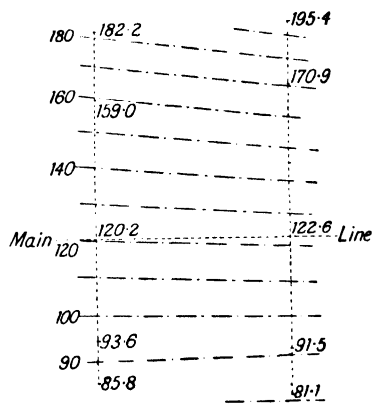


Fig. 6.22

If the horizontal projections are measured directly, the vertical differences are represented $l \tan \alpha$.

The reduced level of the last point of each length is equal to that of the first point $\pm$ the vertical component $l \sin \alpha$ or $l \tan \alpha$.

A number of spot levels are thus deduced, from which the contours may be interpolated as explained in method 2 and Fig. 6.19.[1]

Fig. 6.22 shows the spot levels deduced from Table I.

10. To dispense with chainage operations, the contours may be interpolated from "spot levels", the positions and altitudes of which are determined by means of a tacheometer or omnimeter, as explained in Chapter VIII.

11. Contour lines under water are determined by means of sound-

[1] See also "Contouring with a Clinometer and the Application of Alignment Charts to the Reduction of the Field Notes", by W. N. Thomas, *Proc. Inst. M. and Cy. E.* vol. xlvii. p. 289.

ings, and spot levels, or either parallel or radiating cross-sections may be employed.

The methods of locating and reducing soundings are explained in Chapter XII.

TABLE I

l	$a°$	cos a	sin a	l cos a (by slide rule)	l sin a	Reduced Level of Last Point
700						120·2
110	−14	·970	·242	107	−26·6	93·6
45	−10	·985	·174	44	− 7·8	85·8
150	15	·966	·259	145	38·8	159·0
75	18	·951	·309	71	23·2	182·2
702						122·6
120	−15	·966	·259	116	−31·1	91·5
250	−12	·978	·208	48	−10·4	81·1
165	17	·956	·292	158	48·3	170·9
70	20½	·937	·350	66	24·5	195·4

TABLE II

l	$a°$	tan a	l tan a	Reduced Level of Last Point
700				120·2
110	−14	·249	−27·4	92·8
45	−10	·176	− 7·9	84·9
150	15	·268	40·2	160·4
75	18	·325	24·4	184·8
		etc.		

THE USE OF CONTOURS

General.—A great deal of general information concerning the character of a tract of country is obtainable by inspection of a map upon which contour lines are delineated; whether the country is mountainous, hilly, flat, undulating, gently sloping, intersected by ravines, well watered, etc.

E.g. the sketch map on page 198 shows that near A, where the contour lines are close together, the ground is very steep, and at B, where several contours coincide, there is a vertical cliff; the contours near C, on the other hand, are widely and evenly spaced, indicating that the ground in that vicinity has a gentle and uniform slope; near D the country is undulating and there are a number of small hills.

Dips in the contour lines indicate valleys, as at E, or ridges as at F.

It is sometimes difficult at a glance to distinguish valleys from ridges unless rivers or other natural watercourses are shown; these naturally follow the valleys, and the dip of the contours is towards the source, *i.e.* upstream.

If the values of the contours are noted, the distinction can be easily made, *e.g.* the higher values are inside the loop when a ridge is shown, while in the case of a valley the lower values are inside the loop. If the contours are figured on a definite system, *i.e.* if the figures are always placed on the upper side (say) of the lines, the direction of a slope can be immediately grasped without actually studying the exact values of the contour lines.

Engineering.—The most economical or suitable site for engineering works, such as a road, railway, canal, sewer, water-main, reservoir, etc., may be approximately chosen by the inspection of such a map, as steep gradients and expensive cuttings and embankments may be avoided by approximately following the contour lines.

The contours on a *general* map would of course only be sufficiently close for the general design of a scheme, and the exact route or site would be finally determined after an inspection of the actual district, while the quantities for cuttings, embankments, etc., would be computed from levels specially taken for the purpose.

If, however, the site has been accurately contoured, then the centre line for such a scheme may be definitely laid down, and all data for longitudinal and cross-sections taken direct from the map, thus enabling quantities to be calculated, and the edges of cuttings and embankments, and other details to be determined (see Chapter XI).

Military.—A contoured map is of very great value for military operations. It furnishes a commander with a general idea of the whole district, and enables him to see which positions it is most necessary to occupy; which positions are tenable and which are untenable; the best positions for his guns, reserves, transports, etc.; the probable positions occupied by the enemy's forces, his line of march, and most vulnerable points; which positions are in " dead " ground, and which are exposed, etc.

Hills which are convex, *i.e.* hills on which the lower contours are closer than the upper, furnish dead ground at the foot, and, unless a cross fire can be brought into such ground, provide a convenient sheltered spot in which to rally troops for a final assault on an entrenched position near the top of the hill. Hills which are concave, *i.e.* hills upon which the horizontal intervals diminish towards the summit, provide no such protection for attacking troops.

It is often very important—both for military and other purposes—to know whether one position is visible from another, and this information may often be got by inspection from the map, *e.g.* it is obvious that m is not visible from n, while p is probably clearly to be distinguished from q.

If it is not quite evident from inspection of the map, a section may be drawn from the data furnished by the contour lines, and the matter tested.

For example, a section EH may be constructed, either upon the map itself, *i.e.* upon the dotted line EH, or upon a separate paper.

In the former case ordinates would be erected at right angles to EH at the points at which this line is intersected by the various contour

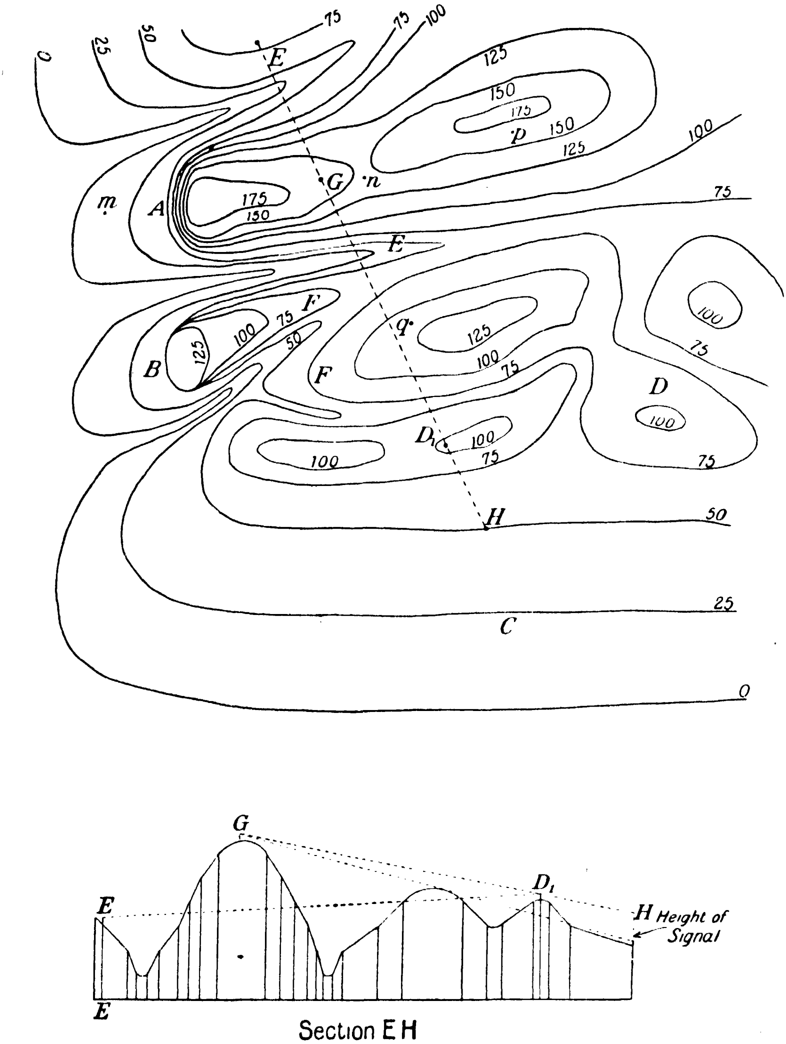


FIG. 6.23

lines, the altitudes of the points marked off along the perpendiculars to any convenient scale, and the section completed.

In the second case the intersections of the various contours may be transferred by means of a scale, or by marks upon the edge of a strip of paper, and a separate section constructed, as in Fig. 6.23.

From this particular section it may be deduced that

(1) Infantry at D_1 would be invisible from E and conversely.
(2) „ H „ „ „ G „ „
(3) „ D_1 „ visible „ G „ „

If, during surveying operations, it was desired to sight from G to H, it would be necessary to erect a signal at H, and the height of this, so that the ray GH shall clear the summit at D_1, may be determined from the section.

For instrumental observations, the ray should be well above the ground at D_1 to prevent the introduction of errors due to uncertain refraction at that point.

Curvature and Refraction.—For long sights, or for very accurate work in levelling operations, allowance must be made for the fact that the line of collimation of a telescope is not a line of equal altitude, *i.e.* is not a level line.

The latter is approximately a circle, its centre being at the centre of the earth, while, if the effect of refraction be neglected, the former line is straight and tangential to this.

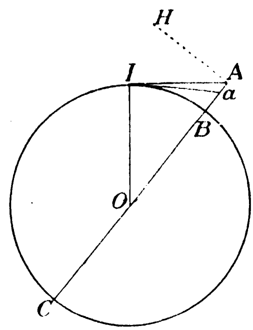


FIG. 6.24
Curvature and Refraction

Thus in Fig. 6.24 let I represent the position of the instrument azis, IA the line of collimation of the telescope, IBC the line of equal altitude through I, and O the centre of the earth.

Now a staff is said to be held vertically at any position, when its length points in the direction of gravity—as indicated by a plumb-bob, so that a staff held vertically at A will lie along the line ABOC, and the difference between the level and the horizontal lines through I is, here, the intercept AB.

But in comparing the altitudes of different stations, the various distances downwards (*i.e.* the staff readings) from the line of collimation IA are compared, whereas, to be strictly accurate, the distances should be measured with reference to the line IBC ; hence in order to avoid error a correction for " curvature " such as AB must be allowed for.

By Euclid III. 36,

$$AI^2 = AB \, . \, AC = AB(AB + BC),$$

or, as AB is generally very small in comparison with the diameter of the earth, BC,

$$AI^2 = AB \, . \, BC,$$

or

$$AB = \frac{AI^2}{BC},$$

$$\therefore \; AB \text{ (in miles)} = \frac{D^2}{7916}$$

where D is the distance from the instrument to the staff station in

miles, and the diameter of the earth is considered to be 7916 miles approximately.

$$\therefore \text{AB in feet} = \frac{5280}{7916} \cdot D^2$$

$$= \cdot 66D^2 \text{ nearly.} \qquad (1)$$

Owing to refraction, however, the line of collimation is not a straight line, but is usually bent downwards as I*a*, the amount varying very considerably as the climatic conditions alter (see Chapter VII).

Taking the correction for refraction to be, on an average, about $\frac{1}{7}$th that for curvature, and in the opposite direction, then the total correction for curvature and refraction (*a*B) is about

$$\cdot 57D^2. \qquad (2)$$

Example (I.C.E.).—In levelling across a river the horizontal web cut the underside of a signboard 14·35 ft. above the level of the ground, the distance being 2 miles 5 chains from the instrument. The back sight to a B.M. close by was 7·25 ft., the level of the bench mark being 52·80 ft. Determine the level of the ground at the signboard.

The correction for curvature and refraction in a distance of 2 miles 5 chains, *i.e.* $2\frac{5}{80} = 2{\cdot}0625$ miles, is $\cdot 57 \times (2{\cdot}0625)^2$ ft. $= 2{\cdot}425$ ft.

The distance of the ground below the edge of the signboard, *i.e.* below the line of collimation, is 14·35 ft. But at the signboard the level line through the instrument is 2·42 ft. below the line of collimation ; so that the ground is there $14{\cdot}35 - 2{\cdot}42 = 11{\cdot}93$ ft. below the level of the instrument axis.

Again, as the reading on the B.M. is 7·25, the reduced level of the axis is $52{\cdot}80 + 7{\cdot}25 = 60{\cdot}05$ ft., so that the ground under the signboard has a reduced level of $60{\cdot}05 - 11{\cdot}93 = 48{\cdot}12$ ft. above datum.

Dip.—It will be noticed that the same formulae (1) and (2) can be applied to determine the distance to the visible horizon I (Fig. 6.24) from a position A (or *a*) at an altitude AB (or *a*B) $= h$ ft., say. *E.g.* by applying formula (2)

$$D = \sqrt{\frac{h}{\cdot 57}} \text{ miles,} \qquad (3)$$

or if h is not negligible compared with the diameter of the earth d, then as

$$AI^2 = AB\ (AB + BC).$$

$$D = \sqrt{h(h+d)}, \qquad (4)$$

where all dimensions are expressed in the same units.

The Dip of the Horizon is ∠HAI, where AH is the horizontal line through A, perpendicular to AO, and the value of ∠HAI in radian measure is the arc BI divided by the radius OB, because ∠HAI = ∠AOI.

The arc BI is approximately equal to D, as given by the above formula, and OB = 4000 miles, say.

Example (I.C.E.).—In connecting the triangulation of Spain with that of Algeria, electric light signals were viewed across the Mediterranean.

Assuming that the Spanish mountains were 9000 ft. high and the Algerian mountains 3000 ft. high, and that the rays from the signal just grazed the sea, how far approximately were the stations apart, omitting the effect of refraction, and taking the diameter of the earth at 8000 miles?

Let M be the position at which the rays graze the sea, then if the distance from M to the Spanish mountains $=D_1$ miles and to the Algerian mountains $=D_2$ miles

$$D_1=\sqrt{\frac{9000}{5280}\left(8000+\frac{9000}{5280}\right)}=\sqrt{1\cdot705\ (8001\cdot7)}=116\cdot7 \text{ miles abt.},$$

$$D_2=\sqrt{\frac{3000}{5280}\left(8000+\frac{3000}{5280}\right)}=\sqrt{\cdot57\ (8000\cdot6)}=\ 67\cdot4 \text{ miles abt.}$$

therefore the total distance is about 184·1 miles.

Or by applying formula (1),

$$D_1=\sqrt{\frac{9000}{\cdot66}}=116\cdot78 \text{ miles (by logs)},$$

$$D_2=\sqrt{\frac{3000}{\cdot66}}=\ 67\cdot4 \text{ miles (by logs)},$$

$$184\cdot2 \text{ miles, about.}$$

Taking into account refraction,

$$D_1=\sqrt{\frac{9000}{\cdot57}}=125\cdot66,$$

$$D_2=\sqrt{\frac{3000}{\cdot57}}=\ 72\cdot55,$$

$$198\cdot2 \text{ miles, about.}$$

Example.—What is the dip of the horizon from the top of the Spanish mountains in the preceding example ?

The distance to the horizon, allowing for refraction, is 125·66 miles, so that the angle in radian measure

$$=\frac{125\cdot66}{4000} \text{ approx.},$$
$$=\cdot0314 \text{ radians},$$
$$=1^\circ\text{-}48' \text{ nearly.}$$

Reciprocal Levelling.—The corrections for curvature and refraction which are given by formulae (1) and (2) are, of course, only mean values, depending upon the latitude of the observer, the shape of the geoid, the state of the atmosphere, etc. ; consequently it is sometimes desirable to eliminate these errors and obviate the necessity of applying a correction formula, the accuracy of which is uncertain.

This may be accomplished by means of the operation known as reciprocal levelling, which in addition eliminates any error due to the line of sight not being exactly parallel with the axis of the bubble.

For instance, let it be required to find the relative altitudes of two positions a and b (Fig. 6.25) separated by a large river or channel.

The instrument is set up at a position A near a, and readings taken upon staffs held at a and at b.

Let Aa_1b_1 represent the horizontal line through the axis of the

instrument and Aa_3b_3 the level line, and Aa_2b_2 the line of collimation, the three points a_1, a_2, and a_3 being practically coincident ($=a'$ say) as the distance Aa is small.

The apparent difference in level of a below b is therefore $a'a - b_2b$, while the true difference of a below b is $a'a - b_3b = d$ say.

$$\therefore\ d = a'a - (b_2b - b_2b_3) = a'a - b_2b + c, \qquad (5)$$

where c, *i.e.* b_2b_3, is the combined error due to curvature, refraction, and want of adjustment of the line of collimation.

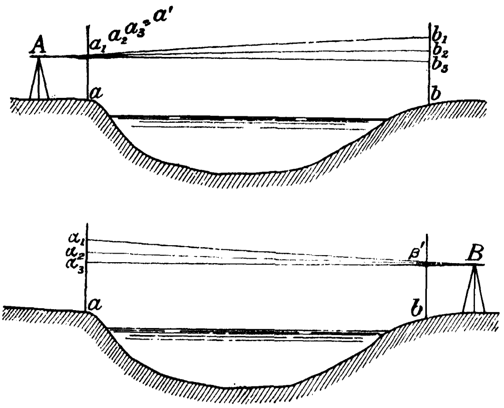


Fig. 6.25.—Reciprocal Levelling

The instrument is now moved to a position B near b and readings again taken on the points a and b.

Preferably the two sets of observations, *i.e.* from A and from B, should be taken simultaneously with different instruments, as otherwise refraction may possibly alter during the interval. In this case, however, special care must be taken to eliminate the collimation errors of both instruments.

Let $\beta' a_1$ be the horizontal line, $\beta' a_2$ the line of collimation, and $\beta' a_3$ the level line in this instance.

The apparent difference in level is now $a_2a - \beta' b$, and the true difference in level is $a_3a - \beta' b = d$,

$$\text{or} \qquad d = (a_2a - a_2a_3) - \beta' b$$
$$= a_2a - \beta' b - c, \qquad (6)$$

because the combined effect of curvature and refraction (and of collimation errors if the same instrument is used) will be equal in each case, *i.e.*

$$b_1b_3 - b_1b_2 = b_2b_3 = c,$$

and $$a_1a_3 - a_1a_2 = a_2a_3 = c.$$

From (5) $$a'a - b_2b + c = d$$

and from (6) $$a_2a - \beta' b - c = d,$$

therefore by addition, and dividing by 2,

$$d = \frac{(a'a - b_2b) + (a_2a - \beta' b)}{2}$$

= half the sum of the apparent differences, the effect of curvature and refraction, etc., being eliminated.

It may be noticed that if a is only slightly below b, *i.e.* if the true difference in level is less than c, then the apparent difference in level, as obtained from B, may indicate that a is *higher* than b unless the correction is applied.

The application of reciprocal levelling to trigonometrical observations is shown on p. 228.

Example.—In levelling across a wide river the following readings were obtained (Fig. 6.25):

From A on to a 3·76, on to b 5·92.
,, B ,, a 6·10, ,, b 8·22.

If the reduced level of $a = 52{\cdot}80$ ft., what is that of b?

From the observations at A, b is $(5{\cdot}92 - 3{\cdot}76) = 2{\cdot}16$ ft. below a.
,, ,, B, b is $(8{\cdot}22 - 6{\cdot}10) = 2{\cdot}12$ ft. below a.

The mean of these, $= \frac{1}{2}(2{\cdot}16 + 2{\cdot}12) = 2{\cdot}14$ ft., may be taken as correct, so that the reduced level of b is

$$52{\cdot}80 - 2{\cdot}14 = 50{\cdot}66 \text{ ft. above datum.}$$

To determine the Sensitivity of the Bubble Tube.—The spirit level consists of an accurately ground cylindrical glass tube of uniform curvature along its longitudinal axis, sealed at the ends, and almost filled with alcohol, ether, chloroform, or other suitable liquid, while the remainder of the tube is occupied by an air or vapour bubble. The top surface of the tube is graduated from the centre outwards, and when the extremities of the bubble lie at equal distances from the centre, they indicate that the tangent to the tube at the central mark is horizontal. If it has previously been arranged that this tangent is parallel to the line of collimation of a level or theodolite telescope, the symmetrical position of the bubble indicates the horizontality of the line of collimation, and, if the instrument is in adjustment, the verticality of the " vertical " axis.

Usually the length of the bubble is liable to change as temperature conditions change, but the Surveyor is only concerned to note that the bubble is symmetrically placed about its central position, or to note the difference between the end readings. Messrs. E. R. Watts & Son do, however, make a bubble tube in which the bubble remains of constant length over a considerable range of temperature (see p. 174), and this enables the position of the bubble to be checked by reading one end of it only.

The " sensitivity " of the bubble tube depends upon the radius of curvature R; the larger R, the greater the distance through which the bubble will be displaced by any specific tilt of the vertical axis, or by any specific displacement of the line of collimation of the telescope from the horizontal, and hence, the greater the sensitivity. Sensitivity is usually expressed in angle per unit division of the bubble-tube scale, and examples are quoted in Question 6, p. 223. See also p. 89.

To determine this constant, a staff station is fixed a convenient distance, say 300 ft., from the instrument, and readings are taken on the staff held there, firstly with the bubble in its central position, and then with the bubble displaced a few divisions in either direction.

[Alternatively, the determination can be made indoors, using a finely divided scale, 20 feet or so from the instrument.]

A number of observations are made, both ends of the bubble being accurately read each time, and the bubble displaced towards the eye-piece for some readings, and for others, towards the object glass—the average displacement of the bubble being, say, n divisions. Let the corresponding average movement of the axial hair line, or preferably of the three webs of a stadia diaphragm, from their normal position on the staff, be s feet, and the distance of the staff station from the instrument d feet, then approximately this displacement of n bubble divisions moves the line of collimation through an angle of $\frac{s}{d}$ radians, or $\frac{s}{d} \cdot \frac{180 \times 60 \times 60}{\pi}$ seconds, and the value of 1 bubble division $= \frac{s}{d} \cdot \frac{180 \cdot 60 \cdot 60}{n \cdot \pi}$ seconds.

Also if l is the length of one bubble division in parts of a foot, and R the radius of the bubble tube in feet, $l/\mathrm{R} = s/d$, so that R is approximately $\frac{l \cdot d}{s}$ ft.

Example.—Find the radius of curvature of the bubble tube and the value of each 2-mm. division, from the following average readings of the ends of the bubble and of a staff 200 feet away.

Staff readings	4·720	4·915
Eye-piece end of bubble, E	20·0	10·0
Object glass end of bubble, O	10·0	20·0

In the first set of readings, the *centre* of the bubble has moved 5 divisions towards the eye-piece end: in the second set, 5 divisions towards the object glass end. The change through 10 divisions has altered the staff reading 0·195 foot,

$\therefore$ 10 divisions represent $0{\cdot}195 \div 200 = 0{\cdot}000975$ radian,

or 1 division represents $0{\cdot}0000975$ radian $= 20''$ of arc nearly.

2 mm. $= 0{\cdot}00656$ foot,

$\therefore \mathrm{R} = 0{\cdot}00656 \div 0{\cdot}0000975$

$= 67{\cdot}3$ feet approximately.

With a very sensitive bubble, it is difficult to ensure that it is kept exactly in its central position while an observation is being made. Often it is simpler to take the readings of the bubble ends and correct the telescope readings accordingly. This procedure is sometimes adopted in precise-levelling (see p. 213, Geodetic Survey of South Africa), and sometimes in making astronomical observations to determine the altitude of a star or the sun. The corrections are known as the bubble corrections or the E and O corrections, the E referring to the eye-piece and O to the object glass end of the telescope.

Example.—The mean apparent altitude of a star was 40°-20′-30″ when the bubble readings were

	E.	O.
F.R.	15	14
F.L.	16	13

1 bubble division $= 10''$.

What correction should be applied to the apparent altitude ?

The corrections for the F.R. and F.L. readings can be calculated separately, or they can be combined. Thus the combined readings E = 31 divs., O = 27 divs., and the combined difference (E - O) = 4 divs.

The average per observation = 2 divs., *i.e.* the average displacement of the centre of bubble was 1 division.

The E end of the bubble tube was too high so that the zero line was tilted downwards, and the apparent altitude is too large by 10″, *i.e.* the corrected altitude is 40°-20′-20″.

Setting Out.—The inverts of channels, conduits, sewers, etc., are set out to their proper gradients by means of sight rails, and boning or travelling rods.

Sight rails consist of long horizontal rails spanning the trench and

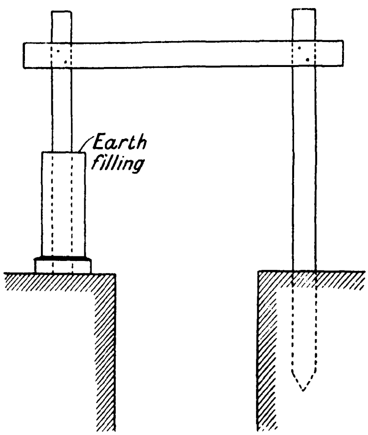


FIG. 6.26.—Sight Rails

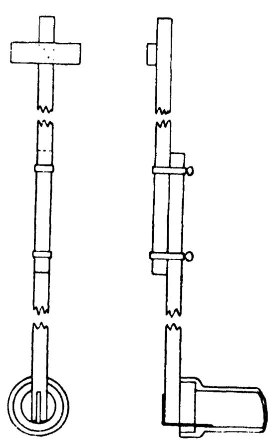


FIG. 6.27.—Travelling Rod

nailed to two or three vertical posts, the lower ends of which are firmly embedded in the earth, or packed with clay or other material into upright pipes (Fig. 6.26).

The travelling rod (Fig. 6.27) consists of a long shaft across which is nailed a small transverse head—much like an ordinary T square.

The upper edge of each sight rail is set truly horizontal with a small hand level, and the line joining the top edges of two consecutive rails is adjusted, by means of a dumpy or other level, to the same gradient that it is required to give the invert of the proposed works.

For concrete conduits, the excavation having been taken down approximately to the required depth, templates are placed temporarily in position. The travelling rod is held vertically upon the centre of each of these in turn (or upon pegs which serve the same purpose), and by sighting from the edge of one sight rail to the next, it is observed whether the head of the rod is in exact alignment. If not, the template must be raised or lowered until the three edges are in line. The length of the travelling rod being constant between any two rails, the invert

is thus set out at the required gradient parallel to the imaginary line joining the edges of the sight rails.

In the case of a long sewer, or channel, the setting out is done in two or more lengths, each sufficiently short to enable the sight rail at one end to be clearly seen from the other end—say 200-250 ft., and the work is generally commenced at the lower or discharge end.

Although it may be required to provide the invert with a uniform gradient, it is not necessary that all the sight rails shall be in one line—*i.e.* that a constant length of travelling rod shall be employed throughout the whole scheme. This length, though constant for any one section, may vary for different sections, which are then entirely independent of each other, except that the invert is made continuous.

The travelling rod is, for that reason, often of an adjustable pattern, as in Fig. 6.27 : in this example an iron shoe is shown to enable the rod to be held on the invert of a sewer pipe, without inclining it from the vertical.

Example.—It is required to fix sight rails for the setting out of a sewer, at an inclination of 1 in 250, the depth at the lower end being 10·5 ft. below a peg A (Fig. 6.28).

A convenient length AB, say 200 ft., is measured in the required direction from A, and uprights fixed at A and B. The fall in the invert from B to A is therefore $\frac{200}{250}$ = ·80 ft.

If the length of the travelling rod is fixed, say 15 ft., the horizontal rail at A is nailed to the uprights at this distance above the discharge level—or 4·50 ft. above the peg A.

If not, the sight rail at A may be fixed at any convenient height above the ground, and the length of the travelling rod adjusted to suit.

The rail at B is now to be fixed at such an elevation that the line joining the top edge to the top edge of the rail at A has an inclination of 1 in 250 towards A, *i.e.* the rail at B is to be ·80 ft. higher than that at A.

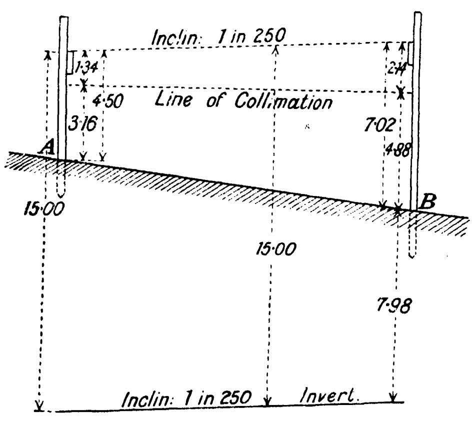


FIG. 6.28

If the ground between B and A is sufficiently elevated, the instrument may be set up about midway between these points and the levelling staff held upon the rail at A, giving a reading of, say, 1·26 ft.

The required reading on the rail at B is evidently 1·26 − ·80 = ·46 ft. The staff is accordingly held against one of the uprights, and raised or lowered until, by trial, this reading is obtained, when the position of the foot of the staff is marked and the horizontal sight rail nailed in place, being levelled with a small hand level. After fixing, the accuracy is checked by holding the staff upon the centre of the rail and observing the reading, which should be ·46 as before. Had the reading on the rail at A been less than ·80 ft., for instance, say, ·26, the staff on the rail at B would not be intersected by the line of collimation, and consequently

it would be necessary to observe a reading—say 3·78 upon a peg below the rail.

Thus, as the rail at B is to be ·80 ft. above that at A, and as the rail at A is to be ·26 ft. below the line of collimation, therefore the rail at B is ·54 ft. above the line of collimation. Again the peg at B is 3·78 ft. below the line of collimation, so that the rail at B must be fixed 3·78 + ·54 = 4·32 feet above the peg at B.

This distance may be measured conveniently with the levelling staff, while the rail is levelled with a hand level and nailed to the uprights.

A more general case is shown in Fig. 6.28, where both rails are above the line of collimation of the telescope.

The reading on the peg at A = 3·16 ft., say, and on a peg at B = 4·88 ft. The length of the travelling rod = 15 ft., say.

The height of the rail A above the peg A is therefore 15·00 − 10·50 = 4·50 ft., or 4·50 − 3·16 = 1·34 ft. above the line of collimation. The rail at B is ·80 ft. higher than that at A, *i.e.* 1·34 + ·80 = 2·14 ft. above the line of collimation. But the peg B is 4·88 ft. below the line of collimation, therefore the rail B is 2·14 + 4·88 = 7·02 ft. above the peg at B.

The depth of the sewer at B is 15·00 − 7·02 = 7·98 ft.

The rail at B in this case is a little too high to be used without a platform of some kind, so that unless all the sighting is to be done from A, both rails should be lowered, say 1·50 ft., and the travelling rod shortened this amount to enable the aligning to be done from either end.

For the prolongation of the sewer the rail at B may remain in its present position, and the same length of travelling rod used for the next section; or it may be raised or lowered to suit the next station C, provided that a corresponding change is made in the length of the travelling rod for the section BC.

Precautions.—A few additional points, the observation of which tends to ensure accuracy in ordinary levelling, will now be mentioned.

The telescope should be correctly focused and parallax eliminated; the large bubble on the telescope should be brought accurately to the centre of its run, and the hand removed from the eye-piece and from the legs of the instrument for all back and fore sight readings; and these turning-points, for any one setting, should be about equidistant from the instrument. The reason for this is that if there is any error due to the line of collimation not being parallel with the axis of the bubble tube (see adjustment (2), p. 162), the effect will be equal for both readings, and consequently the true difference of level between the back and fore sight positions will be obtained. Intermediate points will probably be slightly inaccurate, but an error in these will not carry through and affect all the ensuing work as would be the case if a back or fore sight were in error.

A frequent case in which an error is introduced, due to the back and fore sight distances being unequal, occurs when a line of levels is being run up or down a steep hill.

Thus in Fig. 6.29 let A, B, C be the successive instrument stations and *a*, *b*, *c*, *d* the corresponding back and fore sight stations.

The average height of the telescope above the ground on which the instrument is set up may be taken as about 5 ft., and the length of staff generally employed is 14 ft.

Consequently for points lower down the bank than A the readings from that station will lie roughly between 5·00 and 14·00, while points

higher up the bank than A will give readings between 0·00 and 5·00. The maximum horizontal distances that can be read from A are consequently 5 cot θ up the hill, and 9 cot θ down the hill, where θ is the average inclination of the ground. If the extreme lengths are made equal, each will be very roughly 5 cot θ ft., and only about 10 ft. of the staff will be utilized, while the ground covered by one setting will be about 10 cot θ instead of 14 cot θ ft.; *i.e.* the instrument will need to be moved, roughly, 40 per cent more times.

If the maximum distances are sufficiently small to be unrestricted on account of want of distinctness of the image, in order to save time and labour the natural tendency of a Surveyor will be to work over the whole range of the staff.

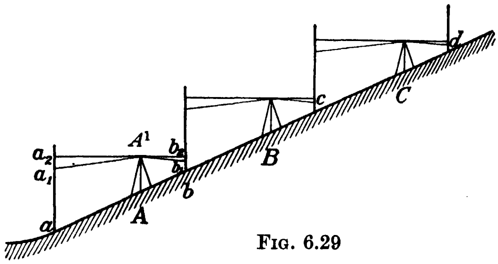


FIG. 6.29

Now suppose that the line of collimation is not parallel to the bubble axis (*i.e.* that adjustment (2), p. 162, is incorrect), but is tilted slightly downwards, so that in levelling up the hill—the instrument being at A—the back-sight reading is aa_1 and the fore-sight reading bb_1, and let the horizontal line through A^1 cut the two positions in a_2 and b_2 respectively, making the error at a equal to a_1a_2 and that at b equal to b_1b_2.

Then as the triangles $a_1A^1a_2$ and $b_1A^1b_2$ are similar, a_1a_2 is much greater than b_1b_2 (approximately $a_1a_2=\frac{9}{5}b_1b_2=1{\cdot}8\ b_1b_2$), so that b, which is actually higher than a by an amount (aa_2-bb_2), is apparently an amount (aa_1-bb_1) higher.

But $(aa_1-bb_1)=(aa_2-bb_2)-(a_1a_2-b_1b_2)$, so that the altitude of b above a so determined, is too small by the amount $(a_1a_2-b_1b_2)$ or approximately $\cdot 8\ b_1b_2$.

Similarly the altitudes of c above b, and of d above c, etc., are too small, and hence the height of the hill is apparently not so great as is actually the case.

It will be seen that the error is exactly the same in levelling *down* the hill, *e.g.* the point a is apparently not as much below the level of b as an instrument in correct adjustment would indicate, and the error is $(a_1a_2-b_1b_2)$ as before, so that it is quite possible by levelling to the top of a hill and down again to the same or another bench mark, to determine wrongly the altitude, although the closing error on the final bench mark may be quite within the prescribed limits of accuracy.

When the line of collimation is in correct adjustment, a_1a_2 and b_1b_2 are each equal to zero, so that no error is introduced when the back and fore sight distances are unequal. It is, however, always advisable to make these distances equal when in *case* of want of adjustment a_1a_2 is equal to b_1b_2 and the final error is eliminated.

An expression for the error in the apparent altitude may be obtained as follows:

Let e be the error in a horizontal sight of 100 ft., due to the line of collimation not being parallel to the bubble axis.

Then $$b_1b_2 = \frac{e \cdot 5 \cot \theta}{100}.$$

But in moving a horizontal distance of 14 cot θ towards the summit of the hill, the error introduced, under the assumptions stated above $= \cdot 8\ b_1b_2$, and therefore the total error, if the horizontal projection of the slope is H or the total height V, becomes roughly

$$\frac{\cdot 8 b_1 b_2 \times \mathrm{H}}{14 \cot \theta} = \frac{\cdot 8 \times 5 \cot \theta \times e \times \mathrm{H}}{100 \times 14 \cot \theta} = \frac{1}{350} \mathrm{H} \cdot e,$$

or $$\frac{1}{350} \mathrm{V} \cdot e \cdot \cot \theta.$$

Thus, if $e = \cdot 02$, $\theta = 10°$ and $\mathrm{V} = 150$ ft., the error is ·05 ft.

The effect of the divergence of the bubble from the centre of its run is to tilt the line of collimation, and so introduce an error the magnitude of which is discussed later.

Back and fore sight readings should always be taken, where possible, upon some firm and substantial object, such as a large stone, a kerb, a peg temporarily driven in for the purpose, etc. It is not necessary that all such positions should be permanent points, but in a long line of levels it is advisable that a certain number should be so: then, in the case of an accidental disturbance of the level and the displacement of a back sight station before the corresponding fore sight reading is taken, it is not necessary to commence levelling again from the starting-point, but operations may be recommenced from any of the "bench" marks so left. Also in the case of a large final error, the bench marks (or change points) may be relevelled, and so separate sections of the work quickly checked, making it possible for the mistake to be approximately located, and so probably limiting the relevelling to a small portion of the ground.

On soft ground and for want of a better change point, the staff may be held upon an arrow pushed up to its head in the ground, the pressure being distributed over a larger area by means of a second horizontal arrow passing through the loop of the vertical arrow.

The instrument station, too, should be on as firm ground as possible, as in marshy and soft ground any movements of the instrument may displace the bubble, and so cause the readings to be in error.

The levelling staff should be held as nearly vertical as possible, and for accurate work a spirit level attached to the back of the staff is often used to indicate to the staff man when this condition obtains. A plumb-bob or pendulum device, arranged to hang suspended exactly over a specified position, or through the centre of a fixed horizontal ring, is also used.

For ordinary work, however, the staff man is left to judge with his eye the verticality of the staff, though the instrument man can observe the truth of this in one direction, by comparison with the vertical

webs of the diaphragm. He may then, if necessary, call, "Top off you", or "Top to you", or motion to the staff man the fact that the staff is out of perpendicular.

If the levelling staff leans directly towards *or* away from him, the instrument man may be unaware of the defect, in which case a reading which is always in excess of the true value is obtained. Thus the error is *cumulative*, and for this reason it is obviously important that due care should be taken to eliminate it. If the true staff reading is l, a deviation of $\theta°$ in *either* direction, from the vertical, will cause the reading to be $l \sec \theta$, which gives an error of $l(\sec \theta - 1)$ ft. (If $l =$ 14 ft. and $\theta = 3°$, the error is ·02 ft.)

An error from this source is very liable to arise when levelling up or down bank, particularly for readings near the top of the staff.

It is often recommended that the staff should be gently waved to and fro by the staff man, through a small angle on either side of the vertical, and the minimum reading booked.

This minimum reading would be the true reading required if the point of rotation at the foot of the staff were in the same plane as the graduated face. This is usually not the case—either the staff is rotated about a point under the centre of the staff, *e.g.* on the head of an arrow—or during the backward motion the point of rotation is the heel of the staff. A reading (S) smaller than the true vertical distance (R) between the line of collimation and the station-point is obtained in each of these cases.

Usually, however, the error is very small, particularly if R is large, and the adoption of this procedure may eliminate errors which are likely to be much larger.

Let θ be the angle the staff is tilted from the vertical when the reading is S, and the true vertical distance is R, and let the distance from the point of rotation to the face of the staff be T, then

$$S \cos \theta + T \sin \theta = R,$$

or

$$S = R \,.\, \sec \theta - T \tan \theta. \qquad (1)$$

For S to be a minimum

$$\frac{dS}{d\theta} = R \,.\, \frac{\sin \theta}{\cos^2 \theta} - \frac{T}{\cos^2 \theta} = 0,$$

i.e.

$$R \,.\, \sin \theta - T = 0,$$

or

$$\sin \theta = \frac{T}{R}. \qquad (2)$$

From (1) and (2) the minimum value of $S = \sqrt{R^2 - T^2}$ which is less than R.

Example.—If $R = 1{\cdot}00$ ft. and $T = 0{\cdot}15$ ft., then

$$\sin \theta = {\cdot}15 \quad \text{and} \quad \theta = 8\tfrac{1}{2}° \text{ about},$$

and

$$S = \sqrt{R^2 - T^2} = 0{\cdot}989.$$

The error in this case, introduced by booking the reading 0·989 instead of 1·00, is 0·011 ft. Similarly if $R = 10{\cdot}00$ the error is 0·0011 ft. about—*i.e.* a negligible amount.

In rough weather it is almost impossible to get accurate results with a level, as gusts of wind are liable to cause the bubble to deviate from its central position, and so displace the line of collimation from the horizontal position, and to bend and shake the levelling staff.

Precise Levelling.—For precise levelling operations, as already mentioned, special instruments and levelling staves are generally employed.

The level is protected from the sun and wind by a screen or umbrella, and the readings corrected, if necessary, for any alterations in the length of the staves due to changes of temperature.

The coefficient of expansion of the staff material is about 0·000004 per degree centigrade, so that for single readings or for work over moderately level country this correction is usually negligible.

If, however, the altitude changes greatly, the sum of the back-sight readings will be either considerably in excess of, or short of, the sum of the fore-sight readings. The cumulative effect of these small corrections then becomes appreciable, and hence it is necessary to apply the temperature correction to this difference in rod lengths; *e.g.* if H is the difference in elevation in metres, the correction to be applied when the average temperature is $t°$ C. above that at which the calibration of the staff graduations is known is $0{\cdot}004 \,.\, t° \,.\, H$ millimetres.

A further correction which it may be necessary to apply is to allow for any variation in the length of the staff graduations from those of the standard.

For instance, if an intercept indicated as 3 metres on the staff is really 3·001 metres long, and if the total apparent difference in altitude of two stations is found to be H metres, the correction to be applied is

$$H \times \frac{\cdot 001}{3} \text{ metres} = \frac{1}{3} H \text{ millimetres.}$$

This must be added if the staff is too long and subtracted if the staff is too short.

The length correction may vary from time to time, and the graduations should be periodically compared with a standard.

A third correction has the nature of an index correction, and is to allow for the fact that the zero of the scale may not coincide with the foot of the staff.

The average wear of the foot of a staff on the New York City Levels was as much as 0·3 mm. per year.

No correction is necessary if the index errors of the two staves are equal or if each staff is used alternately for a back and for a fore sight. Otherwise, if x is the difference between the two index errors, the correction to be applied is Nx where N is the number of settings of the instrument.

Check levels are taken preferably in the opposite direction to the original set, and often by a different surveyor; but one operator may take his own check levels by running an independent line of back and fore sights alongside the primary line.

To ensure complete independence, readings should not be taken on the two lines from one position of the instrument (except for occasional comparisons)—*i.e.* the level should be set up anew each time for the check line, and possibly different staves used.

In order to facilitate operations for a single line two staves are often employed, one, A, being held at the back sight and the other, B at the fore-sight station. For the next setting of the instrument the fore-sight staff B is kept at the same position to furnish a back-sight reading, while the staff A is taken ahead to the next fore-sight station. Thus at any one station the same staff is used for the fore-sight and the back-sight readings there.

To eliminate error due to the possible settlement of the instrument tripod and staff, back sights and fore sights are alternately taken first from successive instrument stations.

Back and fore sight distances are made exactly equal, or a correction is applied to allow for any error due to curvature and refraction. The distances are sometimes measured by reading the three webs of the diaphragm (see Tacheometry): in this case a mean of the three values is taken for the line of collimation instead of the single central reading.

The length of sights is limited (300 ft. say), but the amount depends upon the particular instrument employed.

The work is also confined to times about midday when the atmospheric conditions are favourable and refraction less variable.

The following are abstracted from the General Instructions for Precise Levelling in New York City.[1]

(1) Except when specific instructions are given to proceed otherwise, all lines are to be levelled independently in both the forward and backward direction.

(4) It is desirable that the backward measurement on each section should be made under different atmospheric conditions from those which occurred on the forward measurement. It is specially desirable to make the backward measurement in the afternoon if the forward measurement was made in the forenoon, and *vice versa*. . . .

(5) On all sections upon which the forward and backward measures differ by more than 4·0 mm. $\sqrt{K}$ (in which K is the distance levelled between adjacent bench marks in kilometres), both the forward and backward measures are to be repeated until two such measures fall within the limit.

(7) The programme of observation at each station is to be as follows: Set up and level the instrument. Read the three lines of the diaphragm as seen projected against the front (or rear) rod, each reading being taken to the nearest millimetre (estimated), the level bubble being kept continuously in the middle of the tube. As soon as possible thereafter read the three lines of the diaphragm as seen projected against the rear (or front) rod, estimating to millimetres as before, and keeping the bubble continuously in the middle of the tube.

(8) At each rod station the rod thermometer is to be read to the nearest centigrade degree, and the temperature recorded.

(9) At stations of odd numbers the back sight is to be taken before the fore sight, and at even stations the fore sight is to be taken before the back sight.

(10) The maximum difference in length between a fore sight and the corresponding back sight is to be 10 m. . . .

[1] *Engineering News* ("Precise Levelling in New York City", by F. W. Koop), vol. lxx. No. 10, p. 447.

(11) The recorder shall keep a record of the rod intervals subtended by the extreme lines of the diaphragm on each back sight, together with their continuous sum between bench marks. A similar record shall be kept for the fore sight. The two continuous sums shall be kept as nearly equal as is feasible, without the expenditure of extra time for that purpose, by setting the instrument beyond (or short of) the middle point between the back and front rods. The two continuous sums shall not be allowed to differ by more than a quantity corresponding to a distance of 20 m.

(12) Once during each day of observation the error of the level should be determined. . . .—(*Note.*—This refers to the inclination of the line of collimation to the axis of the instrument.)

(14) The instrument shall be shaded from the direct rays of the sun, both during the observations and the movement from station to station.

(15) The maximum length of sight shall be 150 m., and the maximum is to be attained only under the most favourable circumstances.

(16) At the beginning and end of the season, and at least twice each month during the progress of the levelling, the three-metre interval between metallic plugs on the face of each level rod shall be measured carefully with a steel tape. . . . The purpose of these measures is to detect changes in the length of the rod rather than to determine the absolute lengths. The absolute lengths are determined at the office between field seasons. At least once each month during the progress of the levelling, the adjustment of the rod levels shall be tested. . . . If the deviation from the vertical exceeds 10 mm. on a three-metre length of the rod the rod level must be adjusted.

The following is a short description of the method adopted on the Geodetic Survey of the Transvaal and Orange River Colony.

The levelling staves were made of well-seasoned wood. Each was 3·2 metres long, and consisted of two pieces clamped together to form one staff. One portion was graduated in centimetres (Face I), while the scale upon the second portion (Face II) was arbitrary, the divisions being in the ratio of 11 to 9 to those on Face I. One portion was shod with iron, and rested (at each station) upon a steel pin with a spherical turned head. This pin was passed through a circular iron plate, which rested on the ground, and was forced home.

The staves were compared regularly with a brass scale, and the variations in length of the brass standard, due to changes of temperature, were computed from an attached steel bar, with which it formed a metallic thermometer.

Two staves were used, one on the fore sight and one on the back sight station, and the instrument was set up twice, exactly midway between these points. The three webs were read upon each face, so that 4 sets of 3 readings were taken on each B.S. or F.S. station. In addition, the readings of the object-glass and eye-piece ends of the bubble were noted.

Corrections were then made :

(1) For dislevelment of the bubble. This varied with the distance of the level from the staff and the displacement of the bubble from its central position.

(2) For scale values, as found by comparison with the standard.

The result was then computed separately from Face I and Face II readings, the latter being reduced to centimetres by multiplication with a constant factor.

The Second Geodetic Levelling of England and Wales.[1]—The first primary network of levelling in Great Britain was carried out in the years 1840–1860, and the results are published in the *Abstracts of*

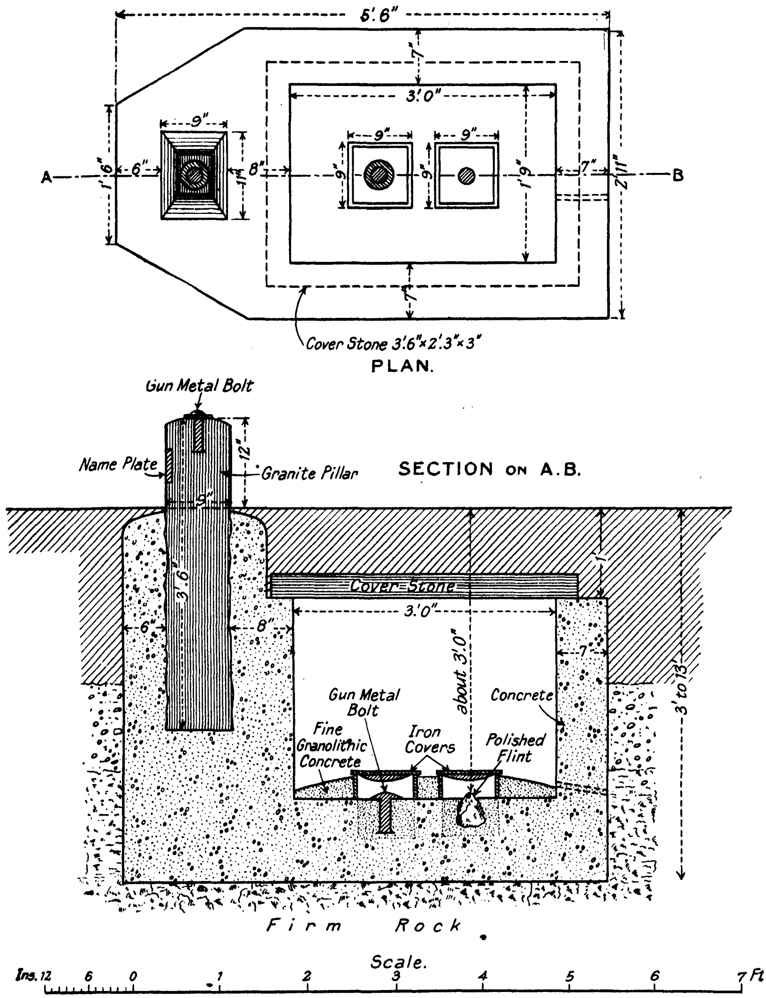


FIG. 6.30.—Fundamental Bench Mark

Levelling. The work, though very good for its date, falls short of the standards of precise levelling of the present day, and difficulties were

[1] *The Second Geodetic Levelling of England and Wales, 1912–1921,* by Colonel Sir Charles Close.

NOTE.—Much of this account follows closely that kindly written for previous editions by Major A. J. Wolff, D.S.O., R.E., the officer in charge of the Levelling Division of the Ordnance Survey, and reproduced here, before the publication of the Report, by the permission of the Director-General, Colonel Sir Charles Close, K.B.E., C.B., F.R.S.

experienced when in the second revision, commenced in 1904, it was attempted to base new accurate work on the existing network. It was therefore decided to carry out a new network of primary levelling, and to connect it with three tidal observatories.

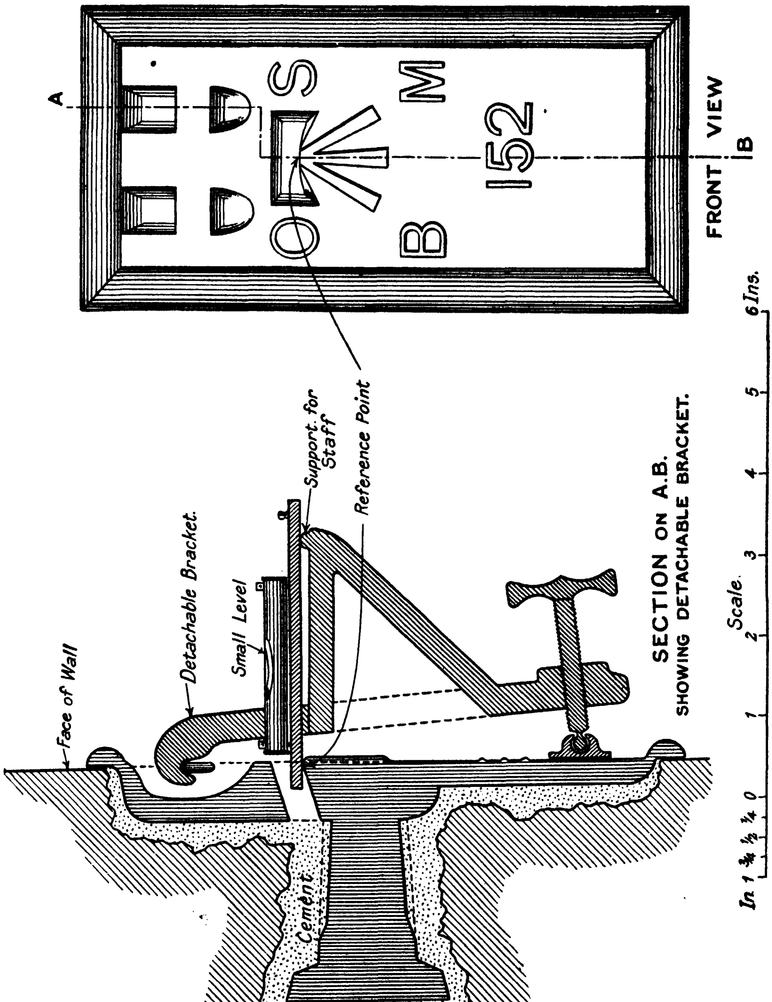


FIG. 6.31.—Flush Bracket

The levelling was commenced in 1912 and completed in 1921. The tidal observations at the three tidal stations—Dunbar, Newlyn, and Felixstowe—were commenced in 1913, 1915, and 1917 respectively, since which time observations to obtain mean-sea-level have been carried out continuously.

The tide gauges are self-registering, and records are abstracted of the height of the tide above the observatory zeros for every hour. From these abstracts the weekly, monthly, and annual means are computed. Barometer readings are taken, corrected for temperature, height above mean sea-level, and latitude, and records are also kept of the temperature and density of the sea water. The immediate response which the level of the sea makes to a rise or fall of the barometer is very marked.

The 1st Class or fundamental bench marks are all on solid rock. They are, on an average, 25 miles apart. It is intended that the precise levelling of these fundamental marks shall serve the double purpose of forming a reliable basis for future branch levelling, and for determining what relative movement, if any, there may be between the levels of the land and sea.

The diagrams show the design of the fundamental bench marks and of the 2nd Class bench marks (Figs. 6.30 and 6.31).

Second Class or flush brackets are spaced about 1 mile apart, and are fixed on the face of old solid buildings, abutments of bridges, and other vertical surfaces with solid foundations.

Each of these flush brackets has a serial number.

The 3rd Class bench marks are copper rivets let into a horizontal surface of stone or brick. They are, on an average, 400 yards apart.

Each line was levelled once in each direction, forward and back, on different days, and no readings were taken less than 0·5 ft. from the ground, so as to reduce the effect of refraction. The levelling staves, at intermediate points, were supported on steel pegs firmly driven in the ground, and these pegs were removed between the forward and back levellings. To eliminate collimation errors, the distances from the level to the staff for back and fore sights were not allowed to differ more than 3 ft.

The Zeiss Level III (see p. 168) was used throughout, and this was provided with a parallel plate arrangement placed in front of the object glass (see p. 174) and a micrometer adjustment enabling the third decimal place of a foot to be read definitely, and the fourth decimal place estimated. An umbrella was used to screen the instrument during the readings.

The original staves were about 10 ft. long, each made from one piece of seasoned yellow pine, with the graduations marked on thin strips of sycamore wood fastened to the face of the staff. For calibration and correction purposes, a bar of invar was incorporated—fixed only at the lower end, and free to expand or contract vertically with reference to the woodwork. In the improved "Cambridge" staves, each again of one piece of seasoned wood, the invar strip, about $\frac{1}{2}$ inch wide, was let into a groove on the face, fixed as before at the bottom only, and graduated to $\frac{1}{50}$ ft. for convenience in use with the parallel plate micrometer on the level. These staves were designed at the Ordnance Survey, and made by the Cambridge Scientific Instrument Company. Each staff was standardised and appropriate staff corrections made to readings.

A full discussion of the subject of the errors of levelling is given by M. Lallemand in his *Nivellement de haute précision*, and the formulae proposed by him were adopted at the Conference of the International Geodetic Association held at Hamburg in 1912. These formulae constitute a standard by which to unify and compare precise level networks, and were employed in the calculations for the English network. The following is a précis of the rules for calculating the probable error per mile[1]:

$$e_1 = \pm \cdot 6745\sqrt{\frac{\Sigma\Delta^2}{4M}},$$

$$e_2 = \pm \cdot 6745\frac{S}{2\sqrt{M}},$$

$$e_3 = \pm \cdot 6745\frac{\sqrt{\frac{c_1^2}{m_1} + \frac{c_2^2}{m_2} + \text{etc.}}}{\sqrt{n}},$$

where Δ = discrepancy between the results of two runnings between two consecutive bench marks;

M = distance in miles;

S = accumulated discrepancy between the two runnings between terminal bench marks of a line;

c_1, c_2, etc. = circuit errors of circuits I, II, etc., respectively;

m_1, m_2, etc. = lengths in miles of circuits I, II, etc., respectively;

n = number of circuits.

The limit for circuit error is four times $e_1\sqrt{M}$ for the circuit.

The formulae for the probable systematic error and probable accidental error are:

(1) For the probable accidental error η_r,

$$\eta_r^2 = \frac{1}{9}\left[\frac{\Sigma\Delta^2}{\Sigma L} - \frac{\Sigma r^2}{(\Sigma L)^2}\Sigma\frac{S^2}{L}\right].$$

(2) For the probable systematic error σ_r, for lines not forming a net,

$$\sigma_r^2 = \frac{1}{9\Sigma L}\Sigma\frac{S^2}{L};$$

or for a network containing at least 10 polygons,

$$\sigma_R^2 = \frac{1}{\Sigma L^2}\left\{\frac{2}{9}\Sigma f^2 - \eta_r^2\Sigma L\right\},$$

where L = length of a line;

ΣL = the accumulated length of a set of lines;

Δ = the discrepancy between the results of two runnings between two consecutive bench marks;

r = the distance between these two consecutive bench marks;

S = the entire systematic discrepancy between the results of the two runnings for a line;

f = the closing error of a polygon.

Limit of Probable Accidental (or Compensating) and Systematic (or Cumulative) Error.—At the Conference of the International Geodetic Association it was resolved that in order to be classified as Levelling of High Precision, then for every line, or set of lines, whether or not they form circuits, run twice, in opposite directions, on different dates as far as possible, the limits for the probable accidental and probable systematic errors, computed by the foregoing formulae, should not exceed

$$\pm 1 \text{ mm. per kilometre}^{\frac{1}{2}} \text{ for } \eta_r.$$
$$\pm 0{\cdot}2 \text{ mm. per kilometre for } \sigma_r \text{ or } \sigma_R.$$

The values of the errors in the various portions of the English Network, and much interesting data, are given in the Report [1] from which the following summary is abstracted :

Below are given the values derived from the International Formulae for the English Network, including σ_r although the lines form a network. E, the probable accidental error revealed by the corrections necessary to the lines in the adjustment, has also been added.

Probable accidental error $\eta_r = \pm 0{\cdot}00182$ ft. per mile$^{\frac{1}{2}}$ $= \pm 0{\cdot}44$ mm. per km.$^{\frac{1}{2}}$
Probable systematic error $\sigma_r = \pm 0{\cdot}00119$ ft. per mile $= \pm 0{\cdot}23$ mm. per km.
,, ,, ,, $\sigma_R = \pm 0{\cdot}00063$ ft. per mile $= \pm 0{\cdot}12$ mm. per km.
$E = \pm 0{\cdot}0077$ ft. per mile.$^{\frac{1}{2}}$

The above values of η_r and σ_r are based upon values of S, the systematic discrepancy in a line of levelling obtained by dividing the line into fragments wherever the nature of the graph seemed to demand it. (See p. 46 in the Report ; also Lallemand, *Nivellement, etc.*, 2nd ed., 1912, p. 717 : " Calcul par tronçons de sections ".)

Taking S for each line as a whole we obtain

$\eta_r = \pm 0{\cdot}00190$ ft. per mile$^{\frac{1}{2}}$ $= \pm 0{\cdot}46$ mm. per km.$^{\frac{1}{2}}$
$\sigma_r = \pm 0{\cdot}00096$ ft. per mile $= \pm 0{\cdot}18$ mm. per km.
$\sigma_R = \pm 0{\cdot}00065$ ft. per mile $= \pm 0{\cdot}12$ mm. per km.

In the final published results account has been taken of the fact that, owing to the spheroidal shape of the earth and the increase in the intensity of gravity from the equator to the poles, a level surface, as defined by the fact that a fluid does not flow from one point to another, is not the same height in feet above mean sea level at different latitudes. On the Ordnance maps the heights are to be given according to the *orthometric* system of measurement, *i.e.* as linear heights above a standard fixed level surface. The values corrected for gravity and referred to a standard parallel of latitude (53°) are termed *dynamic* heights, and these are also given in the Report, where the significance of the corrections is more fully discussed.

The datum to which the new system of levels is referred is the Mean Sea Level at Newlyn, as derived from the mean of the hourly readings taken over a period of six years from 1912 to 1921. The

[1] *Loc. cit.*

new levels do not differ greatly from the old ones near Liverpool; but there exists a maximum difference of about 1·75 feet in the eastern counties, the new level values being there less than the old.

The network is connected to the observatory bench marks by short lines of levels, which are relevelled annually so as to see if there is any appearance of movement of the observatory bench marks.

No such movement has been noticed up to the present.

Accuracy. Spirit-Levelling.—For ordinary work with a Sopwith staff of the usual pattern, graduated to 0·01 ft., it is customary to read to the second decimal place only, and to consider the reading as even if the hair-line falls upon a black, and as odd if the hair-line falls upon a white, division. The limiting error due to reading the staff would therefore be $\pm 0{\cdot}01$ ft., or a p.e. of $\pm 0{\cdot}005$ ft. Assuming the length of sight as 300 ft., there are then $\frac{5280}{600} = 9$ changes of position in 1 mile or 18 readings, so that the p.e. per mile due to incorrect reading only $= \pm 0{\cdot}005\sqrt{18} = \pm 0{\cdot}021$ ft.

The p.e. that might be expected in M miles would therefore be $\pm 0{\cdot}021\sqrt{M}$, but it is impossible to eliminate all other sources of error, such as that due to the displacement of the bubble from the centre of its run, errors of adjustment, etc.

For ordinary levelling a general allowance of $\pm 0{\cdot}10\sqrt{M}$ feet is made, which includes cumulative as well as compensating errors. This figure is the limiting error, and exceeds the p.e. by an amount which is very indefinite, since cumulative errors are more nearly proportional to M than to $\sqrt{M}$.

For more accurate work, when special care is taken to eliminate errors due to unequal back and fore sights, etc., the third decimal place of a foot is sometimes estimated—at any rate to 0·005 ft. A greater degree of accuracy is to be expected here; for instance, on the Ohio River Survey [1] the allowable error was $\pm 0{\cdot}05\sqrt{M}$ ft., while that obtained with the Y levels was $\pm 0{\cdot}03\sqrt{M}$ ft.

Similarly on the Topographical Survey of Cincinnati [2] the averag error in levelling with a Y level was $\pm 0{\cdot}031$ ft. per mile.

For precise levelling, where very great care is taken, and the lines double-rodded, etc., the p.e. is reduced very considerably. A few examples of the accuracy obtained or specified on certain surveys are given below.

The Second Geodetic Levelling of England and Wales [3] has already been described, and the calculated compensating (or accidental) errors and cumulative (or systematic) errors are given on p. 218.

On the Transvaal and Orange River Colony Survey,[4] levels were taken in one direction only, and chiefly along the railway (see p. 213), the p.e. of the results being $\pm 1{\cdot}62$ mm.$\sqrt{M} = \pm 0{\cdot}0053\sqrt{M}$ ft.

[1] *Engineering News*, vol. lxxi. No. 12. "Ohio River Survey." G. G. Graeter.

[2] *Engineering News*, vol. lxix. No. 14. "The Topographical Survey of Cincinnati." H. C. Mitchell.

[3] *The Second Geodetic Levelling of England and Wales, 1912–1921*, by Colonel Sir Charles Close.

[4] *Report on Geodetic Survey of South Africa*, vol. iii.

On the survey of India,[1] *where a check leveller followed closely behind the main leveller, the p.e. was deduced as*

$$\text{p.e.} = \pm\sqrt{(0{\cdot}004)^2M + (0{\cdot}00034)^2M^2}.$$

Some permissible errors that have been adopted in American practice[2] are :

	Millimetres	Feet
U.S. Coast and Geodetic Survey . .	$\pm\ 5\sqrt{2K}$	$0{\cdot}029\sqrt{M}$
U.S. Lake Survey	$\pm 10\sqrt{K}$	$0{\cdot}041\sqrt{M}$
Mississippi River Survey . . .	$\pm\ 5\sqrt{K}$	$0{\cdot}021\sqrt{M}$
U.S. New Coast Survey . . .	$\pm\ 4\sqrt{K}$	$0{\cdot}016\sqrt{M}$
U.S. Geological Survey . . .	$\pm\ 4\sqrt{K}$	$0{\cdot}017\sqrt{M}$
U.S. Catskill Aqueduct . . .	$\pm\ 5\sqrt{K}$	$0{\cdot}02\sqrt{M}$

In the precise levelling operations in New York[3] the following are a few of the actual results :

(1) In the largest level net, which had a perimeter of 118 km. or 74 miles, the closing error amounted to 3·8 mm. or 0·012 ft.

(2) Another circuit of 64 km. or 40 miles failed to close by 6 mm. or 0·02 ft.

(3) Another circuit of 83 km. or 52 miles failed to close by 10 mm. or 0·033 ft.

(4) In the most direct line of 94 km. or 58·7 miles, the probable error was ± 0·30 mm.

The International Geodetic Association decided upon the following standard :

A p.e. of ± 1 mm. per kilometre indicates a very high degree of precision.

A p.e. of ± 2 mm. per kilometre indicates a fair degree of precision.

A p.e. of ± 3 mm. per kilometre indicates a tolerable degree of precision.

A p.e. of ± 5 mm. per kilometre indicates an unsatisfactory degree of precision.

The above remarks refer only to cases in which the ordinary methods of levelling can be applied, equal back sights and fore sights obtained, and the length of sight kept within such limits that the graduations are clearly visible.

The accuracy which can be obtained in continuing a line of levels across a wide, deep river, in the absence of bridges, has been examined in several instances on the Survey of India.

Three methods are in general available :

(1) The tide-pole or water-gauge method.
(2) Levelling with an ordinary level.
(3) Vertical angles with a theodolite.

(1) For the tide-pole method a position is chosen on a straight length

[1] *Report on G.T. Survey of India.* [2] Breed and Hosmer, *Surveying*, vol. ii.
[3] *Engineering News*, vol. lxx. No. 10, p. 452.

of the river where the flow is as even as possible, and the water is not likely to be banked up at one side as would be the case at a sharp bend. The surface of the water is then assumed to be horizontal, and two vertical graduated staves or poles are fastened to stakes driven into the river bed, one near each bank. A number of observations are then taken at different times, and from a mean of the various readings on each staff the relative heights of the zeros of the two scales can be calculated.

The levelling operations on the one bank then terminate on the one pole, while those on the opposite bank are continued from the second pole.

(2) For direct levelling, the method of reciprocal observations explained on p. 201 must be adopted, and if the length of sight is too great to enable the staff graduations to be clearly distinguished, a target rod must be used. That is, the staff is provided with a movable vane which is adjusted on the staff until the instrument man signals that its top edge is bisected by the cross-hairs of his instrument, when the reading on the staff is booked by a competent observer. The mean of a number of readings is adopted. For precise work the observations are made on different days, and under different conditions of lighting, etc.

On the River Manora,[1] where the greatest length of sight was 36 chains, 5 sets of observations were taken on each of 2 days, and the mean of the 10 sets adopted.

(3) In the method of vertical angles a similar vane is sighted, but in this case the vane is stationary and the vertical angle is measured with a theodolite, as explained in the following chapter. The distance across the river is found by triangulation.

The following results were obtained on the River Ganges[1] at Damukdia, where the width from bank to bank was 1·28 miles :

	No. of Observations	C above B	p.e.
(1) By vertical angles with 24″ theodolite	72	2·139 ft.	±0·005
(2) By levelling with 2 standard levels of 21″ focal length .	114	2·132 ft.	±0·016
(3) By water-gauges . .	75	2·212 ft.	±0·001

As a result of these experiments it was recommended that levels should be transferred across large rivers by means of a 12″ theodolite. Usually, however, levelling parties are not provided with theodolites, and reciprocal observations with a level give very good results, as will be seen from the above experiments.

The tide-pole results are very consistent, but the mean varies from that of the other observations, and this method is evidently liable to an appreciable constant error, particularly when a favourable straight reach of river is unobtainable.

The p.e. of a single observation from the arithmetic mean of the set may easily be calculated for each case (see p. 531). It should be noted, however, that the p.e. only refers to "compensating" errors and not to cumulative errors, such as those in the tide-pole results.

[1] *Great Trigonometrical Survey of India*, vol. xix. Appendix V.

On the Indus [1] the differences in height of the two stations were :

By tide-pole, 1·197 ft.
By vertical angles, 0·919 ft.
By levelling, 0·926 ft.

EXAMPLES

1. (I.C.E.) A Dumpy level is set up and levelled with its eye-piece vertically over a peg A.

The height from the top of A to the centre of the eye-piece is measured and found to be 4·62 feet. A level staff is then held on a distant peg B and read. This reading is 2·12 feet. The level is then set over B. The height of the eye-piece above B is 4·47 feet and a reading on A is 6·59 feet :

(1) What is the difference in level between A and B ?
(2) Is the collimation of the telescope in adjustment ?
(3) If out of adjustment can the collimation be corrected without moving the level from its position at B.

2. (U. of L.) A Dumpy level is set up at a point A and readings are taken on the staff when held at points B and C 200 feet to the west and east respectively of the instrument, the bubble having to be readjusted by the footscrews to the zero of its run when turned from B round to C. The readings are 4·2 feet at B and 5·8 feet at C. The instrument is then taken to a point D, 40 feet west from B, and, the bubble being again brought to the centre of its run with the telescope directed in the line of B and C, the readings were 5·6 feet on the staff held at B, and 6·4 feet on the staff held at C.

From these data calculate the inclination of the line of sight to the horizontal, when the instrument was set up at D, and also find what the readings ought to have been with the instrument at D, supposing all the adjustments to have been correct. As it was, which of the permanent adjustments were certainly not in order, and which of them may have been right or wrong.

3. (U. of L.) The following readings were taken with a level on uniformly sloping ground : 1·75, 4·76, 6·94, 11·33, 13·16, 3·32, 6·51, 8·76, 12·92, 4·14, 8·32, 10·75, 12·98.

The reduced level of the first point was 65·81, and the staff was held at points 50 feet apart.

Rule a page of the level-book and enter the above readings, and apply all the usual checks.

Plot the cross-section roughly to scale.

4. (U. of B.) Given the following data :

Distance	Station	B.S.	I.S.	F.S.	H.I.	R.L.
Feet	B.M. 1	9·71	..	..	..	103·62
0	1	..	3·15	..	..	..
100	2	..	1·06	..	..	..
200	3	7·43	..	0·23	..	..
300	4	..	4·17	..	..	..
400	5	11·72	..	3·56	..	..
500	6	..	..	6·39	..	..

Obtain the reduced levels of 1, 2, 3, 4, 5, and 6.

If an even gradient of 1 in 20 starts at 1, at a level of 100 above datum, calculate the heights of bank or depths of cut at the points 1, 2, 3, 4, 5, and 6.

[1] *Great Trigonometrical Survey of India*, vol. xix. Appendix V.

5. (U. of L.) The figure represents the spot levels along a centre line AB, and four cross-sections. Scale, 2 chains to 1 inch.

Draw up a level-book giving possible readings for these levels, showing the order in which you would take them, and the manner of booking, the whole

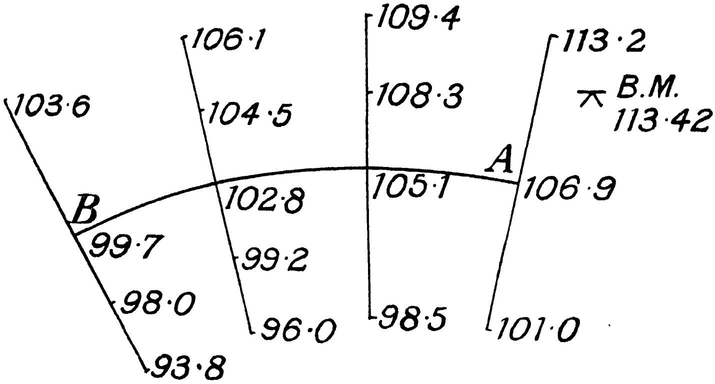


FIG. 6.32

fully made up. Also mark on the plan the approximate positions of your level. (*N.B.—The readings are to be taken with a level and 14-foot staff.*)

6. (*a*) On the 36″ theodolite of the India Survey one division on the bubble tube, measuring $\frac{1}{20}$th of an inch, corresponded approximately to 1 second of arc.

(*b*) On a good 14″ sensitive Y level a similar division corresponds to 5″ of arc.

(*c*) On an ordinary 12″ Dumpy level the value of a $\frac{1}{20}$″ division = 1′ of arc.

(*d*) On the precise level used by the U.S. Geological Survey the value of $\frac{1}{10}$″ division = 4″ of arc.

What would be the radius of curvature, and what would be the difference in the reading on an ordinary levelling staff 300 feet distant, if the bubble were $\frac{1}{2}$″ out of centre in each case ?

7. A sewer 258 feet long is to be laid from A to B at an inclination of 1 in 200, and sight rails are to be erected at A and B.

A travelling rod of 12′ 6″ length is available, and the following data are obtained on the site :

Depth of outfall below a peg at A = 8·20 ft.
Reading of Dumpy level on peg at A = 3·84 ft.
Reading of Dumpy level on peg at B = 2·77 ft.

Find (*a*) at what height above the pegs at A and B the rails are to be fixed, and (*b*) the depth of the sewer at B.

8. (I.C.E.) Let A and B be two stations on the earth's surface, and let B be 1000 feet higher than A. If the horizontal distance of A from B at the level of A is 5 miles, what will be the horizontal distance from A to B at the level of B, assuming the earth's radius at A to be 4000 miles ?

9. The following are abstracts from a contour level-book. Work out results and interpolate 5-feet contours.

3°	2°	65·24	− 1½°	− 1°
70	50	15·00	30	120
5°	3°	57·36	− 2°	+ 3°
85	60	14·00	70	100

See also Questions 14 and 15, p. 242.

CHAPTER VII

TRIGONOMETRICAL AND BAROMETRICAL LEVELLING

Trigonometrical Levelling

To determine the relative altitudes of particular points by means of a theodolite the following methods are available :

Method 1.—Let B (Fig. 7.1) be a position the altitude of which above or below the instrument station A it is required to find.

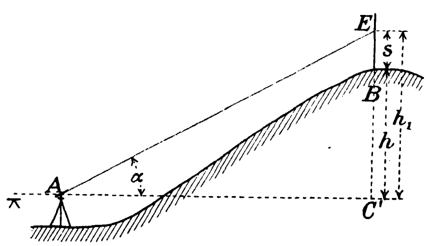


Fig. 7.1

A levelling staff is held at B and a reading BE $=s$ taken upon it ; or, if preferred, a vane fixed at a height s above the foot of the staff may be sighted. At the same time the vertical angle a is observed, and corrected, if necessary, for index error.

If the horizontal projection of the distance from A to B, *i.e.* AC $=$ D, is known from triangulation or other sources, the height CE $=h_1$ may be calculated as

$$h_1 = \text{D} \tan a. \qquad . \quad . \quad . \quad (1)$$

The altitude of B is $(h_1 - s)$, *i.e.* (D $\tan a - s$) above the instrument axis at A.

The height of the axis above the station-point A may be found by direct measurement, or the reduced level of the axis may be deduced from an observation on to a bench mark of known altitude.

Example.—The observed staff readings were 4·00 ft. at B and 1·57 ft. on a B.M. having a reduced level of 81·00. For the reading on the B.M. the telescope was horizontal. The distance D $=350$ ft. and $a = 8°\text{-}35'\text{-}20''$.

$$\text{D} \tan a = 350 \times \cdot 15104 \text{ ft.} = 52\cdot 864 \text{ ft.}$$
$$h = 52\cdot 864 - 4 \text{ ft.} = 48\cdot 86 \text{ ft.}$$

The reduced level of the instrument axis is $81\cdot 00 + 1\cdot 57 = 82\cdot 57$ and that of the point B is therefore $82\cdot 57 + 48\cdot 86 = 131\cdot 43$.

Had the angle a been an angle of depression

$$h = 52\cdot 864 + 4 \text{ ft.} = 56\cdot 86 \text{ ft.}$$
and the reduced level of B $= 82\cdot 57 - 56\cdot 86$ ft.
$= 25\cdot 71$ ft.

Method 2.—A correction for curvature and refraction may be applied to the results of Method 1, when the distance D is moderately large. The nature of the corrections may be seen from Fig. 7.2, where the level line through A is represented by AF and the horizontal line by AC.

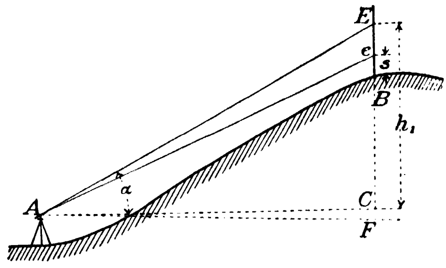


FIG. 7.2

The line of sight is refracted downwards and cuts the staff at *e*, so that B*e* = *s*, say, but the angle observed is EAC = α, say. EBCF [1] is, strictly speaking, normal to the curve AF, but the angle ECA may be considered very nearly a right angle and CE = D tan α, where AF = AC nearly = D unless A and B are at a considerable distance apart and have a large difference in altitude.

The altitude of B above A is the distance FB

$$= CE + CF - eE - eB$$

$$= D \tan \alpha + \text{ correction for curvature } - \text{ correction for refraction } - s$$

$$= D \tan \alpha - s + \cdot 57\left(\frac{D}{5280}\right)^2, \quad . \quad . \quad . \quad (2)$$

as D is here expressed in feet, while in formula (2) (p. 200) D is expressed in miles.

Example.—The vane sighted on a staff is 10 ft. above its foot, the distance, D, to the staff station is 10,130 ft., and the angle of elevation $\alpha = 8°\text{-}35'\text{-}20''$.

$$D \tan \alpha = 10130 \times \cdot 15104 = 1530 \cdot 035$$

$$s = 10$$

$$D \tan \alpha - s = 1520 \cdot 035$$

$$\cdot 57\left(\frac{D}{5280}\right)^2 = 2 \cdot 098$$

and the altitude of B above the instrument axis at A = 1522·13 ft.

Method 3.—The height of a hill, building, or other object B, the distance to which is unknown, may be determined approximately by setting out two instrument stations A and F in line with B on a fairly level stretch of ground, and measuring the intervening distance AF = *d*.

The theodolite is set up and levelled at A and an angle of elevation α observed to a fixed vane or a suitable graduation E upon a levelling staff held vertically at B.

[1] See Method 5.

The instrument is then set up at F and an angle of elevation β observed to the same point E upon the staff at B.

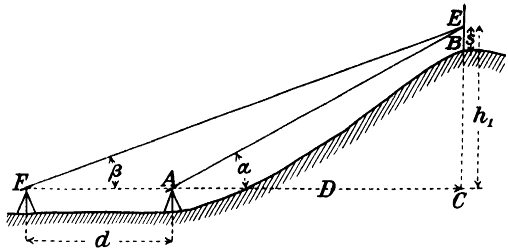


FIG. 7.3

Thus, if the altitude of the instrument axis is approximately the same for each setting, referring to Fig. 7.3,

$$\frac{\mathrm{EC}}{\mathrm{D}} = \tan \alpha, \text{ where AC} = \mathrm{D}, \text{ say ;}$$

also
$$\frac{\mathrm{EC}}{\mathrm{D}+d} = \tan \beta,$$

$$\therefore \mathrm{D} \tan \alpha = (\mathrm{D}+d) \tan \beta,$$

or
$$\mathrm{D} = \frac{d \tan \beta}{\tan \alpha - \tan \beta}, \quad . \quad . \quad . \quad (3)$$

and
$$\mathrm{EC} = h_1 = \mathrm{D} \tan \alpha = \frac{d \tan \alpha \tan \beta}{\tan \alpha - \tan \beta}, \quad . \quad . \quad (4)$$

and the height of B above the instrument axes at A and F

$$= h_1 - s,$$

where s is the height of the vane E above the ground at B.

If F and A are not on the same level, but F is at an altitude h_2 higher/lower than A, then a correction may be applied by increasing/decreasing the distance d by an amount $h_2 \cot \beta$, because this is the distance from F at which the line EF would intersect the horizontal line CA.

Example.—A vane E 4·00 ft. above the ground at B is sighted from two instrument stations A and F at a distance 200 ft. apart.

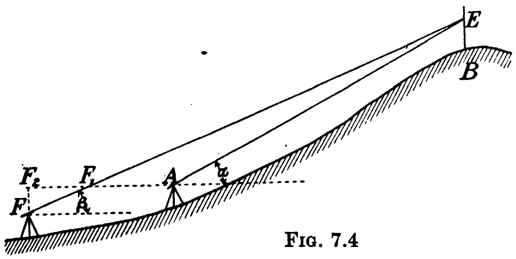


FIG. 7.4

The angles of elevation are 45°-30′ from A and 30°-20′ from F. The height of the instrument axis at A above the ground = 5·10 ft. and at F = 4·76 ft. The staff is held upon the peg at F and a reading 8·48 ft. obtained from the instrument at A, the bubble being in the centre of its run.

Find the horizontal distance from A to B, and the reduced level of B—that of F being 80·00 ft. above datum.

The reduced level of the instrument axis at F = 80·00 + 4·76 = 84·76.
The reduced level of the instrument axis at A = 80·00 + 8·48 = 88·48.
The difference in level FF_2 (Fig. 7.4) = 3·72 ft. and the distance F_2F_1 = 3·72 cot 30°-20′ = 3·72 × 1·709 = 6·36 ft., about. The corrected value of *d* is therefore 200 − 6·36 = 193·64 ft.

From equation (3)

$$D = \frac{193{\cdot}64 \times \tan 30°\text{-}20'}{\tan 45°\text{-}30' - \tan 30°\text{-}20'} = \frac{193{\cdot}64 \times {\cdot}5851}{1{\cdot}0176 - {\cdot}5851}$$

$$= \frac{193{\cdot}64 \times {\cdot}5851}{{\cdot}4325},$$

or, using logarithms, $\qquad D = 261{\cdot}98$ **ft.**

and $\qquad h_1 = D \tan 45°\text{-}30' = 261{\cdot}98 \times 1{\cdot}0176 = 266{\cdot}56$ **ft.**

The reduced level of B is therefore 88·48 + 266·56 − 4·00 = 351·04.

Method 4.—When F and A are at very different levels, the following data are required (Fig. 7.5):

(i) The height of the instrument *a* above the ground at A.
(ii) The horizontal distance *d* between the two instrument stations.

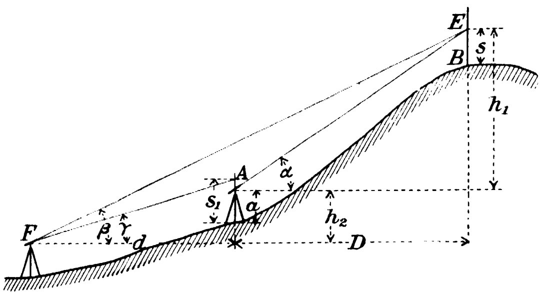


FIG. 7.5

(iii) The angle of elevation α from A to a vane or graduation E, on the staff at B, at a distance *s* above its foot.
(iv) The angle of elevation β from F to the same point.
(v) The angle of elevation γ from F to a point at s_1 ft. above the foot of a staff held on the peg at A.

The height of the peg A above the instrument axis at F is by Method 1,

$$d \tan \gamma - s_1,$$

and the vertical distance between the two instrument axes at F and A is consequently

$$h_2 = d \tan \gamma - s_1 + a.$$

Let the horizontal distance from A to B = D, and let the height of the vane E above the instrument axis at A be h_1 ft. Then from the figure

$$\frac{h_1 + h_2}{D + d} = \tan \beta \text{ and } \frac{h_1}{D} = \tan \alpha,$$

$$\therefore h_1 = (D + d) \tan \beta - h_2 = D \tan \alpha, \quad . \quad . \quad (5)$$

$$\therefore D=\frac{d \tan \beta - h_2}{\tan \alpha - \tan \beta}, \qquad (6)$$

and from (5) $$h_1=\left\{\frac{d \tan \beta - h_2}{\tan \alpha - \tan \beta}\right\}\tan \alpha, \qquad (7)$$

from which the height of B above the peg A can be deduced as h_1+a-s.

Example.—Find the horizontal distance from F to B and the altitude of B from the following data (Fig. 7.5):

$d=200$ ft., $\alpha=45°\text{-}30'$, $\beta=30°\text{-}20'$, $\gamma=10°\text{-}30'$,
$s=4{\cdot}00$, $s_1=4{\cdot}00$, Height of instrument at A $=4{\cdot}76$.

Height of instrument axis at F above datum $=88{\cdot}50$.

The height of the instrument axis at A above that at F is

$$h_2=200 \tan 10°\text{-}30' - 4{\cdot}00+4{\cdot}76$$
$$=200\times{\cdot}18534+0{\cdot}76$$
$$=37{\cdot}83 \text{ ft.}$$

The horizontal distance D from A to B is given by equation (6), *i.e.*

$$D=\frac{200 \,.\, \tan 30°\text{-}20' - 37{\cdot}83}{\tan 45°\text{-}30' - \tan 30°\text{-}20'}$$
$$=\frac{200\times{\cdot}58513-37{\cdot}83}{1{\cdot}01761-{\cdot}58513}$$
$$=\frac{79{\cdot}20}{{\cdot}4325}=183{\cdot}12 \text{ ft.}$$

Also $$h_1=D \tan \alpha=183{\cdot}12\times 1{\cdot}01761$$
$$=186{\cdot}34.$$

The horizontal distance from F to B $=383$ ft. nearly, and the reduced level of B

$$=88{\cdot}50+37{\cdot}83+186{\cdot}34-4{\cdot}00$$
$$=308{\cdot}67 \text{ ft. approximately.}$$

Method 5.—Reciprocal observations to cancel the effects of curvature and refraction for very long sights. Let A and B (Fig. 7.6) be the two points, the relative altitudes of which it is required to determine, and at each of which a theodolite is set up and levelled. And let AA_1 and BB_1 be the two level lines, and AH and BH_1 the horizontal lines through the axes of the instruments at A and B respectively.

A signal at B is sighted from A, and a signal at A simultaneously sighted from B.

Let Ab represent the line of sight from A, curved due to refraction and cutting the signal in b, and let Ab_1, the tangent to this, give the apparent direction at A of the point b.

Similarly, let Ba represent the line of collimation from B, and Ba_1 the apparent direction.

The angle of elevation (or depression) which is measured on the vertical circle of the instrument at A is $HAb_1(=\alpha_1)$, which is equal to $\angle HAb+r$, where $r=\angle bAb_1$,—the correction to be applied due to refraction, or

$$\angle HAb_1=\angle HAB+r+\angle BAb.$$

But if the height Bb is measured, and the distance AA_1 is known

from the triangulation survey, the angle BAb may be accurately computed or may generally be assumed equal to

$$\frac{Bb}{AA_1} \text{ radians} = \frac{Bb}{AA_1} \times \frac{180}{\pi} \text{ degrees} = \phi_1, \text{ say,}$$

$$\therefore \angle HAB = \alpha, \text{ say} = \angle HAb_1 - r - \angle BAb,$$

i.e. $$\alpha = \alpha_1 - r - \phi_1. \qquad (8)$$

Similarly, $$\angle H_1BA = \beta, \text{ say} = \angle H_1Ba_1 + r + \angle ABa$$
$$= \beta_1 + r + \phi_2, \text{ say}. \qquad (9)$$

Let the angle subtended at the centre of the earth, *i.e.* $\angle$AOB, $= \theta$. Then if d—the geodetic distance from A to B, *i.e.* at mean sea-level—

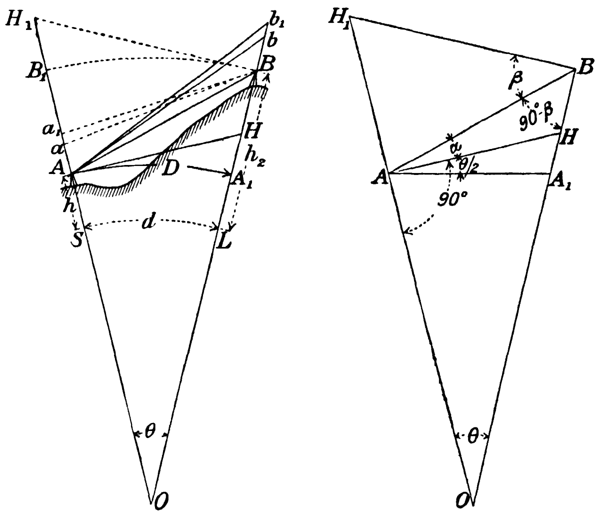


FIG. 7.6.—Reciprocal Observations

and R the radius of the earth at that position, are known, θ can be calculated.

The angle between the tangent AH and the chord AA_1 is equal to half the angle subtended by AA_1 at the centre of the circle, *i.e.* $\angle HAA_1 = \frac{\theta}{2}$, and therefore $\angle BAA_1 = \alpha + \frac{\theta}{2}$.

In the triangle BAA_1, because the angle $ABA_1 = 90 - \beta$.

$$\frac{BA_1}{AA_1} = \frac{\sin\left(\alpha + \frac{\theta}{2}\right)}{\sin\,(90 - \beta)},$$

or $$BA_1 = AA_1 \cdot \frac{\sin\left(\alpha + \frac{\theta}{2}\right)}{\cos\beta}. \qquad (10)$$

But as the three angles of the triangle BAO are together equal to 180°,

$$\therefore \angle BAO + \angle ABO + \angle AOB = 180°,$$

i.e.
$$(90° + a) + (90° - \beta) + \theta = 180°,$$

and
$$a - \beta + \theta = 0,$$

from which
$$\frac{\theta}{2} = \frac{\beta - a}{2}$$

and
$$\frac{\beta}{2} = \frac{\theta + a}{2}, \quad . \quad . \quad . \quad (11)$$

or
$$\beta = \frac{\beta}{2} + \frac{\theta + a}{2},$$

therefore from (10)
$$BA_1 = \frac{AA_1 \,.\, \sin \frac{a + \beta}{2}}{\cos \frac{a + \beta + \theta}{2}}. \quad . \quad . \quad . \quad (12)$$

Strictly, the chord $AA_1 = 2\,(R + h_1)\sin\frac{\theta}{2}$, where h_1 is the altitude of A above sea-level, but may generally be taken equal to the arc $AA_1 = D$, say, and similarly the angle $\frac{\theta}{2}$ in the denominator, being small, may often be neglected, when BA_1, the difference in altitude between A and B, *i.e.*

$$(h_2 - h_1,) = D \tan \frac{a + \beta}{2}. \quad . \quad . \quad (13)$$

Hence by substitution in (12) from (8) and (9)

$$(h_2 - h_1) = D \frac{\sin \frac{1}{2}\{(HAb_1 - r - BAb) + (H_1Ba_1 + r + ABa)\}}{\cos \frac{1}{2}\{(HAb_1 - r - BAb) + (H_1Ba_1 + r + ABa) + \theta\}},$$

from which, on simplifying, the values of r vanish, *i.e.*

$$(h_2 - h_1) = D \frac{\sin \frac{1}{2}\{(a_1 - \phi_1) + (\beta_1 + \phi_2)\}}{\cos \frac{1}{2}\{(a_1 - \phi_1) + (\beta_1 + \phi_2) + \theta\}}. \quad . \quad (14)$$

D may be calculated from the geodetic distance d as explained later. When the difference between the altitudes of A and B is small, both the angles a_1 and β_1 may be angles of depression, when, in a similar manner to the above, it may be shown that

$$BA_1,\ i.e.\ h_2 - h_1, = D \frac{\sin \frac{1}{2}\{(\beta_1 + \phi_2) - (a_1 + \phi_1)\}}{\cos \frac{1}{2}\{(\beta_1 + \phi_2) - (a_1 + \phi_1) + \theta\}}. \quad . \quad (15)$$

Example.—The horizontal distance from A to B deduced

from the geodetic distance d	=	14760 ft.
The reduced level of A	=	780·10 ft.
The height of the instrument at A . . .	=	4·92 „
The height of the instrument at B . . .	=	5·10 „

The height of the signal at A . . .	= 10·00 ft.
The height of the signal at B . . .	= 8·50 ,,
The angle of elevation from A to B . .	$=2°\text{-}39'\text{-}49''$
The angle of depression from B to A . .	$=2°\text{-}39'\text{-}56''$

Determine the reduced level of B.

The height of the signal above the instrument axis at B = 8·50 − 5·10 = 3·40 ft.
The height of the signal above the instrument axis at A = 10·00 − 4·92 = 5·08 ft.

The value of θ_1 is therefore $\frac{3\cdot40}{14760}\times\frac{180}{\pi}\times3600$ seconds $=47\cdot5$ seconds, and the value of θ_2 is $\frac{5\cdot08}{14760}\times\frac{180}{\pi}\times3600$ seconds $=71\cdot0$ seconds (by slide rule).

The value of θ subtended at the centre of the earth is

$$\frac{14760}{20,890,000}\times\frac{180}{\pi}\times3600=145''\cdot7.$$

From equation (14)

$$h_3-h_1=14760\frac{\sin\frac{1}{2}(2°\text{-}39'\text{-}49''-47''\cdot5+2°\text{-}39'\text{-}56''+1'\text{-}11'')}{\cos\frac{1}{2}(2°\text{-}39'\text{-}49''-47''\cdot5+2°\text{-}39'\text{-}56''+1'\text{-}11''+2'\text{-}25''\cdot7)}$$

$$=14760\frac{\sin 2°\text{-}40'\text{-}4''\cdot25}{\cos 2°\text{-}41'\text{-}17''\cdot1}$$

$$\log 14760 = 4\cdot16909$$
$$\log \sin 2°\text{-}40'\text{-}4''\cdot25 = \bar{2}\cdot66788$$
$$\log \sec 2°\text{-}41'\text{-}17''\cdot1 = \cdot00048$$

$$\log (h_2-h_1) = 2\cdot83745$$
$$h_2-h_1 = 687\cdot78 \text{ ft.}$$

The reduced level of B is therefore

$$780\cdot10+687\cdot78+4\cdot92-5\cdot10=1467\cdot7 \text{ ft. nearly.}$$

Terrestrial Refraction.—The value of the angle of refraction r (bAb_1 or aBa_1, Fig. 7.6) is by no means constant: its magnitude varies in different localities, and depends, further, upon the time of day and the climatic conditions which prevail while the observations are being taken.

The greatest value in a given district occurs in the early morning (5 to 6 A.M.); it diminishes—at first quickly and then more slowly—until 9 or 10 A.M., after which it remains fairly constant until about 4 o'clock in the afternoon, when it commences to increase.

The **coefficient of refraction** (K) is the ratio of the angle of refraction (r) to the angle θ subtended at the centre of the earth by the distance over which the observations are taken, *i.e.*

$$K=\frac{r}{\theta}. \qquad (16)$$

The value of this ratio varies roughly between 0·06 and 0·08, and, in the absence of more accurate data, a mean value of 0·07 is often adopted.

To determine the correction (c) in feet which is to be applied to

the apparent difference in altitude, r and θ may be expressed in radian measure, when $r = K\theta$, *i.e.*

$$\frac{c}{5280D} = \cdot 07 \cdot \frac{D}{R},$$

where D is the distance apart of the stations in miles, and R the radius of the earth in miles, from which

$$c = \cdot 09D^2 \text{ ft.}$$

This is about $\frac{1}{7}$th the correction for curvature as previously stated in Chapter VI.

On the Ordnance Survey [1] for rays not crossing the sea, the mean value of K was found to be 0·0750, while for rays crossing the sea the result was 0·0809.

The values adopted on the Massachusetts Survey [1] were somewhat less, viz. 0·0697 in the interior and 0·0784 on the coast.

If the altitudes of two stations A and B have been accurately determined by means of spirit levelling, then the apparent difference in altitude which is found from the reciprocal observations of Method 5 furnishes data from which the values of K from each station may be calculated.

If the true altitudes of A and B have not been determined otherwise than by the reciprocal observations, the mean value only of K can be computed; this value may be slightly different from the individual values at A and B respectively.

In the example on p. 230,

$\theta_1 = 47''\cdot 5$, $\theta_2 = 1'\text{-}11''$, $\theta = 2'\text{-}25''\cdot 7$, $a_1 = 2°\text{-}39'\text{-}49''$, $\beta_1 = 2°\text{-}39'\text{-}56''$,

and from equation (8) $\qquad a = a_1 - r - \theta_1,$

and (9) $\qquad \beta = \beta_1 + r + \theta_2,$

therefore by subtraction $(\beta - a) = (\beta_1 - a_1) + 2r + (\theta_1 + \theta_2)$,

i.e. from equation (11) $2'\text{-}25''\cdot 7 = 7'' + 2r + 1'\text{-}58''\cdot 5$,

$$2r = 20''\cdot 2,$$

$$r = 10''\cdot 1,$$

and

$$K = \frac{10''\cdot 1}{145\cdot 7} = \cdot 0694.$$

Accuracy of Trigonometrical Levelling.—For ordinary work trigonometrical levelling is not capable of such a high degree of accuracy as spirit levelling, but the approximate altitudes of isolated points may be determined much more quickly and economically by this method. The results obtained in continuing a line of levels across the Ganges by trigonometrical observations were mentioned on p. 221. The probable error in any particular case obviously depends upon the special conditions and upon the instruments available.

Thus in Method 1, equation (1), if $\pm \delta a$ is the p.e. in a, the p.e. in h_1 due to this cause is obtained by differentiation, *i.e.*

$$\delta h_1 = D \sec^2 a \,.\, \delta a$$

[1] Clarke's *Geodesy*.

or
$$\frac{\delta h_1}{h_1} = \frac{\delta a}{\sin a \cos a} = \frac{2 \,.\, \delta a}{\sin 2a}.$$

In the example given on p. 224, where D = 350 ft. and $a = 8°\text{-}35'\text{-}20''$, if the p.e. in a is $\pm 10''$, then since $1'' = \cdot 00000485$ radians the p.e. in h_1 will be

$$\pm 350 \times (1{\cdot}0113)^2 \; (10 \times \cdot 00000485), \; = \pm \cdot 017 \text{ ft.}$$

If there is also a fractional p.e. in D of $\pm \frac{1}{1000}$, the total error in h_1 would be

$$\pm \sqrt{(\cdot 017)^2 + (\cdot 053)^2} = \pm \cdot 055 \text{ ft.}$$

Similarly, in Method 3, writing h for h_1, equation (4) becomes

$$\frac{1}{h} = \frac{1}{d} (\cot \beta - \cot a)$$

or
$$h(\cot \beta - \cot a) = d. \qquad . \qquad . \qquad . \quad (17)$$

The probable fractional error in h due to a probable error of $\pm \delta(d)$ in d is therefore

$$\frac{\delta h_1}{h} = \pm \frac{\delta d}{d}.$$

The probable fractional error due to a p.e. of $\pm \delta a$ in a is

$$\frac{\delta h_2}{h} = \pm \frac{\text{cosec}^2 \, a \,.\, \delta a}{\cot a - \cot \beta},$$

and that due to a p.e. of $\pm \delta\beta$ in β is

$$\frac{\delta h_3}{h} = \pm \frac{\text{cosec}^2 \, \beta \,.\, \delta \beta}{\cot \beta - \cot a},$$

and the probable error from all sources is

$$\pm h \sqrt{\left(\frac{\delta d}{d}\right)^2 + \left(\frac{\delta h_2}{h}\right)^2 + \left(\frac{\delta h_3}{h}\right)^2}.$$

Thus in the example on p. 226, where $d = 193{\cdot}64$, $a = 45°\text{-}30'$, $\beta = 30°\text{-}20'$, and $h = 266{\cdot}56$ ft., if $\delta d = \pm \cdot 2$ ft. and $\delta a = \delta \beta = \pm 30''$ or $\pm \cdot 00015$ radians, then by slide rule

$$\delta h_1 = \pm \frac{\cdot 2}{193{\cdot}6} \times 266{\cdot}56 = \pm \cdot 27 \text{ ft.},$$

$$\delta h_2 = \pm 266{\cdot}56 \times \frac{\text{cosec}^2 \, 45°\text{-}30' \times \cdot 00015}{\cot 45°\text{-}30' - \cot 30°\text{-}20'}$$

$$= \pm 266{\cdot}56 \times \frac{(1{\cdot}4020)^2 \times \cdot 00015}{- \cdot 7263} = \mp \cdot 10 \text{ ft.},$$

$$\delta h_3 = \pm \frac{266\cdot56 \times \text{cosec}^2\ 30°\text{-}20'}{\cdot7263} \times \cdot00015$$

$$= \pm \cdot21 \text{ ft.,}$$

and $$\delta h = \pm\sqrt{(\cdot27)^2 + (\cdot10)^2 + (\cdot21)^2} = \pm \cdot36 \text{ ft.}$$

In Method 5 it will be quite sufficient to consider equation (13), *i.e.*

$$h_2 - h_1 = \text{D} \tan \frac{\alpha + \beta}{2} = \text{D} \tan \phi, \text{ say.}$$

A p.e. of $\pm\,\delta\alpha$ in α and $\pm\,\delta\beta$ in β will produce a p.e. of

$$\pm \tfrac{1}{2}\sqrt{(\delta\alpha)^2 + (\delta\beta)^2} \text{ in } \phi,$$

therefore the p.e. in $(h_2 - h_1)$

$$= \pm (h_2 - h_1)\sqrt{\left(\frac{\delta \text{D}}{\text{D}}\right)^2 + \left(\frac{2\,.\,\delta\phi}{\sin 2\phi}\right)^2} \quad . \quad . \qquad . \quad (18)$$

In the example on p. 230, where $\phi = 2°\text{-}40'\text{-}4''\cdot25$ and $\text{D} = 14760$ ft. if the p.e. in $\text{D} = \pm \frac{\text{D}}{5000}$ and in α or β $\pm 5''$, the p.e. in ϕ will be $\pm \left(\frac{5\sqrt{2}}{2}\right)$secs. $= \cdot0000171$ radians.

The p.e. in $(h_2 - h_1)$

$$\text{due to } \delta\text{D} = \pm \frac{1}{5000} \times 687\cdot78 = \pm \cdot1376 \text{ ft.,}$$

$$\begin{aligned} \text{and due to } \delta\phi &= \pm\,\text{D} \sec^2 (2°\text{-}40'\text{-}4''\cdot25) \times \cdot0000171 \\ &= \pm\, 14760 \times (1\cdot0011)^2 + \cdot0000171 \\ &= \pm \cdot253 \text{ ft.} \end{aligned}$$

The total p.e. is then $\pm\sqrt{(\cdot1376)^2 + (\cdot253)^2} = \pm \cdot29$ ft. nearly.

The same result may be obtained by substitution in equation (18) *i.e.*

$$\pm\, 687\cdot78\sqrt{\left(\frac{1}{5000}\right)^2 + \left(\frac{\cdot0000342}{\cdot0929}\right)^2} = \pm \cdot29 \text{ ft.}$$

·Barometric Levelling

As air is a compressible fluid, it follows that strata at a low level will have a greater density than those at a higher altitude.

Consequently, if the difference in pressure between two stations A and B is ascertained by means of a barometer, their relative altitudes can be approximately deduced.

Thus assuming the temperature to be constant, let d represent the density of mercury, and d_a the density of air at any particular station A, h the height of the mercury column of a barometer, H the height of the homogeneous atmosphere (*i.e.* an imaginary value giving the height

of the atmosphere on the assumption that its density is constant throughout and equal to d_a, instead of decreasing at the higher altitudes), and p the pressure at A in absolute units.

Then $$p = \mathrm{H} \,.\, d_a \,.\, g = h \,.\, d \,.\, g, \qquad (19)$$

where g is the acceleration due to gravity.

$$\therefore\ \mathrm{H} = \frac{p}{d_a \,.\, g}.$$

Thus if g is assumed constant, H will be a constant because $\frac{p}{d_a}$ is constant by Boyle's law.

Now let the rise in the barometer reading be δh for a small difference in altitude δl, so small that the densities of the air and mercury are not appreciably altered.

Then from (19), at the point A, $h \,.\, d = \mathrm{H} \,.\, d_a$, and at a distance δl above A

$$(h - \delta h)d = (\mathrm{H} - \delta l)d_a,$$

or $$\delta h \,.\, d = \delta l \,.\, d_a,$$

or $$\delta l = \frac{\delta h \,.\, d}{d_a} = \frac{\delta h \,.\, \mathrm{H}}{h} \text{ from (19)},$$

therefore L, the difference in level between the two stations,

$$= \mathrm{H}\int_{h_1}^{h_2} \frac{dh}{h}$$

where h_1 and h_2 are the barometer readings at the lower and higher stations respectively, *i.e.*

$$\mathrm{L} = \mathrm{H}\ (\log h_1 - \log h_2),$$

the logarithms being to the base e.

Reducing this to common logarithms (*i.e.* base 10), the formula may be written

$$\mathrm{L} = 60158{\cdot}6(\log h_1 - \log h_2) \text{ at } 32^\circ \text{ F. and } 45^\circ \text{ latitude.}$$

The density and pressure of the air, however, vary with the temperature, so that a correction for temperature is applied as follows:

$$\mathrm{L} = 60158{\cdot}6(\log h_1 - \log h_2)\left(1 + \frac{t_1 + t_2 - 64^\circ}{900}\right), \qquad (20)$$

where t_1 and t_2 are the temperatures of the air in degrees Fahrenheit at the lower and higher stations respectively, measured by detached thermometers.

A further correction may be applied to a mercury barometer to allow for any difference of temperature in the mercury.

Thus $$h_2 = h_2'\{1 + \alpha(t_1' - t_2')\}.$$

where h_2' is the actual height of the barometer at the second station, and h_2 is the corrected height substituted in formula (20), a is a coefficient of expansion $=\cdot 00009$ about, and t_1' and t_2' are the mercury temperatures at the lower and higher stations.

The value of gravity (g) is a variable quantity, as it depends both upon the altitude and upon the latitude; it decreases as the altitude increases, or as the latitude increases, being about 32·25 ft. per sec. per sec. at the poles, and 32·09 ft. per sec. per sec. at the Equator, but the corrections are too small to be applied in the case of ordinary surveying work.

There are two types of barometer in general use for the determination of altitudes: (i) the mercury barometer, and (ii) the aneroid barometer.

The **mercury barometer** (Fig. 7.7, by Hicks) is generally on Fortin's principle, and consists of a long glass tube rather more than 30 inches long, dipping into a cistern containing mercury. The height of the mercury column is read by means of a vernier reading to, say, $\frac{1}{500}''$, and an adjusting screw is provided at the lower extremity, by means of which the mercury can be confined to a limited space to render the instrument more portable.

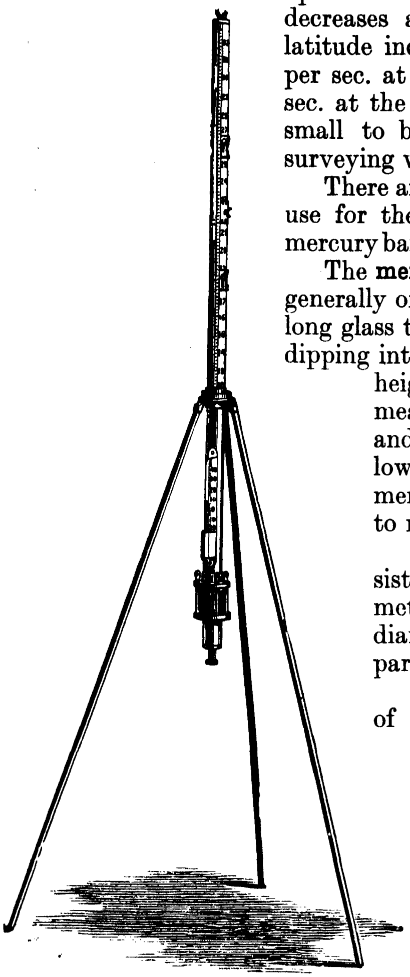


FIG. 7.7.—Mountain Barometer

The **aneroid barometer**[1] (Fig. 7.8) consists of a small cylindrical corrugated metal box of 3 inches to 5 inches or more in diameter, from which the air has been partially exhausted.

The movements of the thin surface of the box, due to changes of atmospheric pressure, are transmitted through a system of levers and springs to the indicating finger of the dial. On the dial are two scales—one to represent the head of a mercury column through a range of, say, 6 inches, and the other to represent altitudes in feet, a magnifying-glass being provided to facilitate the reading.

Usually this instrument is "compensated" so that the effects of changes of temperature on the instrument itself are supposed to be eliminated—the correction for air temperature is, however, applied as with a mercury barometer.

It is a more portable type than a mercury barometer, but is not

[1] See "The Measurement of True Height by Aneroid", by L. N. G. Filon, *J. Scientific Instruments*, Oct. 1923.

so accurate. It should be held in the same position—preferably horizontal—at each station, and should be lightly tapped before taking an observation, to ensure that no part of the mechanism is sticking.

Method of Levelling.—One of the chief difficulties to contend with in the employment of a barometer for the determination of heights is that the atmospheric pressure at any one station is not constant, but is continually varying; so that, if the difference in altitude of two stations A and B is to be deduced, the barometric heights at those stations should be observed simultaneously, and corrected for temperature, etc., as in formula (20).

If one barometer only is available, it is held at A—a datum point of which the reduced level is probably known—and the reading taken. It is then carried to B, and finally back to A, and readings taken at each of these points are corrected for temperature, etc.

The difference between the first and last of the observations at A gives the total variation in pressure at A during the interval; so that if this variation is assumed to be regular, and if the times of the three observations have been noted, the probable pressure at A corresponding to the reading at B can be interpolated.

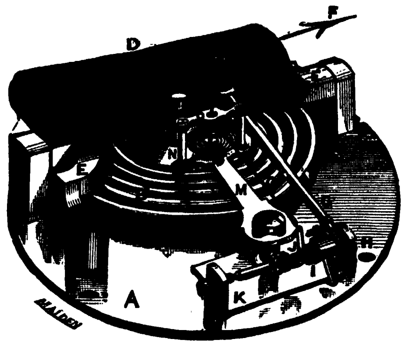


FIG. 7.8.—Aneroid Barometer

If a short stay be made at B or at any intermediate stations *en route*, two observations can be made at each and the rates of change of pressure there reduced. So that, knowing the *total* variation in pressure from the observations at A, and the *rate* of variation over several specific intervals, the probable pressure at A corresponding to the reading at B may be more accurately estimated.

One method of attempting this—if the rate of increase is found by the observations not to be uniform—is to plot by trial a graph of the pressures upon a time base, as the two extremities of this line are fixed, and also the slope of the curve over certain definite horizontal intervals. The probable pressure at A at any given instant may be scaled from this graph.

If two barometers are available, one may be left at the datum station A, where the observer notes a number of readings together with the time of each. The second observer, having compared his barometer with that at A and noted any relative index error, carries the instrument to B, and there observes the reading and the time of observation. On returning to A, the barometers are again compared, and a mean index error computed.

The barometric height at A at the time of the observation at B is then calculated or scaled from a graph plotted from the data obtained at A.

These methods apply equally to either mercury or aneroid barometers.

In the tropics [1] the changes of atmospheric pressure are much more regular than in the temperate zones, though the magnitude of the diurnal change is much more marked.

The diurnal variation in all countries appears to trace two distinct waves in the 24 hours, the maximum pressures occurring about 9 A.M. and 10 P.M., and the minimum about 4 A.M. and 3.30 P.M.

In England the ranges of the night and day waves average about 0·3 and 0·6 mm. respectively, though these values are practically indistinguishable from a single day's observations owing to the extreme irregularity of the atmospheric conditions here.

In the tropics, however, the irregularities are very small, and the range is larger, so that the curve of daily variation may be easily plotted by taking hourly readings for a week (say).

At Java,[1] 7° S. of the Equator, for instance, the ranges are 1·34 and 2·87 mm. respectively.

The curve of variation, however, is not constant for all altitudes; in temperate regions both the range and the phase vary, but in the tropics only the range is affected.

The range appears to vary inversely as the altitude, though the *rate* of change varies in different localities. In Java,[1] the range would appear to be zero at an altitude of 2500 metres, and at Madras at an altitude of 5000 metres or more.

The results obtained from barometer observations are therefore much more reliable in tropical than in temperate regions, for if once the diurnal variation curve is ascertained, and also the rate at which the range decreases as the altitude increases, very trustworthy observations may be made with a single instrument.

Example.—(Single barometer.) Deduce the approximate reduced level of a station B from the following data:

Barometer reading at A = 30·27″ at 10.0 A.M., temp. 58° F.
,, ,, B = 29·72″ at 11.30 A.M., temp. 44° F.
,, ,, A = 30·32″ at 12.30 P.M., temp. 62° F.

The reduced level of A = 80·00.

The total variation of pressure in $2\frac{1}{2}$ hours (10 A.M. to 12.30 P.M.) = 30·32 − 30·27 = ·05″, therefore in $1\frac{1}{2}$ hours, *i.e.* at 11.30 A.M., the change is $\frac{3}{5} \times \cdot 05 = \cdot 03''$.

The probable reading at A at 11.30 A.M. is therefore 30·30″.

$$t_1 \text{ (mean)} = \frac{58+62}{2} = 60^\circ \text{ F.},$$
$$t_2 = 44^\circ,$$
$$\therefore\ t_1 + t_2 - 64 = 40^\circ,$$

therefore applying formula (20)

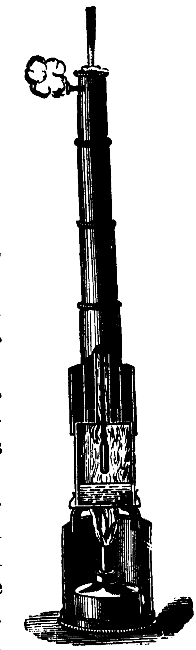
$$(h_2 - h_1) = 60158\cdot6\ (\log 30\cdot30 - \log 29\cdot72)\left(1 + \frac{40}{900}\right)$$
$$= 60158\cdot6\ [1\cdot48144 - 1\cdot47305]\left(1 + \frac{4}{90}\right)$$
$$= 527 \text{ ft. nearly.}$$

The reduced level of B would therefore be about
80 + 527 = 607 ft.

[1] Gribble, *Proc. Inst. C.E.* vol. clxxi.

Hypsometry.—The "boiling" point of water depends upon the pressure to which it is subjected, being lowered as the pressure decreases or as a higher altitude is attained. Consequently, if the boiling-point temperatures are determined at various stations, the results are a guide as to the atmospheric pressures there, and therefore as to the altitudes. The boiling-point is 212° F. when the pressure is 29·92 inches of mercury.

The instrument used for the purpose of determining altitudes is shown in Fig. 7.9 (by Hicks).


FIG. 7.9
Hypsometer

The water in the boiler is heated by means of a spirit lamp, but as the temperature of the *water* at the boiling-point is found to be influenced by other factors than the pressure, *e.g.* the presence and amount of any dissolved impurities, the thermometer is not allowed to touch the liquid, but is so placed as to record the temperature of the issuing steam, which is more independent of such conditions and yields more consistent results.

The temperature of the air—in the shade—is simultaneously observed with a detached thermometer, and a correction applied, if necessary, as on p. 235.

The pressures corresponding to the various boiling-points, *i.e.* to the temperatures of the saturated steam, can be obtained from tables, and if, from these, the corresponding barometer heights are deduced, the barometric formula may be applied. One inch of mercury at a temperature of 0° C. corresponds to a pressure of 0·491 lb. per sq. inch, and at a temperature of 20° C. to a pressure of 0·489 lb. per sq. inch.

Various empirical formulae have been suggested at different times to express the relationship between pressure and saturated steam temperature.

One such formula,[1] for the range 70° F. to 212° F. is

$$\log p_s = 7{\cdot}4397 - 4003{\cdot}9/T_s, \qquad (21)$$

where p_s is in inches of mercury and T_s = absolute temperature *i.e.* $460 + t°$ F.

The application of such formulae is not very convenient, and preferably the altitude would be derived directly from specially prepared tables.[2] Another[3] empirical formula which gives good agreement with the tables is

$$h = t(521 + 0{\cdot}75t)a, \qquad (22)$$

where h is the height above datum and t is the number of degrees F.

[1] Schule, *Technical Thermodynamics*.
[2] Close and Winterbotham, *Text Book of Topographical and Geographical Surveying*.
[3] Clark, *Plane and Geodetic Surveying*, vol. ii.

below 212° F. at which the water boils. The air temperature correction a has to be applied to the result, as in formula 20, p. 235.

Example.—The boiling-point at the lower station = 210·8° F., air temperature = 60° F.; the boiling-point at the upper station = 206·4° F., air temperature = 56° F. Applying formula (22), the lower station readings give a value of $h = 626$ ft.; the higher $h = 2941$ ft.; the difference being 2315 ft.

The temperature factor is $1 + (60 + 56 - 64)/900 = 952/900$, ∴ difference in altitude = 2448 ft. approximately.

The tables give a value 2438 ft.

Formula (21) gives the two values of log p_s as 1·4709 and 1·4314. Formula (20) then gives $h_2 - h_1 = 2514$ ft., differing from the former estimates by about 70 feet.

Accuracy of Barometric Levelling.—Barometric levelling cannot in any way replace spirit levelling or trigonometrical observations for accurate work, but with proper care it may be usefully employed to determine relative altitudes in such cases where only approximate values are required, or where the results are not sufficiently important to justify the expenditure of time and money necessary for more accurate determinations. Usually barometric observations are made with an aneroid on account of its greater portability, but as a rule the results cannot be relied upon to nearer than a few feet—although the scale may be graduated to single feet in some cases. In fact, the value of the constant (which is given as 60158·6 in the previous pages) in the reduction formula varies from ½ to 1 per cent as given by different authorities.

Gribble, in some experiments published in his *Preliminary Survey and Estimates*, shows that with a small aneroid the values derived for the height of a hill a little over 110 ft. high varied as much as 30 or more feet on the up and down hill journeys.

He states that, in general, the "up" hill values are the more reliable, and he also states that three "ups" will usually give a mean result correct to 3 ft., or two "ups" to 6 ft., in a difference of level ranging from 50 to 500 ft.

The boiling-point thermometer gives still less approximate figures, as 1° F. difference in the boiling-point temperature corresponds to a difference of about 550 ft. in altitude. A small error in the estimation of the temperature thus has a large effect upon the derived result. In addition, the published tables and empirical formulae vary very considerably and yield by no means consistent results, so that a discrepancy of 100 ft. or more from the true altitude may easily be obtained in many cases.

EXAMPLES

1. An instrument was set up at a distance of 500 ft. from a tower and the angle of elevation to a point on the parapet was 9°-39′, while the angle of depression to the foot of the walls was 2°-52′. The staff reading upon a B.M. with the telescope horizontal was 2·48 and the reduced level of the B.M. was 86·60.

What was the approximate height of the tower and the reduced level of the parapet?

2. If there was a p.e. of ± 1 ft. in the distance of the instrument from the

tower, and of $\pm 30''$ in each of the observed angles, what is the probable error in the two values deduced in question (1) ?

3. A vane 15 ft. above the foot of a staff was sighted at a point 6000 ft. away from the instrument.

The reduced level of the instrument axis was 286·60 and the angle of depression 1°-18′-00″. Allowing for curvature and refraction, calculate the approximate reduced level of the staff station.

4. (I.C.E.) The top of a hill subtends an angle of 41°-2′-10″ from a point on a plane at its base; a base line 264 ft. long is measured directly towards the base, and the hill then subtends an angle of 55°-40′-17″. Find the height of the hill.

5. What is the p.e. in the result of question 4 if the p.e. in each of the observed angles is $\pm 5''$ and in the virtual length of the base line $\pm 1{\cdot}0$ ft.? the latter error being due to differences in the heights of the instrument axis and to errors in linear measurement (cf. F_2F_1, Fig. 7.4).

6. Referring to Fig. 7.5, find the horizontal distance from F to B and the altitude of B from the following data :

$d = 500$ ft. $\alpha = 40°\text{-}36'$ $\beta = 28°\text{-}24'$ $\gamma = 10°\text{-}30'$.
$s = s_1 = 4{\cdot}00$ ft. Height of instrument at A $= 4{\cdot}92$ ft.
Height of instrument axis at F above datum $= 88{\cdot}50$ ft.

7. Assuming a p.e. of $\pm 30''$ in each of the angles α, β, and γ, and of $\pm{\cdot}5$ ft. in d in question 6, deduce the p.e. in the results.

8. Referring to Fig. 7.6., determine the reduced level of B from the following data :

Geodetic distance from A to B . . .	15240 ft.
Reduced level of A	1250 ,,
Height of instrument at A . . .	5·00 ,,
Height of instrument at B . . .	4·95 ,,
Height of signal at A	7·00 ,,
Height of signal at B	7·00 ,,
Angle of depression from A to B . .	1′-00″
Angle of depression from B to A . .	15″

Find also the mean angle of refraction and the coefficient of refraction.

9. The following data were obtained from reciprocal observations for altitude at A and B (Method 5, p. 228).

The reduced level of A	= 100·00 ft.
The height of instrument at A	= 5·05 ,,
The height of instrument at B	= 4·95 ,,
The height of signal at A	= 25·00 ,,
The height of signal at B	= 10·00 ,,
The angle of elevation from A to B . . .	= 4°-34′-59″
The angle of depression from B to A . . .	= 4°-33′-38″
The horizontal distance from A to B as determined by triangulation	= 20520 ft.

What was the reduced level of B ?

Find also the mean value of the angle of refraction and of the coefficient of refraction.

10. Assuming that the p.e. in the angles α and β is $\pm 5''$, what will be the p.e. in the reduced level of B ?

If there is also a p.e. of $\frac{1}{10000}$ in the horizontal distance D, what is then the p.e. in the result ?

11. (U. of L.) A theodolite is set up at two stations A and B at the water's edge of a lake which is 1240 ft. above sea-level. A staff on a hill at C is sighted from each station. From A the vertical angle of C is 15°-14′ and the horizontal

angles CAB, CBA are 59°-10′ and 71°-48′ respectively. The length of the line AB is 820 yds. Find the height of C above sea-level.

12. Deduce the approximate reduced level of a station B from the following data:

Barometer reading at A = 29·92 at 12.0 noon, temp. 60° F.
,, ,, B = 30·38 at 1.15 P.M., temp. 72° F.
,, ,, A = 29·97 at 3.50 P.M., temp. 55° F.

The reduced level of A = 250·00 ft.

13. (U. of B.) From the ends of a base line AB, 500 ft. in length, two points P and Q were observed with a transit theodolite. The following angles were recorded:

Bearing of line AB 45° E. of N.
,, ,, AP 45° W. of N.
,, ,, BP 97° W. of N.
,, ,, AQ 15° E. of N.
,, ,, BQ 20° W. of N.

Altitude of P as observed from A, 20°.
,, Q ,, ,, A, 15°.
The point A is 200 ft. above datum.

Determine the distance PQ and the heights of P and Q above datum.

14. In levelling across the River Manora, the greatest length of shot was 36 chains. (*a*) What would be the correction for curvature in this distance, and (*b*) what would be an approximate allowance for curvature and refraction?

15. In levelling across a river by reciprocal observations the following results were obtained (Fig. 6.25).

Reduced level of instrument axis at A—obtained from a reading upon a B.M. near = 85·02 feet.

Staff reading of *a* from A = 5·56 feet.
,, ,, *b* ,, A = 6·25 ,,
,, ,, *a* ,, B = 4·39 ,,
,, ,, *b* ,, B = 4·40 ,,

What are the reduced levels of *a*, *b*, and the instrument axis at B, and hence what is the reduced level of a point C near B, upon which the staff reading from B is 2·24 feet?

Assuming the instrument to be in adjustment, what would be the approximate distance from *a* to *b*?

CHAPTER VIII

TACHEOMETRY AND RANGE-FINDING

TACHEOMETRY

A TACHEOMETER is essentially a transit theodolite, the diaphragm of which is furnished with two (or four) horizontal webs or points (known as stadia webs, wires, lines, or points) in addition to that which marks the horizontal axis or line of collimation of the telescope (Fig. 2.15, *e*, *f*, *g*, and *h*).

The stadia rod or staff used in conjunction with the tacheometer may be the usual type of levelling staff reading to ·01 ft. (Fig. 6.13), or it may be of special design painted with easily distinguishable markings (Fig. 8.1) to enable observations to be made over much longer distances than are possible with the ordinary graduations of the Sopwith pattern. Further, to render the staff graduations more clear at a distance, the magnifying power of the eye-piece is generally rather greater than that of the ordinary theodolite.

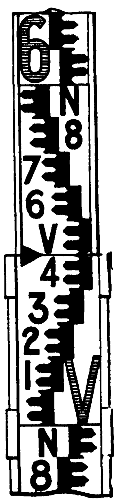


FIG. 8.1
Stadia Rod

The object of tacheometry is to enable horizontal and vertical distances to be computed from readings upon a stadia rod, and thus to render chaining operations unnecessary.

The observations that are required for the complete location of a point A with reference to the instrument station O are :

(1) The bearing of the line OA from some fixed meridian through O.

(2) The angle of elevation or depression recorded on the vertical arc of the instrument.

(3) The readings of the three diaphragm webs upon a stadia rod at A sighted through the telescope. The difference between the two outer web readings gives the intercept (s) upon the staff which is employed in the calculations as explained below.

The field work can be done very quickly, and so long as the country is fairly open, any roughness or unevenness of the ground does not affect the accuracy of the work as would be the case with direct linear measurements.

Tacheometry is thus particularly suitable for preliminary location surveys, and for the filling in of detail on topographical maps : its use in conjunction with the plane table is treated in Chapter IX.

The **principle of the tacheometer** is as follows :

Let C be the optical centre of the object-glass, A_1 and B_1 the stadia webs spaced at a distance i apart, and O_1 the axial web of the diaphragm. Then, when the telescope is correctly focused, these webs coincide with the image of the staff which is being observed, and the portion of the image intercepted by A_1B_1 corresponds with the portion AB ($=s$) of the staff, where A_1CA and B_1CB are straight lines through C (Fig. 8.2, and Fig. 2.8, p. 51). O_1CO is the line of collimation of the telescope.

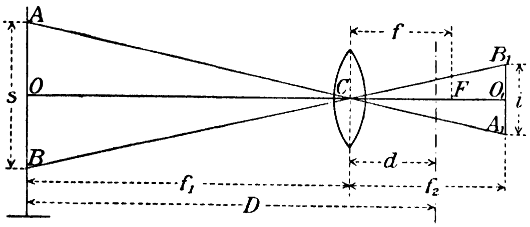


FIG. 8.2

Referring to Fig. 8.2, let the distances CO and CO_1, which are conjugate focal lengths of the lens, be represented by f_1 and f_2, and let the distance from C to the principal focus of the lens F be f, and that from C to the central vertical axis of the instrument be d.

Then, because the triangles ABC and A_1B_1C are similar,

$$\frac{AB}{A_1B_1}=\frac{OC}{O_1C} \text{ or } \frac{s}{i}=\frac{f_1}{f_2}. \qquad (1)$$

But as was shown in Chapter II,

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2} \qquad (2)$$

or

$$\frac{f_1}{f}=1+\frac{f_1}{f_2},$$

therefore by substitution for $\frac{f_1}{f_2}$ in (1)

$$\frac{s}{i}=\frac{f_1}{f}-1,$$

or the distance of the staff from the object-glass, *i.e.*

$$f_1=s\,.\frac{f}{i}+f. \qquad (3)$$

But the distance D of the staff from the vertical axis of the instrument is equal to f_1+d,

$$\therefore\ D=s\,.\frac{f}{i}+f+d. \qquad (4)$$

This is the formula to be applied for ordinary stadia observations when the telescope is horizontal or when the staff is perpendicular to the line of collimation.

Usually $\frac{f}{i}$ is given a convenient value such as 100—occasionally

50 or 200—while $(f+d)$ has a value of 1 to 1·5 or 2, depending upon the particular instrument.

Example.—The constants $\frac{f}{i}$ and $(f+d)$ for a certain instrument were 100 and 1·5 respectively; the readings of the three diaphragm webs, on a staff held at a distant point, were found to be 2·16, 3·30, and 4·44, the telescope being horizontal.

The intercept between the outer webs is thus $(4{\cdot}44-2{\cdot}16)=2{\cdot}28$ ft., so that by formula (4) the distance from the instrument to the staff

$$\begin{aligned} D &= 2{\cdot}28 \times 100 + 1{\cdot}5 \\ &= 229{\cdot}5 \text{ ft.} \end{aligned}$$

If the reduced level of the instrument axis was, say, 76·76 ft., that of the staff station $=76{\cdot}76-3{\cdot}30=73{\cdot}46$ ft. above datum.

Determination of the Tacheometric Constants.—The values of the constants $\frac{f}{i}$ and $(f+d)$ for a given instrument may be determined experimentally as follows:

(*a*) To find f the cross-hairs are focused on to a far distant object, and the distance along the top of the telescope between the object-glass and the diaphragm screws is measured with an ordinary rule.

From equation (2) $$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2},$$

so that if f_1 is very large, then as f and f_2 are each less than the length of the telescope, $\frac{1}{f_1}$ is negligible compared with $\frac{1}{f}$ or $\frac{1}{f_2}$, and consequently f is very nearly equal to f_2, *i.e.* the distance between the object-glass and the diaphragm.

As explained in Chapter II, the relative distance between the object-glass and the diaphragm is altered to correspond with any alteration in f_2 either.

(1) By moving the object-glass forward, or

(2) By moving the eye-piece and diaphragm backward relatively to the telescope tube.

In the latter case the value of d is constant for different lengths of sights, while in the former case d is a variable quantity. It is, however, sufficiently accurate to adopt a mean value, and this is measured directly on the instrument.

The results obtainable with an ordinary instrument are theoretically only accurate to the nearest foot, owing to the fact that the finest graduation on the staff is 0·01 ft., and that generally $\frac{f}{i}=100$, *i.e.* a difference of one of the smallest readable divisions corresponds to a difference in distance of 1 ft. With $\frac{f}{i}=50$ the results are estimated to the nearest 0·5 ft. Consequently the sum of $(f+d)$ is, as a rule, expressed merely to the nearest foot or half foot—*e.g.* 1, 1·5, or 2 ft.—but occasionally to the nearest ·25 ft.

The distance i between the stadia webs is too small to be measured

with ease very accurately, so that the value of $\frac{f}{i}$ is found by chaining or taping a horizontal distance D_1 from the instrument and noting the corresponding intercept s_1 on the staff. The figures are then substituted in equation (4), giving

$$D_1 = s_1 \cdot \frac{f}{i} + (f+d),$$

from which, as $\frac{f}{i}$ is the only unknown quantity, it may be calculated, *i.e.*

$$\frac{f}{i} = \frac{D_1 - (f+d)}{s_1}.$$

An average of several determinations is adopted.

Example.—The value of $(f+d)$ having been determined by direct measurement as 1·5 ft., and the staff readings at a measured distance away of 301·5 ft. being 1·07, 2·57, and 4·07, the value of $s_1 = 3{\cdot}00$ ft., and $\frac{f}{i}$ is found from equation (4), *i.e.*

$$301{\cdot}5 = 3{\cdot}00 \cdot \frac{f}{i} + 1{\cdot}5,$$

$$\frac{f}{i} = \frac{300}{3{\cdot}00} = 100.$$

(*b*) An alternative method is to measure out two definite distances D_1 and D_2 and find the corresponding intercepts, s_1 and s_2, on the staff held at these positions.

By substituting these values in equation (4) two simultaneous equations are obtained :

$$D_1 = s_1 \frac{f}{i} + (f+d), \quad . \quad . \quad . \quad (5)$$

$$D_2 = s_2 \frac{f}{i} + (f+d), \quad . \quad . \quad . \quad (6)$$

which may be solved to find the two unknown quantities $\frac{f}{i}$ and $f+d$; *e.g.* by subtracting (5) from (6),

$$\frac{f}{i} = \frac{D_2 - D_1}{s_2 - s_1}, \quad . \quad . \quad . \quad (7)$$

and by substitution in either (5) or (6), or by multiplying (5) by s_2 and (6) by s_1 and subtracting, we find

$$f+d = \frac{s_2 D_1 - s_1 D_2}{s_2 - s_1}. \quad . \quad . \quad . \quad (8)$$

Example.—Two distances of 50 and 300 ft. were accurately measured out, and the intercepts on the staff between the outer stadia webs were ·49 at the former distance and 2·99 at the latter.

The two equations are thus

$$50 = {\cdot}49 \frac{f}{i} + f + d, \quad . \quad . \quad . \quad . \quad (9)$$

$$300 = 2{\cdot}99 \frac{f}{i} + f + d. \quad . \quad . \quad . \quad . \quad (10)$$

By subtracting (9) from (10)

$$250 = 2{\cdot}50\frac{f}{i}$$

or

$$\frac{f}{i} = 100.$$

By substitution in (9) $50 = 49 + (f+d)$,

$$\therefore (f+d) = 1.$$

Anallatic Lens.—To eliminate the constant $(f+d)$ from the formulae for telemetric telescopes, an additional lens, known as an anallatic lens, is sometimes provided in the telescope between the object-glass and eye-piece. The advantage of such a device is that the calculation of heights and distances from the field notes is very much simplified—particularly for inclined sights. The disadvantage—apart

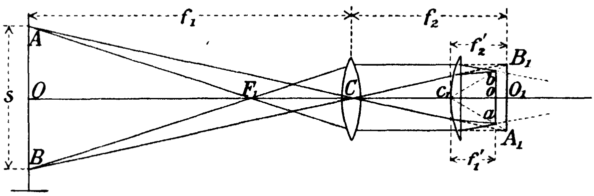


FIG. 8.3.—Anallatic Lens

from the additional initial expense incurred—is that an appreciable proportion of light is thereby intercepted. Sometimes, to compensate for this, the telescope tube is made of a slightly larger diameter than that of an ordinary instrument.

The theory of the lens may be explained by reference to Fig. 8.3. Let AB be the position of the staff, C the optical centre of the object-glass, B_1A_1 the position of the image which would be formed were no anallatic lens provided, so that OC and O_1C are conjugate focal lengths, $=f_1$ and f_2, say.

In the figure two rays only are shown from each point A and B—one passing through the outside principal focus of the object-glass and emerging parallel with the telescope axis, and one passing in a straight line (approximately) through the optical centre C of the lens (*cf.* Fig. 2.9, p. 52).

The effect of the anallatic lens—which is placed between the object-glass and the image A_1B_1—is to intercept these rays, refract them, and cause the image to be formed at some position *boa*, where if c_1 be the optical centre of the anallatic lens, c_1o $(=f_1')$ and c_1o_1 $(=f_2')$ are conjugate focal lengths.

The equations from the laws of lenses (p. 53) are

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} \quad . \quad . \quad . \quad (11)$$

and

$$\frac{1}{f'} = \frac{1}{f_1'} - \frac{1}{f_2'}, \quad . \quad . \quad . \quad (12)$$

a result similar to (11), which may be proved from Fig. 8.3. Also if $AB = s$, $A_1B_1 = i$, and $ab = i'$, where i' is a fixed value, *i.e.* the distance between the stadia points,

$$\frac{s}{i} = \frac{f_1}{f_2} \quad \text{and} \quad \frac{i}{i'} = \frac{f_2'}{f_1'}$$

because the triangle ABC is similar to the triangle B_1A_1C, and the triangle $B_1c_1A_1$ is similar to the triangle bc_1a.

Therefore by multiplication

$$\frac{s}{i'} = \frac{f_1}{f_2} \cdot \frac{f_2'}{f_1'}. \qquad (13)$$

By substitution from (11) and (12)

$$\frac{s}{i'} = \frac{f_1 - f}{f} \cdot \frac{f' + f_2'}{f'}$$

$$= \frac{f_1 - f}{f} \cdot \frac{f' + f_2 - d'}{f'},$$

where d' is the distance between the lenses, *i.e.* Cc_1,

$$= \frac{f_1 - f}{f} \cdot \frac{f' + \dfrac{f \cdot f_1}{f_1 - f} - d'}{f'}$$

$$= \frac{ff_1 + (f_1 - f)(f' - d')}{ff'}$$

$$= \frac{f_1(f + f' - d') + f(d' - f')}{ff'},$$

$$\therefore f_1 = \frac{s}{i'} \cdot \frac{ff'}{f + f' - d'} - \frac{f(d' - f')}{f + f' - d'}, \qquad (14)$$

or $f_1 = K \cdot \frac{s}{i'} - c$, where $K = \frac{ff'}{f + f' - d'}$ and $c = \frac{f(d' - f')}{f + f' - d'}$, each being constant for any particular instrument as f, f', and d' are all fixed quantities.

Therefore D, the distance from the instrument axis to the staff station, $= f_1 + d$, where d is the distance from the instrument axis to the object-glass as in equation (4), *i.e.*

$$D = K \cdot \frac{s}{i'} - c + d. \qquad (15)$$

By suitably arranging the values f, f', d', and i', the value of $\frac{K}{i'}$ can be made equal to 100 (or some other convenient number such as 50 or 200), while at the same time c can be made equal to d, thus reducing the formula (4) to

$$D = 100\, s,$$

or more generally

$$D = ms, \qquad (16)$$

where m is a suitable multiplying factor.

Alternative Proof of Formulae.—The formula for the ordinary telemetric telescope and that for an instrument provided with an anallatic lens may also be derived geometrically.

Let A_1B_1 (Fig. 8.4) represent the wires of the diaphragm. Then those rays proceeding from A and B which travel parallel to the principal axis of the object-glass lens after refraction at A_{11} and B_{11} must pass through F_1, the outside principal focus of the lens.

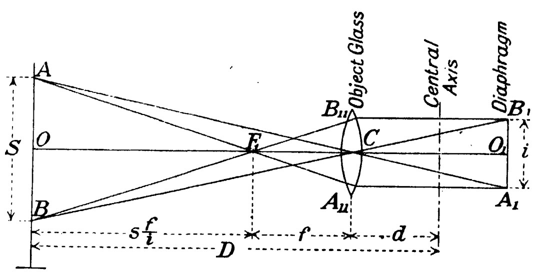


FIG. 8.4

But if A_1 and B_1 are at constant distances from the axis, A_{11} and B_{11} are fixed points, and consequently, as F_1 is also fixed, the angle AF_1B is fixed in magnitude.

From the similar triangles ABF_1 and $A_{11}B_{11}F_1$

$$\frac{AB}{A_{11}B_{11}} = \frac{OF_1}{CF_1},$$

or

$$\frac{s}{i} = \frac{OF_1}{f},$$

where i is the distance apart of the stadia webs $A_1B_1 = A_{11}B_{11}$, f is the principal focal length CF_1 of the object-glass, and s is the intercept AB on the staff,

$$\therefore\ OF_1 = s\,.\frac{f}{i}.$$

The distance of the object-glass from $AB = OF_1 + f$, and if the instrument axis is at a distance d from the object-glass, the total distance is given as before by the formula

$$D = s\,.\frac{f}{i} + f + d.$$

Anallatic Lens.—To consider the effect of the anallatic lens let ba (Fig. 8.5) be the positions of the diaphragm webs, and let $BB_{11}F'b'b$ represent the path of the particular ray from B which becomes parallel to the axis of the telescope after emerging from the anallatic lens at C_1 and which therefore passes through the principal focus F′ of this lens. Then as b remains at a constant distance from the longitudinal axis OCo during focusing, the position of the point b' is fixed when the length $ab = i'$ is decided upon.

The slope of the line $b'F'B_{11}$ depends upon the value of the principal focal length f' of the anallatic lens (*i.e.* upon the position of F′) and also upon the position of b', *i.e.* upon the value of i'. The position of

B_{11} is further dependent upon the distance d' between the two lenses, and the direction of $B_{11}B$ is governed by the principal focal length (f) of the object-glass and by the position of B_{11}.

Let the directions of BB_{11} and the corresponding ray which proceeds from A to a, viz. AA_{11}, meet when produced at a point D from symmetry on the horizontal axis of the telescope.

Then the position of D and the ratio of AB to OD, depending as they do upon the slope of BB_{11} and AA_{11}, and the positions of A_{11} and B_{11}, are dependent upon the values of f, f', d', and i'.

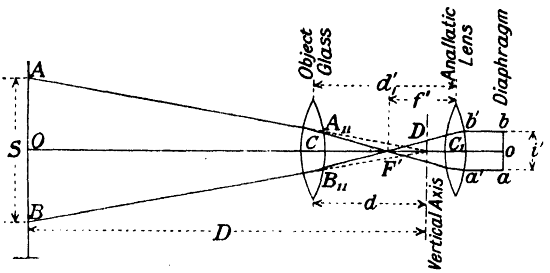


FIG. 8.5.—Anallatic Lens

These values are so arranged that D lies upon the vertical axis of the instrument, and the ratio $\frac{AB}{OD}$, *i.e.* $\frac{s}{D}=\frac{1}{100}$, or some other suitable fraction.

The value of s is given by the reading intercepted between the cross-hairs at ab, and D is easily calculated from the formula $D=ms$, where m is usually 100.

The Internal Focusing Telescope.—The theory of the internal focusing telescope has been discussed on p. 57-58. There equation (8) was derived, giving the relationship between the various quantities d, f, f', d_1 and f_1. For a particular instrument f and f' are fixed, being the principal focal lengths of the two lenses. As in the Zeiss instrument described on p. 168, d_1, the distance between the object-glass and the cross-hairs, is also kept a constant value, consequently as f_1, the distance to the staff, varies, d must be altered by means of the focusing screw until equation (8) is satisfied. The relationship between the size of the object s and the size of the image i is obviously complex, and the ordinary tacheometric formulae (4) and (16) are not strictly applicable. The suitability of internal focusing telescopes for tacheometric surveying was investigated by Major E. O. Henrici, R.E.[1] He found that though the amount of the discrepancy incurred when these formulae were adopted depends upon the size of the telescope and the properties of the lenses, yet normally, if an additive constant of 5 to 8 inches is applied, the results are correct to within a fraction of 1 per cent, except for unusually short distances, under 15 feet or so. For many purposes, as the additive constant is so small, it may be neglected.

Inclined Sights.—When the staff station and the instrument station are at greatly different altitudes, an observation with the telescope

[1] "The Use of Telescopes with Internal Focusing for Stadia Surveying", *Trans. Opt. Soc.* vol. xxii, 1920–21.

axis in its horizontal position becomes impossible, and therefore an inclined sight must be taken.

To do this the staff is held either in a vertical position or in a position at right angles to the line of collimation.

(*a*) *Staff Vertical.*—When this method is adopted, a spirit level or pendulum device is sometimes provided to ensure verticality, though for general work it is left to the judgment of the staff holder. The telescope is tilted until the three horizontal webs intersect the staff, when the three stadia readings are taken, and also the angle (θ) of elevation or depression of the telescope and the bearing in azimuth.

In Fig. 8.6 let C be the position of the instrument, CO the line of collimation, and A, O, B the positions on the staff cut by the webs of the diaphragm. Let AB $= s$, the distance CO $= l$, the horizontal distance CE $= h$, and the vertical distance EO $= v$.

Then in the figure, if A′B′ be the projection of AB perpendicular to the line of sight, as AB and A′B′ are at right angles to CE and CO

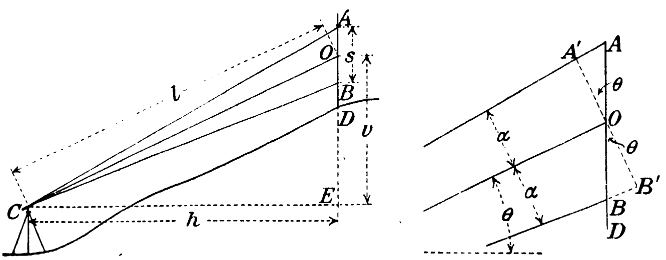


FIG. 8.6.—Inclined Sights

respectively, the angles AOA′ and BOB′ are each equal to θ, so that if the angles AA′O and BB′O are assumed to be approximately 90°,

$$\text{A}'\text{O} = \text{AO} \cos\theta \text{ and } \text{B}'\text{O} = \text{BO}\cos\theta,$$

i.e.

$$\text{A}'\text{B}' = \text{AB}\cos\theta,$$

or writing s' for A′B′

$$s' = s \,.\, \cos\theta. \qquad (17)$$

A′B′ may be determined more rigorously if we assume the angles ACO and BCO are each equal to α, say.

Then the exterior angle AA′O of the triangle A′CO is equal to the sum of the interior and opposite angles A′CO and A′OC, *i.e.*

$$\text{AA}'\text{O} = 90^\circ + \alpha,$$

also as B′OC is 90° the remaining two angles of the triangle B′CO are together equal to 90°, and consequently

$$\text{BB}'\text{O} = 90^\circ - \alpha.$$

From the triangle AOA′

$$\text{AO} = \frac{\text{A}'\text{O} \,.\, \sin \text{AA}'\text{O}}{\sin \text{A}'\text{AO}}$$

$$= \frac{\text{A}'\text{O} \,.\, \sin(90^\circ + \alpha)}{\sin\{90^\circ - (\theta + \alpha)\}}, \qquad (18)$$

and similarly $$BO=\frac{B'O \sin (90°-\alpha)}{\sin \{90°-(\theta-\alpha)\}}. \qquad (19)$$

By addition AO + BO = s

$$=\frac{A'B'}{2}\left\{\frac{\sin (90°+\alpha)}{\sin \{90°-(\theta+\alpha)\}}+\frac{\sin (90°-\alpha)}{\sin \{90°-(\theta-\alpha)\}}\right\} \text{as } A'O=B'O=\frac{A'B'}{2},$$

i.e. $$s=\frac{s'}{2}\cdot\left\{\frac{\cos \alpha}{\cos (\theta+\alpha)}+\frac{\cos \alpha}{\cos (\theta-\alpha)}\right\}$$

$$=\frac{s'}{2}\cdot \cos \alpha\left\{\frac{\cos (\theta-\alpha)+\cos (\theta+\alpha)}{\cos (\theta+\alpha) \cos (\theta-\alpha)}\right\}$$

$$=\frac{s'}{2}\cdot\frac{\cos \alpha \cdot 2 \cdot \cos \theta \cos \alpha}{(\cos \theta \cos \alpha-\sin \theta \sin \alpha)(\cos \theta \cos \alpha+\sin \theta \sin \alpha)}$$

$$=\frac{s'}{2}\cdot\frac{\cos^2 \alpha \cdot 2 \cdot \cos \theta}{\cos^2 \theta \cos^2 \alpha-\sin^2 \theta \sin^2 \alpha},$$

$$\therefore s'=s \cdot\left\{\frac{\cos^2 \theta \cos^2 \alpha-\sin^2 \theta \sin^2 \alpha}{\cos \theta \cos^2 \alpha}\right\}$$

$$=s \cdot \cos \theta-s \cdot \frac{\sin^2 \theta}{\cos \theta} \tan^2 \alpha,^1 \qquad (20)$$

but as α is very small, $\tan^2 \alpha$ is small, and the second term is negligible.

The distance l along the line of collimation CO by formula (4) is equal to

$$A'B' \cdot \frac{f}{i}+f+d,$$

i.e. $$l=s \cdot \frac{f}{i} \cos \theta+(f+d), \qquad (21)$$

and the horizontal projection $=l \cos \theta$, *i.e.*

$$h=s \cdot \frac{f}{i} \cos^2 \theta+(f+d) \cos \theta. \qquad (22)$$

Similarly the vertical height of O above the instrument axis is $l \sin \theta$, *i.e.*

$$v=s \cdot \frac{f}{i} \sin \theta \cos \theta+(f+d) \sin \theta, \qquad (23)$$

or $$v=s \cdot \frac{f}{i} \frac{\sin 2 \theta}{2}+(f+d) \sin \theta, \qquad (23a)$$

or $$v=h \tan \theta. \qquad (23b)$$

The altitude of the staff station D is thus equal to (the reduced level of the instrument axis) $\pm v$ − (the axial reading DO), according to whether the vertical angle is an angle of elevation or depression.

In taking the observation, sometimes it is so arranged that the vertical angle shall be a whole number of degrees or minutes or be of some other definite value which will enable the functions to be conveniently abstracted from tables: or sometimes the lower stadia

[1] Breed and Hosmer, *Surveying*, vol. ii.

reading is taken at an even foot so that the stadia interval (s) is more easily calculated.

The values of $\cos^2 \theta$ and $\sin \theta \cos \theta$ may be deduced from ordinary mathematical tables, or from one of the specially prepared works in which these functions are tabulated. Diagrams are also published to facilitate the computations, and special forms of slide rules are manufactured.[1]

When the telescope is fitted with an anallatic lens, the same formulae are applied, but the term containing $(f+d)$ is omitted in each case.

NOTE.—The formulae for the reduction of horizontal distances and vertical heights may be dispensed with entirely, by the use of a direct-reading tacheometer such as that devised by Prof. H. H. Jeffcott.[2] In this instrument, by means of a simple cam and lever mechanism, the stadia points are automatically made to move relatively to one another as the angle of the telescope is altered, and this enables horizontal distances and vertical heights to be read off directly from the staff intercepts.

Example 1.—The readings on a staff held upon a bench mark of reduced level 80·00 were 1·00, 2·56, 4·14, while the vertical angle was $-5°\text{-}32'\text{-}00''$. The readings at a station N were 2·00, 3·88, 5·76, while the vertical angle was $4°\text{-}18'\text{-}00''$.

Deduce the horizontal distance from the instrument station to N, and its reduced level if $\frac{f}{i}=100$ and $(f+d)=1$.

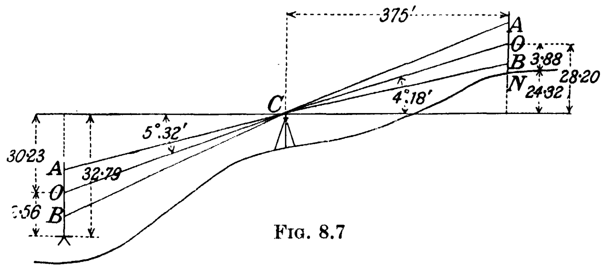


FIG. 8.7

The vertical distance of the axial web below the instrument axis in the first observation (Fig. 8.7) is by (23):

$$\begin{aligned} v &= (3{\cdot}14 \times 100 \times \sin 5°\text{-}32' \times \cos 5°\text{-}32') + (1 \times \sin 5°\text{-}32') \\ &= (314 \times {\cdot}0964 \times {\cdot}9953) + {\cdot}10 \\ &= 30{\cdot}13 + {\cdot}10 = 30{\cdot}23 \text{ ft.} \end{aligned}$$

The B.M. is thus $30{\cdot}23+2{\cdot}56=32{\cdot}79$ ft. below the instrument axis. The reduced level of the instrument axis is therefore

$$32{\cdot}79+80{\cdot}00=112{\cdot}79 \text{ ft.}$$

The horizontal distance to the second staff station, *i.e.* N, is given by equation (22), *i.e.*

$$\begin{aligned} h &= (3{\cdot}76 \times 100 \times \cos^2 4°\text{-}18') + (1 \times \cos 4°\text{-}18') \\ &= (376 \times {\cdot}9944) + 1 = 375 \text{ ft. nearly.} \end{aligned}$$

The vertical distance of the axial reading O above the instrument is, from (23*b*),

$$v = h \tan \theta = 375 \times {\cdot}07519 = 28{\cdot}20 \text{ ft.,}$$

[1] See *Proc. Inst. M. and Cy. E.* vol. lii. for a Nomograph by the Author.

[2] "A Direct-Reading Tacheometer", by W. H. Connell. *Journal of Scientific Instruments*, June 1926.

or, from (23),

$$v = (3{\cdot}76 \times 100 \times \sin 4°\text{-}18' \times \cos 4°\text{-}18') + (1 \times \sin 4°\text{-}18')$$
$$= (376 \times {\cdot}0750 \times {\cdot}9972) + {\cdot}08$$
$$= 28{\cdot}12 + {\cdot}08 = 28{\cdot}20 \text{ ft.}$$

The station N is thus 28·20 − 3·88 = 24·32 ft. above the axis, and the reduced level is 112·79 + 24·32 = 137·11 ft. above datum.

Example 2.—A tacheometer fitted with an anallatic lens gave the following readings from an instrument station C :

(1) On B.M. of reduced level 80·00, angle of elevation 1°-30′, stadia readings 2·00, 3·28, 4·57 ;

(2) On point P, angle of depression 2°-48′, stadia readings 1·00, 3·12, 5·25.

Determine the distance from C to P, and the reduced level of P.

The vertical distance of the axial web above the instrument axis by (23*a*), omitting the term containing $(f+d)$, is

$$v = 2{\cdot}57 \times 100 \times \frac{\sin 3°}{2} = 257 \times {\cdot}0262$$
$$= 6{\cdot}73 \text{ ft.}$$

The reduced level of the instrument axis is therefore

$$80{\cdot}00 + 3{\cdot}28 - 6{\cdot}73 = 76{\cdot}55 \text{ ft.}$$

The horizontal distance to P by equation (22)

$$h = s\,.\,\frac{f}{i}\,.\,\cos^2 2°\text{-}48' = 425 \times ({\cdot}9988)^2 = 424 \text{ ft. nearly.}$$

The vertical distance of the axial reading below the instrument axis is

$$v = 424\,.\,\tan 2°\text{-}48' = 424 \times {\cdot}0489 = 20{\cdot}74 \text{ ft.,}$$

or, from (23*a*), $$v = 425 \times \frac{\sin 5°\text{-}36'}{2} = 20{\cdot}74 \text{ ft.}$$

The reduced level of P is therefore 76·55 − 20·74 − 3·12 = 52·69 ft. above datum.

(*b*) *Staff Inclined.*—When the method of holding the staff at right angles to the line of collimation is adopted for inclined sights, the correct inclination of the staff is secured by the staff man, who sights to the instrument along a special rule or small telescope fixed to the side of the staff for this purpose.

The instrument man can also see this sight through the tacheometer telescope, and he can judge as to the accuracy of the position of the staff, whereas when the staff is held vertically the truth of the verticality depends entirely upon the judgment of the staff man. The inclination of the staff should, of course, be directly towards or away from the instrument ; and as in ordinary levelling operations, any error at right angles to this direction is easily seen by the instrument man, who may signal the information to the staff man.

The distance along the line of collimation is given by formula (4), *i.e.*

$$l = s\,.\,\frac{f}{i} + (f+d), \qquad . \qquad . \qquad . \quad (24)$$

or $$l = s\,\frac{f}{i} \text{ for an anallatic lens,} \qquad . \quad (25)$$

and the horizontal distance

$$h = l \cos \theta \quad . \quad . \quad . \quad (26)$$

and the vertical distance

$$v = l \sin \theta. \quad . \quad . \quad . \quad (27)$$

Strictly speaking, the value of h given above is the horizontal projection of CO, *i.e.* the length CE_1 (Fig. 8.8), while the horizontal distance required is CE, where E is vertically below the foot of the staff D.

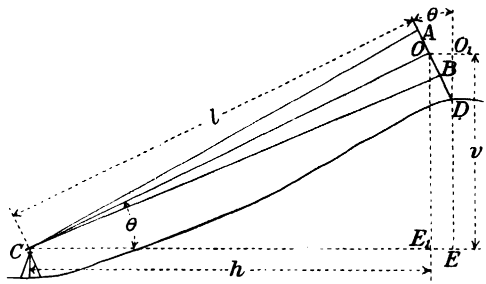


FIG. 8.8.—Inclined Sight

This error $OO_1 = DO \sin \theta$ is generally negligible, as O is kept as near the foot of the staff as possible, and DO is only a few feet in length, while $\sin \theta$ is very small when θ is, as is usually the case, only a few degrees in magnitude.

Example.—In an ordinary case when DO = 5·00 ft. and $\theta = 4°$, say,

$$DO \sin \theta = 5 \times \cdot 0698 = \cdot 35 \text{ ft.},$$

which is less than the probable error of observation.

If DO = 5·00′ and $\theta = 30°$—a value which would be less seldom obtained in practice—the error is more appreciable, and a correction should be applied, *i.e.*

$$DO \sin \theta = 5 \times \cdot 5 = 2 \cdot 5 \text{ ft.}$$

Similarly, in computing the reduced level of D, the usual procedure is to add or subtract, as the case may be, the axial reading DO from the reduced level of O, instead of the more correct value of $DO \cos \theta$, *i.e.* DO_1.

The error is thus $DO\ (1 - \cos \theta)$, which again is negligible under ordinary circumstances.

Example.—If DO = 5·00 and $\theta = 4°$, the error in altitude, unless a correction is applied, is only 5·00 (1 − ·9976) = ·012 ft., while in the more exceptional case when $\theta = 30°$ the error is more appreciable, *i.e.* = 5·00 (1 − ·866) = ·67 ft.

Errors due to Incorrect Inclination of the Staff.—Though the staff is generally held vertically, the errors introduced by the incorrect inclination of the staff in each case are now considered.

Let θ = the inclination of the line of collimation.

s = the intercept on the staff when truly perpendicular to the line of collimation.

s_1 = the intercept on the staff when inclined at ϕ from the true position and away from the instrument.

s_2 = the intercept on the staff when inclined at ϕ from the true position but towards the instrument.

s' = the intercept on the staff when held vertically.

s_1' = the intercept on the staff when held at ϕ from the vertical and away from the instrument.

s_2' = the intercept on the staff when held at ϕ from the vertical and towards the instrument.

α = the angle subtended by half the intercept at the instrument.

For angles of depression s_1 and s_1' are the intercepts when the staff is inclined towards the instrument, and s_2 and s_2' when inclined away from it.

Staff Inclined.—If a point near the foot of the staff is bisected by the lower wire, for angles of elevation,

$$(a)\ \frac{s}{s_1}=\frac{\sin 90-\phi-\alpha}{\sin 90+\alpha} \quad \text{and} \quad (b)\ \frac{s}{s_2}=\frac{\sin 90-\phi+\alpha}{\sin 90-\alpha}$$

$$=\frac{\cos \alpha+\phi}{\cos \alpha} \qquad\qquad =\frac{\cos \alpha-\phi}{\cos \alpha}$$

$$\text{and } s_1=\frac{\cos \alpha}{\cos \alpha+\phi}\,.\,s \qquad \text{and } s_2=\frac{\cos \alpha}{\cos \alpha-\phi}\,.\,s.$$

The true horizontal distance CE (Fig. 8.8) is

$$\left(s\,.\,\frac{f}{i}+f+d\right)\cos\theta+\frac{s}{2}\sin\theta,$$

and in (a) the apparent horizontal distance is

$$\left(s_1\frac{f}{i}+f+d\right)\cos\theta,$$

which is too small by an amount

$$\frac{s}{2}\sin\theta-(s_1-s)\frac{f}{i}\cos\theta,$$

i.e.

$$\frac{s}{2}\sin\theta-s\,.\,\frac{f}{i}\cos\theta\left(\frac{\cos\alpha}{\cos\alpha+\phi}-1\right), \quad . \quad . \quad (28)$$

and in case (b) the apparent length is too small by an amount

$$\frac{s}{2}\sin\theta-s\,.\,\frac{f}{i}\cos\theta\left(\frac{\cos\alpha}{\cos\alpha-\phi}-1\right). \quad . \quad . \quad (29)$$

For small angles, *i.e.* until $\tan\theta > 2\frac{f}{i}\left(\frac{\cos\alpha}{\cos\alpha+\phi}-1\right)$ or $> 2\frac{f}{i}\left(\frac{\cos\alpha}{\cos\alpha-\phi}-1\right)$, the values given by equations (28 and (29) are negative, and the errors are consequently positive.

If a correction is made as explained above, the apparent distance will be in case (a)

$$\left(s_1\frac{f}{i}+f+d\right)\cos\theta+\left(\frac{s_1}{2}\sin\theta\right),$$

and the error is now positive and equal to

$$\left\{\frac{f}{i}\cos\theta+\frac{\sin\theta}{2}\right\}(s_1-s) \;=s\left(\frac{\cos\alpha}{\cos\alpha+\phi}-1\right)\left(\frac{f}{i}\cos\theta+\frac{\sin\theta}{2}\right), \quad (28a)$$

and in case (b) $$= s\left(\frac{\cos\alpha}{\cos\alpha - \phi} - 1\right)\left(\frac{f}{i}\cos\theta + \frac{\sin\theta}{2}\right). \qquad (29a)$$

Similarly the actual height of the staff station above the instrument axis is

$$\left(s\frac{f}{i} + f + d\right)\sin\theta - \frac{s}{2}\cos\theta,$$

while in case (a) the apparent height is

$$\left(s_1\frac{f}{i} + f + d\right)\sin\theta - \frac{s_1}{2},$$

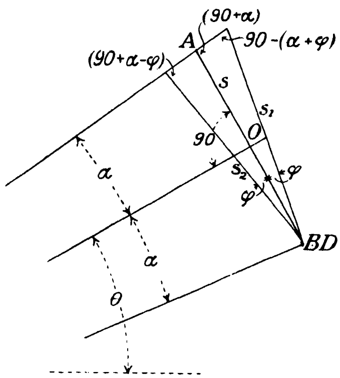


FIG. 8.9.—Inclined Staff

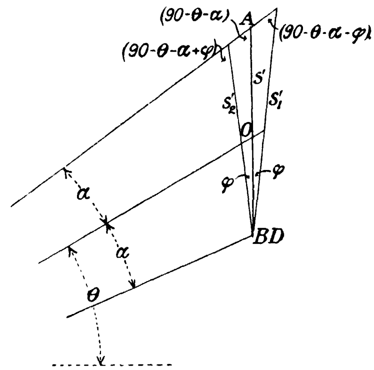


FIG. 8.10.—Vertical Staff

and the resulting error (positive) is

$$\frac{s}{2}\cos\theta + \frac{f}{i}\sin\theta\,(s_1 - s) - \frac{s_1}{2},$$

i.e. (a) $$s.\frac{f}{i}\sin\theta\left(\frac{\cos\alpha}{\cos\alpha + \phi} - 1\right) - \frac{s}{2}\left(\frac{\cos\alpha}{\cos\alpha + \phi} - \cos\theta\right), \qquad (30)$$

and in (b) $$s.\frac{f}{i}\sin\theta\left(\frac{\cos\alpha}{\cos\alpha - \phi} - 1\right) - \frac{s}{2}\left(\frac{\cos\alpha}{\cos\alpha - \phi} - \cos\theta\right). \qquad (31)$$

If, however, the more correct expression $\frac{s_1}{2}\cos\theta$ is deducted from v instead of $\frac{s_1}{2}$, the errors are

(a) $$s.\left(\frac{\cos\theta}{2} - \frac{f}{i}\sin\theta\right)\left(\frac{\cos\alpha}{\cos\alpha + \phi} - 1\right), \qquad (30a)$$

(b) $$s.\left(\frac{\cos\theta}{2} - \frac{f}{i}\sin\theta\right)\left(\frac{\cos\alpha}{\cos\alpha - \phi} - 1\right). \qquad (31a)$$

Each of these expressions is zero for $\theta = 17'$, and gives a negative error below and a positive error above this value.

Staff Vertical.—In order to compare the two conditions it will be necessary to express s' in terms of s.

By (20), using the new notation, $\frac{s}{s'}=\left(\cos\theta-\frac{\sin^2\theta\tan^2 a}{\cos\theta}\right)$,

and the ratio (*c*) $\frac{s'}{s_1'}=\frac{\sin 90-\theta-a-\phi}{\sin 90+\theta+a}=\frac{\cos(\theta+a+\phi)}{\cos(\theta+a)}$ and (*d*) $\frac{s'}{s_2'}=\frac{\sin 90-\theta-a+\phi}{\sin 90-\theta-a}=\frac{\cos(\phi-\theta-a)}{\cos(\theta+a)}$.

The error in length in a positive direction is, in (*c*)

$$(s_1'-s')\frac{f}{i}\cos^2\theta,$$

and in height $\quad (s_1'-s')\frac{f}{i}\sin\theta\cos\theta-\frac{1}{2}(s_1'-s')$,

or substituting for s' in terms of s, the positive error in length in (*c*) is

$$s\,.\,\frac{f}{i}\left(\frac{\cos\theta}{\cos^2\theta-\sin^2\theta\tan^2 a}\right)\left(\frac{\cos\theta+a}{\cos\theta+a+\phi}-1\right)\cos^2\theta, \quad . \quad (32)$$

and negative error in length in (*d*) is

$$s\,.\,\frac{f}{i}\left(\frac{\cos\theta}{\cos^2\theta-\sin^2\theta\tan^2 a}\right)\left(1-\frac{\cos\theta+a}{\cos\phi-\theta-a}\right)\cos^2\theta, \quad . \quad (33)$$

[The error in (*d*) is positive for values of θ less than $\left(\frac{\phi}{2}-a\right)$, at which angle the error is 0 and the error in (*c*) is negative for values of θ less than $\left(\frac{\phi}{2}+a\right)$, at which angle the error is 0.]

and the positive error in height in (*c*) is

$$\frac{s\,.\,\cos\theta}{\cos^2\theta-\sin^2\theta\tan^2 a}\left(\frac{f}{i}\sin\theta\cos\theta-\tfrac{1}{2}\right)\left(\frac{\cos\theta+a}{\cos\theta+a+\phi}-1\right), \quad (34)$$

and the negative error in height in (*d*) is

$$\frac{s\,.\,\cos\theta}{\cos^2\theta-\sin^2\theta\tan^2 a}\left(\frac{f}{i}\sin\theta\cos\theta-\tfrac{1}{2}\right)\left(1-\frac{\cos\theta+a}{\cos\phi-\theta-a}\right). \quad (35)$$

An approximation may be obtained by neglecting a which is $\tan^{-1}\frac{1}{200}=17'$ nearly—*i.e.* writing $\tan^2 a=0$, $\cos a=1$, and $\sin a=0$, when the above expressions will be much simplified.

Example.—The following table (Table I, p. 259) shows the values of the errors neglecting the sign, when $\phi=2°$ for different values of θ up to 30°, $s=4{\cdot}00$, and $\frac{f}{i}=100$. The lower stadia reading, as in the above equations, is assumed as nearly zero as possible.

A great number of the values shown are, of course, negligible, *e.g.* errors in length of 1 foot and under, as the errors of observation are of this order, and similarly with errors in height of ·01 or thereabouts.

It will be noticed that much better results are obtainable with the inclined staff, *i.e.* columns (28)-(31*a*).

TABLE I

(Analysis of Errors : See Example, p. 258)

θ	Staff Inclined								Staff Vertical			
	Distance		Height		Distance		Height		Distance		Height	
	(28)	(29)	(30)	(31)	(28a)	(29a)	(30a)	(31a)	(32)	(33)	(34)	(35)
0°	·32	·18	− ·002	− ·001	·32	·18	− ·002	− ·001	− ·16	+ ·18	− 002	− ·001
			0 at 17′	0 at 17′			0 at 17′	0 at 17′	0 at 1°-17′	0 at 43′		
2°	·25	·11	·008	·004	·32	·18	·009	·005	·80	− ·32	·024	− ·01
5°	·14	·00	·018	·007	·32	·17	·025	·014	1·54	− 1·05	·126	− ·086
	0 at 9°-6′	0 at 5°-2′		0 at 9°-47′								
10°	− ·04	− ·17	·023	− ·001	·31	·17	·053	·030	2·75	− 1·76	·471	− ·385
			0 at 17°-27′									
20°	− ·39	− ·52	− ·013	− ·061	·30	·17	·106	·060	5·15	− 4·58	1·845	− 1·643
30°	− ·72	− ·85	− ·112	− ·181	·27	·15	·157	·087	7·45	− 6·79	4·250	− 3·840

Method of Procedure.—The usual method of carrying out a tacheometrical survey is as follows :

The instrument is set up and levelled at some station P, from which a number of sights can be taken to prominent or governing positions and objects at the commencement of the survey ; and the height of the instrument axis above the peg at P is accurately measured.

The zero of the vernier is adjusted to and clamped at the zero of the horizontal circle of the instrument, which is then turned about its lower or outer axis until the telescope lies in the magnetic meridian as indicated by the compass needle, or in the true meridian as determined by astronomical bearings, or is directed to any other arbitrary point chosen as a referring object. The instrument is then clamped and the bearings of all other points referred to this meridian as a datum. Usually the northerly direction (magnetic or astronomical) is taken as representing a bearing of 360° (*i.e.* zero), but in American practice the south is often referred to as the origin.

The telescope is then directed to a bench mark, the reduced level of which has been previously determined, and the horizontal bearing, the vertical angle of elevation or depression, and the axial and stadia readings are observed. From these observations the position and reduced level of P can be computed as already explained.

From P a number of " side shots " are taken to prominent points and objects within range, and similar data to the above are booked for each.

To plot a complete plan from the stadia observations only would require an enormous number of readings, so that to facilitate the work a great deal of information and detail should be noted and sketched in the field book.

On these sketches the positions of the various staff stations should be indicated, and the stadia readings supplemented by chain and tape measurements in the case of buildings, widths of roads, etc.

When all the required " side " sights have been taken from P, a final complete set of observations is made to the next instrument station Q, and the instrument moved to and set up at Q.

The vernier of the horizontal scale is kept clamped at the reading which from P recorded the bearing PQ ; the telescope is transitted and directed back to the staff held at P—rotating the instrument about its lower axis and adjusting with the lower clamp and tangent screw.

The stadia readings and the vertical angle are then recorded, and the height of the instrument axis above Q measured. From these observations the distance from P to Q and the reduced level of Q can be calculated *and checked.* The telescope is then transitted to its normal position and a further number of side shots taken from here before proceeding to the next station R.

The method of procedure is thus similar to the " Fast-needle " method of traversing, described in Chapter V. If the telescope cannot be transitted, after directing back to P, with the vernier still fixed at the bearing of Q *from* P, the same results may be obtained

by reading all the bearings required from Q upon the opposite vernier to that used at P.

Example.—In the example in Table II, on page 263, the data in columns 1-6 and 10 are obtained from observations in the field, while the remaining columns are filled in by calculation. The reduced level of the B.M., *i.e.* 86·60, is known from a map or from previous surveys.

The calculations for columns 7, 8, and 9 are similar to those already described, and will afford a useful exercise for the student, while the following notes will explain the object of the remaining columns :

The height of the first station above the instrument axis (⊼) is 19·50 − 3·61 = 15·89 feet, and consequently this figure is entered in the " Rise " column : the height of the instrument axis is therefore 86·60 − 15·89 = 70·71 ft. above datum. From this figure the altitude of the second point is deduced—*i.e.* 70·71 − 2·43 = 68·28 ft. The third staff station is 16·99 + 4·87 = 21·86 feet *below* the instrument axis, and hence its reduced level is 70·71 − 21·86 = 48·85 ft. above datum. The value 21·86 is entered in the " Fall " column as shown.

Similarly the staff station B is found to have a reduced level of 70·71 + 20·96 = 91·67, hence when the instrument is set up there, the reduced level of the axis is 91·67 + 4·82 = 96·49 ft., and the height of the staff station A is 96·49 − 30·80 = 65·69 ft. above datum. As a check the reduced level of A calculated from the first line is 70·71 − 5·01 (*i.e.* the height of the instrument at A) = 65·70 ft., which agrees nearly with the value 65·69 obtained above.

Modifications.—Several modifications of the usual method described above have been suggested :

(1) Mr. Dempster [1] in a preliminary traverse for a railway in the Eastern Transvaal used an ordinary 5-inch Troughton-Simms tacheometer, with the extremity of one of the vertical circle vernier arms extended about 35° above and below the horizontal. This arc was engraved in both directions from the horizontal, with a sin-cos scale, while upon the vertical circle was an arrow mark. The opposite vernier arm was provided with a vernier as usual, for reading angles of elevation or depression, but was not used for the tacheometric work.

When the telescope was horizontal, the arrow was in coincidence with the zero of the scale ; and when a reading 0·03, say, was obtained, the telescope was inclined at an angle whose sin × cos = 0·03.

Parallel and attached to the side of the usual telescope was a supplementary tube fitted with a prismatic lens. The operator was thus enabled to sight through the tube with his left eye, and bring the arrow mark into conjunction with any convenient graduation on the sin-cos scale, while at the same time with his right eye he sighted through the main telescope to the staff and read the three web intercepts.

That is, instead of fixing the vertical circle vernier to an even angle, or fixing the lower stadia web at an even foot, a convenient value of sin cos was obtained, and the staff readings taken which then coincided with the cross-hairs. Thus in formula 23 the work of reduction is much simplified.

Example.—If the stadia readings were 12·28, 7·51, 2·75, and the sin-cos scale reading was −·02, then the vertical component $v = -·02 \times 100\ [12·28 - 2·75] = -2 \times 9·53 = -19·06$ ft., and the reduced level of the staff station, if that of the instrument axis was 3781·0, was 3781·0 − 19·06 − 7·51 = 3754·4 ft. nearly.

[1] *Proc. Ins. C.E.* vol. clxxxvii. p. 241, " Notes on a System of Surveying with a Tacheometer ".

In the same paper Mr. Dempster describes a set of scales for executing the calculations mechanically; by the use of these he dispenses with a great deal of the drudgery of the reductions. There is also a short table giving the amount per cent to be deducted from $100 \times s$, for different divisions of the sin-cos arc, in order to compute the horizontal distance (h) from the instrument to the staff station. For the majority of sights, the correction is negligible unless the vertical angle is of considerable magnitude.

It may be interesting to note that in the total length of 115 miles of this survey, 41,248 points were observed, which is equivalent to 360 points per mile. On the heaviest section, however, 18 miles in extent, the average number was over 500 per mile.

(2) In America, *e.g.* on the Minnesota River Survey,[1] where approximately 175 points per square mile were observed, the values of small elevations have been ascertained by the "Interval" method. In this method the telescope is first truly levelled and some point is noted on the landscape, which coincides with the central web of the diaphragm. By means of the vertical circle tangent screw, the telescope is next tilted until the lower stadia is raised to this point. A point coinciding with the upper wire is now noted, and the lower web brought up to this, the process being repeated, and the telescope tilted more and more until both webs are seen to intersect the staff. That is, the vertical distance from the horizontal to the final intercept on the staff is measured in terms of a number of stadia intervals, and consequently the elevation of the staff station is approximately (height of instrument axis) + (number of stadia intervals × value of one interval) − lower web reading.

Or, if the lower web is raised to occupy the original position of the upper instead of the middle web in the first instance, the axial reading is subtracted instead of the lower web reading. Angles of depression may be similarly measured.

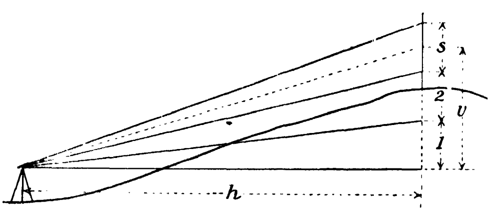


FIG. 8.11.—"Interval" Method

Example.—If in Fig. 8.11 the stadia interval $s = 9{\cdot}5$ ft. say, and the telescope has been tilted through two such intervals, then the reduced level of the staff station is

$$86{\cdot}5 + (2 \times 9{\cdot}5) - 1{\cdot}5 = 104{\cdot}0 \text{ ft.}$$

if 86·5 is the reduced level of the instrument axis and 1·5 and 11·0 the final staff readings.

For the small angles to which this method is applicable the horizontal distance is $100 \times s \times \cos\theta$ approximately, where $\theta = 3 \times$ the angle whose chord is $\frac{1}{100} = 3 \times 34'\text{-}23''$, *i.e.*

$$h = 950 \times \cos 1°\text{-}43'\text{-}9'' = 950 \text{ ft. nearly.}$$

[1] *Engineering News*, vol. lxiv. No. 9.

TABLE II

Tacheometric Field Book. (Example—see p. 261) $\frac{f}{i}=100, \quad f+d=1.$

Instrument Station	Staff Station	Height of Instrument Axis	Azimuth	Vertical Angle θ	Stadia Readings	Distance along Line of Collimation	Horizontal Distance h	Vertical Component v	Axial Reading	Rise	Fall	Reduced Level: Instrument Axis	Reduced Level: Staff Station	Remarks
A	$\overline{\Lambda}$	5·01	65°-30′	+ 3°-28′	5·22 / 2·00	322·4	322	19·50	3·61	15·89		70·71	86·60	
	①		2°-14′	0°-00′	3 87 / 1·00	288	288	0	2·43		2·43		68·28	
	②		16°-48′	− 2°-36′	6·74 / 3·00	374·6	374	− 16·99	4·87		21·86		48·85	
	⋮ B		198°-20′	+ 4°-18′	5·28 / 2·00	328	327	24·60	3·64	20·96			91·67	
B	A	4·82	18°-20′	− 4°-24′	7·28 / 4·00	328	327	− 25·16	5·64		30·80	96·49	65·69	

This method is, of course, only suitable for very small angles, and for heights of more than three intervals a correction must be made to allow for the increase in the stadia intercept as the slope increases.

Mr. Meyer [1] gives as an approximate rule : " When more than three intervals are read, reduce their rod equivalent by ·1 per cent of itself, for each interval above three ".

It is claimed that, when the inclination is small, this method is quicker and more accurate than the usual method of observing the vertical angles. The calculations are also simplified.

The Subtense Theodolite.—The principle of this type of instrument is the same as that of the tacheometer already described, except that for each reading the cross-hairs are adjusted by means of finely threaded micrometer screws to intercept some constant distance s on the staff.

That is to say, for horizontal sights the formula

$$D = s \cdot \frac{f}{i} + f + d$$

still holds, but s has a constant value—usually 10 ft.—while i or $\frac{f}{i}$ is variable.

The distance apart of the webs (i) is measured in terms of the pitch (p) of the micrometer screws, the whole number of turns being indicated on a scale in the field of view, while the decimal parts of a revolution are indicated on the two micrometer drums which are situated one above and one below the eye-piece of the telescope.

An expression for $\frac{f}{i}$ in terms of the micrometer scale can be best determined experimentally, by measuring $(f+d)$ on the instrument itself, chaining out distances D_1 and D_2 ft., and taking micrometer readings m_1 and m_2 for $s=10$ ft. say. That is, the staff has two fixed vanes, *e.g.* discs marked with a black cross on a white ground, spaced 10 ft. apart, and the subtense or stadia webs are manipulated by means of the micrometer screws until this distance is exactly intercepted by the two wires.

The reading m expresses the number of revolutions and parts of a revolution of the screw and the corresponding movement of the webs through a distance of m multiplied by the pitch, *i.e.* mp.

Or if there is an index error e, then $i=(m_1-e)\,p$ in the first case and $(m_2-e)\,p$ in the second.

Then $$D_1 - (f+d) = \frac{s \cdot f}{(m_1 - e)p} \quad . \quad . \quad . \quad (36)$$

and $$D_2 - (f+d) = \frac{s \cdot f}{(m_2 - e)p}, \quad . \quad . \quad . \quad (37)$$

so that $$\{D_1 - (f+d)\}(m_1 - e) = \{D_2 - (f+d)\}(m_2 - e),$$

from which e may be determined.

[1] *Engineering News*, vol. lxiv. No. 9.

Then by substitution in (36)

$$\frac{f}{p}=\frac{\{D_1-(f+d)\}(m_1-e)}{s}=c \text{ say}, \qquad (38)$$

where c is the coefficient of the instrument for an intercept of s on the staff.

The general formula then becomes

$$D=\frac{c \,.\, s}{m-e}+(f+d), \qquad (39)$$

or if there is no index error,

$$D=\frac{cs}{m}+(f+d). \qquad (40)$$

The values of c and e should be constant for any other values of D unless the screw is cut inaccurately, and with a screw of 50 threads to 1 inch, and a focal length of 12 inches, c has a value of about $\frac{1}{600}$. Inclined sights are calculated by equations (21)-(27), exactly as in the case of the telemetric telescope before described.

The micrometer wires are often capable of being turned into a vertical position, so that the graduated staff can be read horizontally instead of vertically.

In this case the micrometer drums are brought to the right and left sides respectively of the eye-piece, and the modified formulae for inclined sights are then

$$h=\left\{\frac{c \,.\, s}{m}+(f+d)\right\}\cos\theta, \qquad (41)$$

and

$$v=\left\{\frac{cs}{m}+(f+d)\right\}\sin\theta, \qquad (42)$$

instead of the formulae given in equations (21)-(27).

When the staff is to be used horizontally, it should be provided with two sights to enable the staffman to place it exactly at right angles to the line of sight from the telescope. The reduction of the readings of the subtense theodolite, it will be seen, is much more laborious than for the usual tacheometer, as the reading m is in the *denominator* of the fraction $\frac{cs}{m}$. The field work for long sights is, however, probably more accurate, as it is much easier to intersect a fixed mark or vane with a movable wire than it is to read the staff graduations. Very accurate results can and have been obtained with this instrument, but the ordinary type is much to be preferred for general work.

Example.—The constant for an instrument being 610, the value of $(f+d)=$ 1·5 ft., and the intercept $s=10$ ft. Calculate the distance from the instrument to the staff when the micrometer readings are 4·238 and 4·242.

Here $m=4{\cdot}238+4{\cdot}242=8{\cdot}480,$

$$\therefore D=\frac{610\times 10}{8{\cdot}480}+1{\cdot}5=721 \text{ ft.}$$

Tangential Method.—In the tangential method of tacheometry the horizontal and vertical distances from the instrument to the "staff" station are computed from the observed vertical angles to two vanes fixed at a constant distance "s" apart upon the staff. Thus if C (Fig. 8.12) represents the axis of the instrument, D the staff station, A and B the two vanes, and α and β the two angles of elevation to the vanes A and B respectively, then the horizontal distance

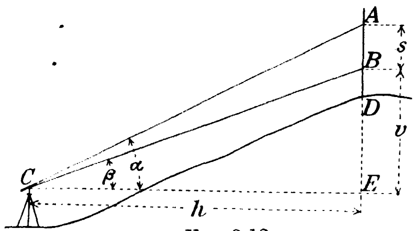


FIG. 8.12
Tangential Method of Tacheometry

$$CE = EB \cot\beta = EA \cot\alpha. \qquad (43)$$

Therefore if

$$EB = v \text{ say,}$$

$$v \cot\beta = (v + s)\cot\alpha,$$

$$\therefore\ v(\cot\beta - \cot\alpha) = s\cot\alpha,$$

or

$$v = \frac{s\tan\beta}{\tan\alpha - \tan\beta}, \qquad (44)$$

and from (43)

$$CE = h = v\cot\beta,$$

$$h = \frac{s}{\tan\alpha - \tan\beta}. \qquad (45)$$

When α is an angle of elevation and β an angle of depression, or when both α and β are angles of depression, formulae (44) and (45) still apply if the correct signs for $\tan\alpha$ and $\tan\beta$ are used.

Formulae (44) and (45) may also be transformed into expressions which are more convenient for logarithmic computations. (See Example 9, p. 286.)

Example.—The vertical angles to vanes fixed at 2 ft. and 12 ft. above the foot of the staff held vertically at a station D were 2°-14′ and 5°-36′ respectively.

Deduce the horizontal distance, and the reduced level of D if the height of the instrument was determined from observations on to a bench mark to be 137·14 ft. above datum.

By substitution in formula (45)

$$h = \frac{10}{\tan 5°\text{-}36' - \tan 2°\text{-}14'}$$

$$= \frac{10}{\cdot 09805 - \cdot 03900} = 169{\cdot}3 \text{ ft.,}$$

while

$$v = h\tan 2°\text{-}14' = 6{\cdot}60 \text{ ft.}$$

The reduced level of D is therefore

$$137{\cdot}14 + 6{\cdot}60 - 2{\cdot}00 = 141{\cdot}74 \text{ ft.}$$

Had the angle to the lower vane been an angle of depression

$$\alpha = +5°\text{-}36',\ \beta = -2°\text{-}14', \text{ then}$$

$$h = \frac{10}{0{\cdot}09805 + 0{\cdot}03900} = 73 \text{ ft. nearly,}$$

$$v = 73 \times -0{\cdot}03900 = -2{\cdot}85 \text{ ft.,}$$

so that the reduced level of D would have been

$$137{\cdot}14 - 2{\cdot}85 - 2{\cdot}00 = 132{\cdot}29 \text{ ft.}$$

Had both angles been depression, *i.e.* $\alpha = -2°\text{-}14'$ to the upper vane and $\beta = -5°\text{-}36'$ to the lower vane,

$$h = \frac{10}{-0{\cdot}03900 - (-0{\cdot}09805)} = 169{\cdot}3 \text{ ft.},$$

$$v = h \tan -5°\text{-}36' = -16{\cdot}60 \text{ ft.},$$

and as can be seen from a sketch, the reduced level of D would have been

$$137{\cdot}14 - 16{\cdot}60 - 2{\cdot}00 = 118{\cdot}54 \text{ ft.}$$

Accuracy.—The degree of accuracy which is obtainable in the measurement of linear distances by tacheometry depends not only upon the length of sight but also upon the power of the telescope, the type of stadia rod, the type of diaphragm webs, etc., and upon the inclination of the line of collimation. If an ordinary levelling staff divided into ·01 ft. graduations is employed, it is usual to read any black division upon which the hair-line falls as even and any white division as odd, so that the limiting error should be about 0·01 ft. for each web reading, and the probable error would be equal to the average error of $\pm 0{\cdot}005$ ft.

The value of the intercept might be taken to have a p.e. of $\pm 0{\cdot}005 \sqrt{2} = \pm 0{\cdot}007$ ft. say, which corresponds to an error in the distance L of $\pm 0{\cdot}70$ ft. For very short sights, when the graduations are very clearly visible, this error might easily be halved if the divisions were estimated to ·005 ft. There are, however, other sources of error, *e.g.* a small inaccuracy in the value $\frac{f}{i}$ or in the additive constant $(f + a)$; a small displacement of the staff from its exact vertical position. Some of these errors would be cumulative and others compensating. The chief source of error mentioned above would appear to be independent of the length of sight, provided that the distance was not so great as to render the graduations very indistinct. Above this limit of distance the error would probably be very erratic.

Mr. R. E. Middleton,[1] M.Inst.C.E., found by experiment that with the particular instrument he employed, the limit of distance was about 800 ft., but for an ordinary theodolite fitted with stadia webs a much lower limit, *e.g.* 400 ft., would be expected.

Some of the results he obtained, reading the three hair-lines, are given on p. 268. The staff was graduated to ·01 ft., and provided with a small spirit level to ensure verticality.

If the values of the average error be plotted it will be seen that they lie very approximately upon a straight line for distances up to 800 ft. so that, contrary to expectations, the average error appears to vary directly as L, according to the equation av.e. $= \pm 0{\cdot}0024$ L.

Thus in a stadia traverse of N sides each of length L feet, a closing

[1] *Proc. Inst. C.E.* vol. cxvi.

error—assuming the angular dimensions to be correct—might be expected of $0{\cdot}0024\ L\,.\sqrt{N}$, though if the usual practice is adopted of

Length of Line	0 to 100 ft.	100 to 200 ft.	200 to 400 ft.	400 to 600 ft.	600 to 800 ft.	800 to 1000 ft.	Above 1000 ft.	Total
Number of lines	5	13	41	39	31	10	1	140
Average length per line in feet	55·94	163·01	328·88	498·96	676·97	912·49	1136·19	475·64
Average error per line	0·20	0·57	0·72	1·22	1·51	2·82	2·07	1·16

observing one backward and one forward reading between each station, this might be reduced in the ratio of $1 : \frac{1}{\sqrt{2}}$ so that the average error due to errors in length would be then about $0{\cdot}0017\ L\sqrt{N}$.

Comparing this with the results obtained by good chainage with an av.e. of say 1 in 2000 (the limiting error being 1 in 1000 say), the accuracy of the two methods would be about equal for sights of 400 ft., if the traverse had about 12 sides, *i.e.*

$$0{\cdot}68\sqrt{N} = {\cdot}0005\,.\,400\,.\,N.$$

For inclined sights additional errors would be introduced owing to small errors in reading the vertical angle, and in the holding of the staff.

For inclined sights, by the differentiation of equation (22), neglecting the last term, since $(f+d)$ is small and the error induced in it negligible,

$$h = s\,.\frac{f}{i}\cos^2\theta + (f+d)\cos\theta,$$

$$\frac{\delta h_1}{h} = \frac{\delta s}{s}, \qquad . \quad . \quad . \quad . \quad (50)$$

$$\delta h_2 = -s\,.\frac{f}{i}\,2\cos\theta\sin\theta\,.\,\delta\theta,$$

$$\frac{\delta h_2}{h} = -2\tan\theta\,\delta\theta, \qquad . \quad . \quad . \quad (51)$$

$$\therefore \frac{\delta h}{h} = +\sqrt{\left(\frac{\delta s}{s}\right)^2 + 4\,(\tan\theta\,\delta\theta)^2}. \qquad . \quad . \quad (52)$$

In this expression the value of $\frac{\delta s}{s}$ depends upon

(1) The inaccuracy in reading—as considered for level sights ;
(2) The inaccuracy in holding the staff—*vide* table, p. 259.

Other errors which enter into inclined sights have also been mentioned in the previous pages.

Space will not allow of a further examination into these errors, but

the student will be able to compare results for inclined sights by assuming a p.e. in the vertical angle of say $\pm 10''$, and in the verticality of the staff of say $\pm 30''$ to $1°$.

The Subtense instrument, fitted with a micrometer eye-piece, should give rather more accurate results than the telemetric telescope considered above.

The Tangential Method.—For the tangential system, the error may be studied as follows :

Let δh_1 be the error in length due to an error δs in the distance apart of the vanes.
δh_2 be the error in length due to an error $\delta \alpha$ in the angle α.
δh_3 be the error in length due to an error $\delta \beta$ in the angle β.

Then as $h = \dfrac{s}{\tan \alpha - \tan \beta}$,

$$\delta h_1 = \frac{\delta s}{\tan \alpha - \tan \beta} \quad \text{or} \quad \frac{\delta h_1}{h} = \frac{\delta s}{s}. \qquad (53)$$

Also

$$\delta h_2 = -\frac{s \cdot \sec^2 \alpha \cdot \delta \alpha}{(\tan \alpha - \tan \beta)^2} \quad \text{or} \quad \frac{\delta h_2}{h} = -\frac{\sec^2 \alpha\, \delta \alpha}{\tan \alpha - \tan \beta}. \qquad (54)$$

$$\text{or} \quad -\frac{\sec^2 \alpha \cdot \delta \alpha \cdot h}{s}. \qquad (55)$$

Also

$$\delta h_3 = \frac{s \cdot \sec^2 \beta \cdot \delta \beta}{(\tan \alpha - \tan \beta)^2} \quad \text{or} \quad \frac{\delta h_3}{h} = \frac{\sec^2 \beta \cdot \delta \beta}{\tan \alpha - \tan \beta} \qquad (56)$$

$$\text{or} \quad \frac{\sec^2 \beta \cdot \delta \beta \cdot h}{s}, \qquad (57)$$

or, if $\pm \delta s \pm \delta \alpha$ and $\pm \delta \beta$ are probable (or average) errors, the p.e. (or av.e.) in h, *i.e.*

$$\delta h = \pm \frac{h}{s}\sqrt{(\delta s)^2 + (h \sec^2 \alpha\, \delta \alpha)^2 + (h \sec^2 \beta\, \delta \beta)^2}. \qquad (58)$$

The intercept s may, as a rule, be considered fairly correct except in so far as it is affected by the inclination of the staff from its correct position, so that, assuming the resulting error $\pm \delta h$ is chiefly due to the errors $\pm \delta \alpha$ and $\pm \delta \beta$, and that $\delta \alpha = \delta \beta$,

$$\delta h = \pm \frac{h^2 \delta \alpha}{s}\sqrt{\sec^4 \alpha + \sec^4 \beta}. \qquad (59)$$

Thus, for example, if $h = 400$ ft. $\alpha = -\beta = 43'$, and $s = 10$ ft., a p.e. in distance might be expected of ·54 ft., when $\delta \alpha = \delta \beta = \pm 5$ seconds say, *i.e.*

$$\delta h = \frac{(400)^2}{10} \cdot \cdot 000024 \times 1{\cdot}4145 = \cdot 54 \text{ ft.}$$

If $\delta \alpha = \pm 10$ seconds this error would be doubled, *i.e.* 1·08 ft. in 400.

The figures [1] tabulated below are due to experiments by Mr. R. E. Middleton, a single FR and FL observation being taken on a 5-inch

[1] *Proc. Inst. C.E.* vol. cxvi.

theodolite graduated to 20′ for each point (*i.e.* the angles are the average of 4 readings). The conclusions drawn from this and the previous table are not quite the same as those deduced by Mr. Middleton.

The theoretical expression (equation 59) indicates that the resulting

Length of Line	0 to 100 ft.	100 to 200 ft.	200 to 400 ft.	400 to 600 ft.	600 to 800 ft.	800 to 1000 ft.	Above 1000 ft.	Total
Number of lines	5	12	32	28	20	10	13	120
Average length of line in feet	55·94	164·23	323·42	498·36	682·23	902·60	1388·42	560·62
Average error per line	0·16	0·15	0·84	1·91	3·13	4·28	17·17	3·43

error is proportional to h^2, as for horizontal sights the value of $(\sec^4 \alpha + \sec^4 \beta)$ will differ only very slightly from 2.

This result is roughly confirmed by Mr. R. E. Middleton's experiments quoted above, as may be seen by plotting the average errors against the squares of the lengths.

An approximate formula may be deduced from the straight line which coincides most nearly with the plotted points, *i.e.*

$$\text{av. error} = \pm 0{\cdot}00000785\, h^2, \qquad (60)$$

where h is the length of sight in feet. This result is equivalent to an average error of $\pm 11{\cdot}5$ secs in α and β.

If a micrometer theodolite is employed the value of the coefficient of h^2 will be much reduced.

The results tabulated do not show that there is a limit, above which the formula is unreliable, as the last point (*i.e.* $h = 1388{\cdot}42$) falls moderately close to the mean straight line of the graph.

A comparison of the two systems shows that rather more accurate results were obtained by the tangential system for distances up to 400 ft., but that above this distance the stadia method was more reliable.

The following results are taken from actual stadia surveys :

(1) On the U.S. Lake [1] Survey, where the limit of error was fixed at 1 in 300, the average error on 141 lines was found to be 1 in 650. The average length of sight was 800 to 1000 ft.

(2) On the Mexican Boundary Survey,[2] the error in distance upon a number of lines was as follows :

Number of Lines	Length in Miles	Error
1	12·3	1 in 28500
1	13·2	1 ,, 765
1	8·1	1 ,, 600
1	13·5	1 ,, 700
1	14·0	1 ,, 3214
1	61·1	1 ,, 1116
1	67·0	1 ,, 2200

[1] Johnson, *Theory and Practice of Surveying.*

[2] *Engineering News*, " The Stadia and Stadia Surveying ", vol. lxiii. No. 17

Chain measurements over parts of the same lines gave errors varying from 0 to 1 in 827 from the triangulation distances—the average discrepancy being about 1 in 1440.

(3) In a Stadia Traverse with a 30-inch Bausch and Lomb instrument, round Mille Lac Lake,[1] which is 73·2 miles in circumference, the closing errors were 144 ft. Eastings, and 106 ft. Southing, or 179 ft. total. The azimuths were checked to Polaris every 10 miles or less.

Tacheometric Levelling.—*Stadia Method.*—By the differentiation of equation (23*a*), neglecting the last term as the error induced in $(f+d)$ must be inappreciable.

$$v = s \,.\, \frac{f}{i} \frac{\sin 2\theta}{2},$$

$$\frac{\delta v_1}{v} = \frac{\delta s}{s}, \qquad (61)$$

$$\frac{\delta v_2}{v} = 2 \cot 2\theta \,.\, \delta\theta, \qquad (62)$$

and

$$\delta v = \pm v \sqrt{\left(\frac{\delta s}{s}\right)^2 + (2 \cot 2\theta \; \delta\theta)^2}, \qquad (63)$$

where $\pm \delta v$, $\pm \delta s$, and $\pm \delta\theta$ are the probable errors in v, s, and θ respectively.

For horizontal sights $\theta = 0$ and $v = 0$, and the errors are the same as in ordinary levelling operations, p. 219.

For inclined sights the value of $\frac{\delta s}{s}$ depends upon

(i) The error in reading the axial web (p.e. = ± ·005 say);

(ii) The error introduced if the staff is not held correctly (see p. 255).

Other sources of error due to approximations, etc., have also been discussed previously.

Tangential Method.—By differentiating equation (44), p. 266, *i.e.*

$$v = \frac{s \tan \beta}{\tan \alpha - \tan \beta},$$

and proceeding as in the above examples we may show that

$$\frac{\delta v_1}{v} = \frac{\delta s}{s}, \qquad (64)$$

$$\frac{\delta v_2}{v} = -\frac{\sec^2 \alpha \,.\, \delta\alpha}{(\tan \alpha - \tan \beta)}, \qquad (65)$$

$$\frac{\delta v_3}{v} = \frac{\tan \alpha \sec^2 \beta \,.\, \delta\beta}{\tan \beta \,.\, (\tan \alpha - \tan \beta)}. \qquad (66)$$

Thus if $\pm \delta s$, $\pm \delta\alpha$, $\pm \delta\beta$ are the p.e.'s in s, α, and β respectively the p.e. in v

$$\delta v = \pm v \sqrt{\left(\frac{\delta v_1}{v}\right)^2 + \left(\frac{\delta v_2}{v}\right)^2 + \left(\frac{\delta v_3}{v}\right)^2}. \qquad (67)$$

[1] *Ibid.* "The Mille Lac Lake Survey", vol. lxiii. No. 14.

An example is given (No. 6, p. 285) to be worked by the student.

In traverse surveying it is usual to make backward and forward observations except for side shots, so that for main stations the p.e. would be $\pm\frac{\delta v}{\sqrt{2}}$, and at the end of a traverse of N stations the resulting error would be $\pm\frac{\delta v}{\sqrt{2}}\sqrt{N-1}$.

The following results are taken from actual stadia surveys :

1. On the Mexican Boundary Survey,[1] the average closing error in height on 14 traverses of an average length of 8·2 miles was 0·17 ft.

As might be expected, however, the error was considerably greater when the average vertical angle increased, so that on 17 traverses of 4·1 miles average length in rough country the closing error was ·59 ft. in height.

2. On the St. Louis Topographical Survey [1] the closing error in height in 40 miles of traverse was 0·64 ft. At 20 miles from the commencement the error was nil ; at 27 miles it was double the final closing error.

3. In a Survey in British Guiana,[2] the maximum closing error in stadia levels over 24 miles of traverse was ± 4 ft.

In this case the height of the instrument—a 4-in. transit—was measured at every station, and a vertical angle to the nearest minute observed to this height on a 14-ft. Sopwith staff repainted in link units. For sights up to 15°, the staff was held vertically, a small hand level being used for adjustment. Above 15°, duplicate readings were taken (1) with the staff vertical, (2) with the staff perpendicular to the line of sight. The first was used for the reduction of levels, and, in a comparison between the two for distances, preference was given to the latter.

4. On subsidiary transit and stadia traverses for topography on the Ohio River Survey,[3] an allowance of 0·4 ft. per mile was allowed in levels, but the actual closure was much less.

Eckhold's omnimeter is very similar to a theodolite in construction, and may be used as such, but the addition of a microscope at right angles to the telescope enables distances and elevations to be deduced from observations upon a staff.

The axis of the microscope intersects exactly the horizontal transverse axis of the telescope, so that when the line of collimation of the telescope is horizontal the microscope tube is vertical, and any movement of rotation of the telescope in a vertical plane is accompanied by a corresponding movement of the microscope.

On the upper scale plate of the instrument, and between the supports, is a straight finely divided horizontal scale, 4 in. in length, upon which the microscope is focused.

This scale is fitted in a slide and is capable of movement through a small distance in either direction by means of a micrometer screw having 50 threads to the inch.

[1] Johnson, *Theory and Practice of Surveying*.
[2] *Proc. Inst. C.E.* [3] *Engineering News*, vol. lxxi. No. 12.

The principle of the instrument may be explained by reference to Fig. 8.13.

The telescope at O is directed to a graduated staff, or to a rod provided with vanes b and c at a definite distance apart, held at some point A, the position of which it is required to locate.

Then a represents the position at which the horizontal line of collimation would cut the vertical through A, and a_1 represents the point at which the microscope cross-hairs intersect the horizontal scale, Oa_1 being at right angles to Oa.

If the telescope be now tilted so that the cross-wires intersect the staff at some definite graduation or vane b, the microscope cross-wires will intersect the graduated scale at b_1, and similarly when the telescope is directed to c the microscope reading is at c_1, where Ob_1 and Oc_1 are perpendicular to Ob and Oc respectively.

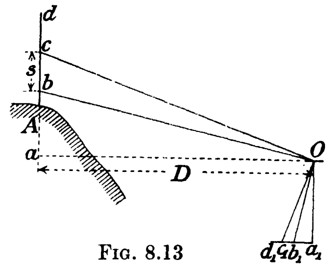


FIG. 8.13

The Principle of the Omnimeter

Either or both of the points b and c may be below a, when the vertical arc would register an angle of depression; and in such a case the corresponding readings b_1 or c_1 would be less than a_1.

From Fig. 8.13 it can easily be shown that

$$Oa = Oa_1 \,.\, bc/b_1c_1,$$

i.e.

$$D = Oa_1 \,.\, s/b_1c_1.$$

Thus as Oa_1 and s are constants, D, the horizontal distance to the staff, is inversely proportional to the intercept b_1c_1 on the microscope scale. Similarly the difference in level $ab = s \,.\, a_1b_1/b_1c_1$, so that it can be calculated from the observed quantities a_1b_1 and b_1c_1.

In previous editions of this book, the method of using the instrument was described and the adjustments and accuracy discussed. This matter is now omitted since the instrument has been superseded by more modern types of tacheometer, though many omnimeters must still be in use.

Example.—The readings on the microscope scale were

$$r_a = 50025$$
$$r_b = 58862$$
$$r_c = 61725$$

The intercept on the staff was 10 feet, the lower vane being 2 feet above the ground.

The height Oa_1, corresponded to 150,000 micrometer divisions. Then, referring to Fig. 8.13,

$$\begin{aligned} D &= 150{,}000\, s/(r_c - r_b) \\ &= (150{,}000 \times 10)/(61725 - 58862) \\ &= 524 \text{ ft. approx.} \\ ab &= 10 \times (r_b - r_a)/(r_c - r_b) \\ &= 10 \times 8837 \div 2863 \\ &= 30{\cdot}8 \text{ feet.} \end{aligned}$$

If the reduced level of the instrument axis is 86·60 feet, that of the staff station at A = 86·60 + 30·8 − 2·0 = 115·4 ft.

Range-Finding

There are several distinct methods by which the distance from an observer to a distant object may be determined. Each method depends upon the solution of a triangle having its altitude equal to the desired distance. This is sometimes solved directly, and sometimes by comparison with a similar triangle.

(1) In the ordinary tacheometer (p. 243) the triangle to be solved has its apex at a distance f in front of the object-glass (Fig. 8.4), or $f+d$ in front of the instrument station, and the angle at this point has a constant value $= 2 \tan^{-1}\frac{i}{2f}$.

The base (s) of the triangle is at the distant object and is the variable quantity which is to be measured, and from which the range is deduced.

(2) In the subtense method of tacheometry (p. 264) the base of the triangle, which has a constant value s, is at the distant object, while the angle at the apex of the triangle, at a distance f in front of the instrument station, is the variable, and has the value $2 \tan^{-1}\frac{i}{2f}$ where i is now the variable and the quantity to be measured.

(3) In the tangential method of tacheometry (p. 266) the base of the triangle is at the distant object, and has a constant value s, while the apex of the triangle is at the instrument station, and the angle there is measured directly upon the vertical circle of the instrument.

(4) With the omnimeter the conditions are similar to those in the tangential method of tacheometry, but the value of the apical angle of the triangle to be solved is determined from the reading on the horizontal scale. As in methods (1) and (2), however, the actual value of the angle is not deduced, but the triangle is solved by a comparison with the properties of a similar triangle.

The above-mentioned instruments are not usually included in the term range-finders, though they can be used to determine ranges. Most range-finders proper belong to one or other of the groups (5) or (6) mentioned below.

(5) With many of these instruments the object whose range is required is situated at the apex of the triangle to be solved, and a base of known length, s, is measured at the observer's position ; while the two base angles (one of which may be a right angle) are measured by the range-finding instrument or instruments employed. The instruments are usually graduated to give the range directly for a standard base. The telemeter and the mekometer are instruments of this class, and so too are the more elaborate military range-finders used to determine the altitude of hostile aircraft from the extremities of long base lines.

(6) In other instruments of the single observer type, such as in the Barr and Stroud range-finders, the base of the triangle to be solved is contained in the instrument itself, and its length accurately determined by the instrument makers.

One of the base angles is a right angle, and the amount that the other base angle differs from a right angle is measured by the observer. As the three angles of a triangle are equal to two right angles, this measurement gives the value of the angle subtended at the vertex of

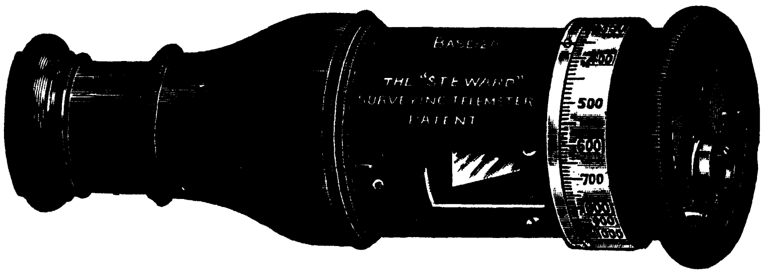


FIG. 8.14.—Telemeter

the triangle (*i.e.* at the observed object) by the instrument base. This angle is known as the parallax of the object. The magnitude of this angle cannot, of course, be measured directly, as could be done in the case of a long base line. Different instrument makers employ different devices for determining its value indirectly, or for determining the value of some other dimension which is proportional to it. In some instruments the stereoscopic principle is adopted, in others rotary end reflectors with a multiplying gear as in the Zeiss range-finder, in others a travelling prism as described below in the description of the Barr and Stroud instruments.

The **telemeter** as made by Messrs. Steward is shown in Fig. 8.14. It is about $4\frac{1}{2}$ in. long and $1\frac{1}{2}$ in. in diameter, and consists of a metal tube which encloses two mirrors corresponding to the index and horizon glasses of a sextant.

At one end of the tube is a small telescope, while at the opposite end is a graduated collar connected to the index mirror by means of a metal arm. The index mirror can be moved in azimuth by the rotation of this ring, and the horizon glass can also be turned through a few degrees by means of the small toothed wheel on the front of the instrument. The front of the tube is provided with a sliding shutter to keep out dust, etc.

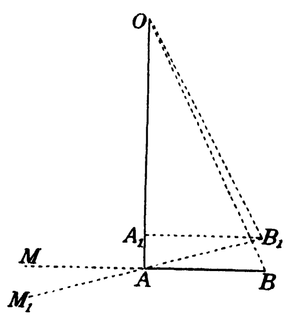


FIG. 8.15

The principle of the instrument depends upon the same law that was stated on p. 74 for the sextant, viz. that if a ray is successively reflected from the surfaces of two plane mirrors, the angle between the initial and final directions of the ray is twice the angle between the mirrors.

By the application of this principle, distances may be deduced as follows: Let O be an object, the range of which from A is required (Fig. 8.15). The zero of the scale on the graduated collar is made to

coincide with the fixed index mark on the body of the tube, and the arrow upon the small toothed wheel is brought into coincidence with the arrow mark upon the small fixed stop on the front of the instrument; if the instrument is in adjustment, the mirrors now include an angle of 45°. The observer stations himself at A, points the telescope in a direction AM approximately at right angles to AO, and turns until he sees the image of O in the lower part of the horizon glass after reflection from the index mirror, the ray passing through the rectangular opening in the side of the tube (Fig. 8.14).

Above this image O_1, say, through the front of the tube, he obtains by direct vision a view of the landscape, and notes any well-defined natural object M which coincides with O_1. If no such object is available, a ranging rod or other object may be ranged into the desired position.

The observer then moves to a second station B, which is in line with MA, and at a distance of 20 units (*e.g.* yards) from A. For accurate work this distance AB should be accurately taped, but for approximate ranges pacing may be sufficiently precise.

At B he directs the telescope to M and rotates the graduated collar—thus moving the index mirror—until the image of O again coincides with M. As with the sextant, the angle between the mirrors is now $\frac{1}{2}\angle OBA$, but instead of this angle being indicated on the scale, the graduations, though determined experimentally for each range of an instrument, are proportional to the tangents of $\angle OBA$, with reference to a base AB of 20 units, *i.e.* the distance AO is indicated directly on the scale.

Thus when $\angle OBA = \alpha$ degrees, the mirrors include an angle of $\frac{\alpha}{2}$ and the graduation opposite the fixed mark on the tube is $20 \tan \alpha$. The maximum reading on the scale is 1000 for a base of 20 units, so that for greater distances the length AB must be increased. Evidently the maximum range is $50 \times$ length of base, so that if AB $=50$ yds., ranges up to $50 \times 50 = 2500$ yds. may be observed, etc.

If the object O cannot be seen from the end of a 20-unit base, the length may be extended or diminished and the scale reading altered in the same ratio, *e.g.* if O is not visible from a point 20 yds. from A but may be seen from a point 25 yds. from A, then if the reading is 680, the range AO is $\frac{680 \times 25}{20} = 850$ yds. There are several modifications of the above method that may be adopted in practice.

If there is no natural distinctive object exactly at right angles to OA, but a convenient point M_1 is found to be nearly in such a position, the small wheel in the front of the instrument may be rotated until the image of O is made to coincide with M_1. The effect of this movement is to rotate the horizon glass, and, consequently, to alter the angle included between the two mirrors, without altering the reading on the scale, which only records movements of the index glass.

Thus when the observation is made from B_1, which is in the line

M_1A and at a distance of 20 units from A, the reading recorded is 20 tan $\angle OB_1A_1$, where B_1A_1 is parallel to BA, because the portion $\angle A_1B_1A$ of the angle between the rays OB_1, AB_1, which is equal to $\angle M_1AM$, has been taken up by the *horizon* glass and is not recorded.

But as $AB = AB_1$ and $\angle B_1AB$ is very small, *i.e.* not more than a few degrees, the displacement of B_1 from the line BO is very small.

Consequently, $\angle OB_1A_1$ is very nearly equal to $\angle OBA$, and the recorded reading is a close approximation to AO, *i.e.* 20 tan OBA.

An alternative method of obtaining a range is by sighting from A directly towards the object O, and by reflection finding or setting out the point M at right angles to AO (Fig. 8.16).

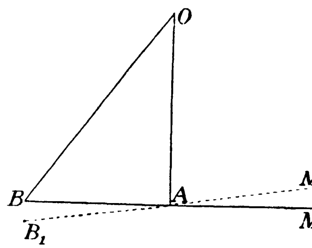


FIG. 8.16

The base AB, in the line MA produced, is then measured or paced according to the degree of accuracy required. O is sighted from B, and the graduated wheel rotated until the points A or M are seen by reflection to coincide with O. The range is indicated by the reading as before.

If desired, the range may be determined by setting out M at right angles to AO from A, viewing O by reflection as already explained, then fixing the scale at the 1000 reading and finding by trial a point B on MA produced, at which the image of O coincides with M. The distance AO is then $50 \times AB$.

Or if the point O is sighted directly from A, then at B the *image* of M is made to coincide with O as seen by direct vision, and AO = 50 AB as before.

Adjustment of the Telemeter.—A point M is accurately set out by any of the usual methods (*e.g.* box sextant, theodolite, etc.), so that AM is at right angles to AO.

The zero of the graduated collar is brought into coincidence with the fixed mark on the tube, and the arrow mark on the collar of the small toothed wheel is made to coincide with the fixed arrow on the front of the instrument.

The operator then stands directly over A, steadies the instrument against a ranging rod if he wishes, and directs the telescope to O, when the image of M should coincide with O on the horizon glass. If not, the instrument requires to be adjusted.

To complete the adjustment, if it is found necessary, the small toothed wheel is rotated until coincidence is obtained between M and O, when the mirrors should be inclined at 45°, the scale still reading zero. The arrow mark on the collar of the toothed wheel is thus moved relatively to the fixed arrow, and it is required to bring it back into coincidence without altering the relative positions of the mirrors. Consequently, the toothed wheel itself must not be disturbed, but by loosening a small set screw, the collar upon which is engraved the arrow mark may be rotated relatively to the wheel until the two

arrows again coincide ; the set screw is then tightened and the collar secured in position.

The operation should be repeated until the adjustment is perfect.

The **accuracy** of the results obtained depends very largely upon the care with which the length of the base is determined.

It is claimed that, with care, results giving a p.e. of ± 1 in 100 can be obtained.

The **Mekometer** was at one time used as a military range-finder, and though it has now been superseded for this purpose by a Barr and Stroud instrument (see Fig. 8.17) it is illustrative of a two-operator arrangement. It consists of two instruments connected by a silk-covered hemp base-cord 25 (or 50) yds. in length. The left-hand instrument has two fixed mirrors inclined at 45° like those of an optical square, while the right-hand instrument has one mirror fixed and the other capable of rotation as in a box sextant.

The left-hand operator sights to the point A whose range is required and moves into such a position that the reflected image of the right-hand instrument coincides with the direct image of A, *i.e.* he arranges that the angle ALR is 90°, where L and R are the positions of the left- and right-hand operators respectively.

Now, from the other end of the measured base, the right-hand man sights directly to the same object A, and rotates the movable index mirror by means of a graduated drum until he brings the reflection of a white strip on the left-hand instrument into coincidence with A. That is, he measures the angle ARL as with a sextant, but, instead of reading the result in degrees, the graduations on the drum are so arranged that the readings are proportional to the tangents of the angles subtended at R, and for a constant length of base of 25 units the range from L to A is indicated.

With the general service instrument it was specified that the mean of six determinations should give a maximum error of 1 per cent at 1000 yds., and 5 per cent at 5000 yds., with a 25-yd. base.

The **Barr and Stroud Range-finders** are extensively used for Naval, Navigational, Military, and Surveying purposes, and may be of any base length from 30·5 cm. (1 foot) to about 30 metres (100 feet). Most of the larger instruments (from about 2 metres upwards) can be used for high-angle observation and many of them incorporate means for the measurements of heights as well as ranges. Fig. 8.17 gives two views of one of the smaller instruments of 80 cm. or 1 metre base, while Fig. 8.18 is a diagrammatic sketch showing the general arrangement of the internal parts. The instrument consists essentially of two telescopes provided with object-glasses O_1 and O_2, and having a common eye-piece E_1, the magnification in this case being 14 diameters. Those rays of light from the observed object, which enter the left-hand window W_1, are successively reflected from two opposite faces of the silvered pentagonal prism R_1. The two reflecting surfaces are inclined to one another at 45°, as in the optical square, so that the prism deflects the rays through a constant angle of 90°. This angle would not be affected by a slight rotation of the prism, due to, say, temperature

causes, as it would were a single plane mirror to be used. With the prism, the effect produced by the rotation of one reflecting surface would be counterbalanced by the effects produced by the equal rotation of the other.

After being deflected at right angles to their original course, the

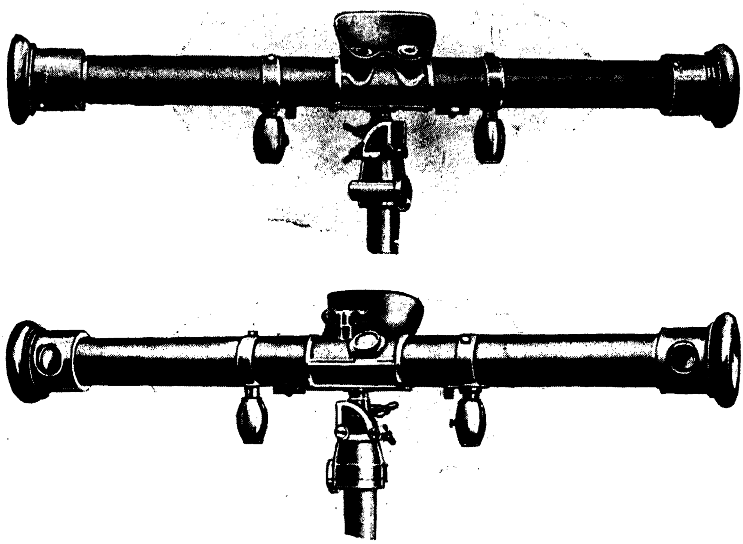


FIG. 8.17.—Two Views of a Barr and Stroud Range-finder

rays pass through the object-glass O_1, and are then diverted by the combination of prisms CP through the eye-piece E_1 to the eye. The rays which pass through the window W_2 are similarly reflected by the pentagonal prism R_2, pass through the object-glass O_2, and are reflected

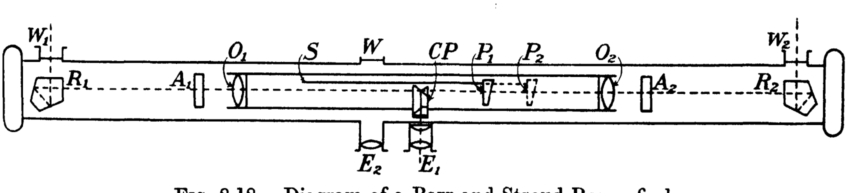


FIG. 8.18.—Diagram of a Barr and Stroud Range-finder
(from *The Modern Range-finder*, by F. J. Cheshire—by permission)

by the prisms CP through the eye-piece E_1 to the eye. In addition to the rays being deflected through 90° horizontally by the prisms CP, they are turned upwards through about 60°. The observer thus views the images by looking downwards at this angle through the eye-piece, and this is a much more convenient and comfortable arrangement than viewing them horizontally, as in most other instruments. The prisms also separate the images obtained through the left- and right-hand

windows. The arrangement, as seen through the eye-piece, may show the field of view divided by a sharply defined horizontal line, when the lower portion would be the object as seen through the left-hand window, and the upper portion would be that seen through the right-hand window.

Often the lower image is erect and the upper one erect as in the field of view of an ordinary or a box sextant.

Sometimes, however, the upper image is inverted and the lower erect, and occasionally the opposite is the case.

Another arrangement which is sometimes adopted is to divide the field of view into three portions by two sharply defined horizontal lines near the middle. The upper and lower portions show views from the left window, while the image on the central band is that through the right-hand window. The two outer images are erect, while the central strip may show either an erect or an inverted image.

" Coincidence " is obtained by bringing well-defined points of the one image into exact coincidence with the corresponding points on the opposite image—exactly as with the box sextant and other instruments.

In order that the two images may be made to coincide in this way—whatever be the distance of the observed object from the instrument —a movable prism [1] P_1 is interposed in the path of the rays from R_2.

The angular amount of deflection produced by this prism P_1 is constant whatever its position, and, consequently, the linear displacement of the picture at the focal plane will increase or diminish as the distance of P_1 from that plane is increased or diminished respectively.

The displacement due to P_1 counterbalances the displacement due to the fact that the rays through W_2 are not parallel to those through W_1 when the range is a measurable amount.

The movement of P_1 necessary to obtain coincidence of the images is therefore a measure of the range, and the magnitude of this movement is indicated by the scale S attached to P_1.

The scale which appears vertical is viewed through the left-hand eye-piece E_2, and is illuminated through the window W, opposite which is a fixed pointer to mark the reading.

The movement is regulated by means of a milled-head near the right-hand handle, shown in the figure.

The Astigmatisers A_1 and A_2 are negative cylindrical lenses with horizontal axes, which may be moved into the path of the rays when required by means of a lever near the left-hand handle. The effect is to convert the image of a point, such as a star, into a thin vertical line, and so permit of coincidence being more easily and accurately obtained.

Adjustments.—There are two adjustments which may be necessary with this instrument : (1) the Halving Adjustment, and (2) the Coincidence or Zero Adjustment.

The adjusting heads for correcting each of these are situated at the left hand of the instrument and are protected by a movable cover ring.

[1] Compare the use of the prism M in the optical reading system of the Tavistock theodolite, Figs. 4.10 and 4.11.

(1) The Halving Adjustment is to enable the observer to move one of the images up or down by a small amount so as, in the case of both images being erect, to secure that the picture shall form a complete whole without duplication or deficiency.

This adjustment is not necessary when ranging on a line whose image is perpendicular to the line of separation in the field of view.

If, however, the image forms an angle with this line, accurate halving is essential. In the case when one of the fields is erect and the other inverted, the adjustment is effected by seeing that the two images of a point of the target are equidistant from the separating line, and this is most readily attained by moving the instrument slowly in altitude and altering the adjustment until the images of the point reach the separating line simultaneously.

(2) The Coincidence Adjustment is to ensure that, when exact coincidence is obtained, the correct range of the object is indicated on the scale.

The accuracy of the adjustment can be tested by several methods, such as : (*a*) the observation of a very far distant object (*e.g.* the moon or a star), when an infinity reading should be obtained ; or (*b*) the observation of a well-defined object at a measured distance away, when the reading on the scale should agree with the range determined by direct measurement ; or (*c*) observations on a pair of marks, 300 to 500 yds. distant, the distance between the marks being exactly equal and parallel to the base of the range-finder. The necessary corrections are made with the adjusting heads already mentioned. That for the halving adjustment imparts a slight tilt as required to the left pentagonal prism, just as in the corresponding adjustment of a box-sextant a tilt is given to the horizon glass (p. 76).

The method of completing the coincidence adjustment is ingenious. The left window has its surfaces slightly inclined to one another, so that beams of light passing through the glass are displaced, exactly as are those passing through a deviating prism (*e.g.* P_1). As the glass is rotated—and the amount of rotation is indicated on a scale—the displacement of the rays is altered, and the image formed by these rays is raised or lowered as required until coincidence with the image from the right prism is effected.

The necessary corrections are made with the adjusting heads already mentioned.

Accuracy.—Let D be the correct range of an object, when the length of the instrument base is s, and the angle of parallax is a.

Then $$s = \mathrm{D} \tan a = \mathrm{D}a \text{ nearly,}$$

or $$a = \frac{s}{\mathrm{D}}$$

and $$\delta a = -\frac{s}{\mathrm{D}^2} \cdot \delta\mathrm{D},$$

or $$\pm \delta\mathrm{D} = \mp \frac{\mathrm{D}^2}{s} \cdot \delta a.$$

where δD is the error in distance caused by an error of $\delta \alpha$ in obtaining coincidence of the images.

Thus if the eye, unaided, can distinguish differences of $\delta \alpha'$ in angle, then with a telescope of magnification m it can distinguish differences of $\delta \alpha'/m$.

Substituting this expression for $\delta \alpha$ in the above equation and omitting the signs,

$$\delta D = \frac{D^2}{s} \cdot \frac{\delta \alpha'}{m},$$

i.e. the uncertainty in the range is proportional to the range squared, since $\frac{\delta \alpha'}{m}$ and s are constants.

The value of $\delta \alpha'$ is normally well under 12 seconds or ·000058 radians, so that, assuming this value, with the Barr and Stroud instrument described above, when $s = 39{\cdot}4$ in. or 1·1 yds.,

$$\delta D = \frac{D^2}{1{\cdot}1} \cdot \frac{{\cdot}000058}{14} = {\cdot}0000037 D^2.$$

Thus if $D = 1000$ yds., $\delta D = 3{\cdot}7$ yds. about.

The makers state that under favourable conditions the uncertainty of an observation with this instrument at 1000 yds. is 4 yds., and this includes errors in reading the scale, etc.

For a range of 2000 yds. δD will be four times the amount for 1000 yds., *i.e.* nearly 15 yds.—as the uncertainty varies as D^2.

The probable error would be about half these values, though much depends upon the conditions obtaining and the skill of the observer. The p.e. may be much reduced by taking the mean of several independent readings, and still further reduced if two readings are taken for each observation ; one when the images are apparently just not in coincidence on the one side, and the second when the images are apparently just not in coincidence on the opposite side. A mean of these two values is found by experience to be usually more accurate than the single direct reading as generally taken.

Stereoscopic Range-finders have the same basis of measurement as instruments of the "coincidence" type, which have already been described : the range in each case is determined from the measurement of the angle of parallax, *i.e.* the angle at the object, subtended by the base length of the instrument.

In stereoscopic range-finders the measurement of the angle is effected by stereoscopic comparison between the image of the object and a reference mark or marks in the field of view. By means of the range gear, the image is adjusted until it and the reference mark appear stereoscopically to be the same distance away. Actually the reference mark is fixed, and the image is moved, but to the observer it seems that the object remains stationary and that the mark moves. The instrument is thus, rather misleadingly, referred to as the "moving mark" instrument.

Many of the Barr and Stroud instruments are arranged to be either

of the coincidence type or the stereoscopic type, and the accuracy of the two types is comparable, some users preferring the one, and some users the other. Fig. 8.19 illustrates the field of view in such an instrument.



FIG. 8.19.—Appearance of the Field of View in a Barr & Stroud Stereoscopic Range-finder of the "moving mark" type

The ortho-pseudo stereoscopic system is a development in which the ordinary stereoscopic image is moved until it appears to be exactly the same distance away from the observer as a secondary image in pseudo-stereoscopic relief—*i.e.* one in which the stereoscopic effects are reversed, concavities appearing as convexities, etc. The effect is to give increased stereoscopic relief between the image and the reference mark, and hence to allow increased accuracy of setting. The scale arrangements, as before, are similar to those of the "coincidence" instrument.

In another, though less accurate, type of stereoscopic range-finder there are no moving parts. A series of fixed marks, seen stereoscopically in the field of view, furnishes a scale of distances, and the apparent position of the stereoscopic image of the object, relative to this scale, defines the range, which is read directly by the observer.

EXAMPLES

1. (U. of L.) How may the ordinary telescope of a theodolite be arranged to give distances by observing readings on a staff?

In a telescope the solar focal length of the object-glass is 12 in. and the vertical axis of the theodolite is midway between the object-glass and the solar focus. When the staff is at a distance of 301·5 ft. from the axis of the theodolite, the intercepted height on the staff was found to be 3 ft. What is the distance between the pair of webs on the diaphragm? If, in a second case, the intercepted reading was 4·63 ft., what is the distance of the staff from the theodolite?

2. To determine the values of the constants for a tacheometer, the following readings were observed:

Distance Chained	Readings
50	2·16, 2·65
100	2·24, 3·23
200	2·45, 4·43
300	3·04, 6·01

What would you say were the values of the constants?

3. The following readings are abstracted from the field book of an actual survey. Find the distance of each of the points 1, 4, 9, . . . from D and the

reduced levels of each. The staff was held vertically, and the reduced level of A = 146·20 ft. $\frac{f}{i} = 100$, $f + d = 1$.

Instrument Station	Staff	Height of Instrument	Azimuth	Stadia Readings	Axial Readings	Vertical Angle	Reduced Level
A	B	5′·00	179°-59′	7·06 / 3·35	5·20	− 10°-17′	
B	A	4′·54	360°-00′	9·27 / 5·55	7·40	+ 10°-47′	
B	C	,,	45°-4′	5·80 / 4·00	4·90	+ 10°-32′	
C	B	4′·3	225°-4′	9·77 / 8·00	8·88	− 8°-59′	
C	D	,,	327°-32′	3·88 / 2·00	2·94	− 2°-15′	
D	C	3′·35	147°-32′	4·88 / 3·00	3·94	+ 2°-2′	
	1	,,	211°-9′	5·65 / 5·00	5·32	− 17°-9′	
	4	,,	255°-28′	3·19 / 2·00	2·59	− 6°-2′	
	9	,,	82°-5′	8·70 / 7·00	7·85	+ 20°-18′	

4. What error would be introduced in the determination of the altitude of station 9 above that of the instrument axis at D if the staff were held 1° from its true vertical position ?

What would you say would be the p.e. in the reduced level of D if the reduced level of A is assumed correct ?

5. (U. of L.) A tacheometrical survey is made by running a traverse A, B, C, D, E along one side of a valley and occupying the points A, B, C, D, and E as theodolite stations from which to observe on to 12 other stations, *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *k*, *l*, *m*, and *n*, on the opposite side of the valley. The observations are taken by reading the angles of altitudes or depression of the upper and lower vanes of a 10 ft. vane staff (or target rod) held successively at stations *a*, *b*, *c*, *d*, . . . *m*, *n*. The directions of the sights are fixed by azimuthal bearings with the line AB, which is assumed as the meridian or zero direction.

Table I gives particulars of the traverse from A to E which fixes the relative positions of the theodolite stations.

Table II gives the booked observations taken at stations A, B, C, D, and E on the staff when held at *a*, *b*, *c*, *d*, . . . *m*, *n*.

The lower vane of the staff is 2 ft. above the ground, and the upper vane 12 ft. above the ground.

(*a*) Calculate the horizontal distances from the stations A, B, C, D, and E to the corresponding staff positions *a*, *b*, *c*, *d*, . . . *m*, *n*.

(*b*) Calculate the reduced levels of the stations *a*, *b*, *c*, *d*, etc., at which the staff was held.

(*c*) Plot to a scale of 100 ft. to an inch the positions of *all* the stations in plan.

(*d*) Draw to a horizontal scale of 100 ft. to one inch, and a vertical scale of

10 ft. to one inch, a longitudinal section of the route through *a*, *b*, *c*, *d*, . . . *k*, *l*, *m*, *n*, working to a datum of 250 ft.

(*e*) What is the average gradient between the staff stations *a* and *n* ?

Note.—The observations are taken by what is commonly known as the "tangential method."

TABLE I

Theodolite Traverse A to E

Line	Length in Feet	Whole Circle bearing with Meridian AB	Reduced Level in Feet of
AB	356	0°-0′	A = 280·4
BC	314	17°-40′	B = 278·8
			C = 282·0
CD	405	348°-30′	D = 286·7
DE	329	325°-20′	E = 292·3

TABLE II

Field Book Observations at Theodolite Stations A, B, C, D, E

Theodolite Station	10 ft. Staff at	Upper Vane		Lower Vane		Whole Circle bearing with Meridian AB	Height of Instrument
		Elevation	Depression	Elevation	Depression		Ft.
A	*a*		2°-48′		4°-3′	270°-0′	4·9
	b		1°-24′		2°-31′	288°-30′	
B	*c*		1°-3′		2°-21′	258°-20′	5·0
	d	0°-28′			0°-50′	277°-50′	
	e	1°-42′		0°-31′		297°-30′	
C	*f*	1°-45′		0°-37′		268°-40′	4·8
	g	3°-3′		1°-52′		286°-3′	
		3°-23′		2°-16′		300°-0′	
D	*k*	4°-32′		3°-0′		259°-30′	4·9
	l	4°-36′		3°-8′		274°-0′	
E	*m*	5°-25′		3°-28′		229°-40′	5·1
	n	6°-47′		4°-54′		251°-40′	

6. What would be the p.e. in the position of (*a*) in Question (5) if the angles are liable to a p.e. of $\pm 20''$ and the height of the instrument axis to a p.e. of $\pm 0{\cdot}05$ ft. ?

7. If the constant for a subtense theodolite is 595, the intercept on the staff 10 ft., and the value of $(f+d)$ is 1·5 ft., what is the distance from the instrument to the staff if the sum of the micrometer readings is 9·256 (*a*) when the telescope is horizontal, (*b*) when inclined at 3°-30′ to the horizontal ?

8. (U. of L.) A tacheometer has a diaphragm with three cross-hairs spaced at distances apart of $\frac{1}{40}$ in. The focal length of the object-glass is 9 in. and the distance from the object-glass to the trunnion axis is $4\frac{1}{2}$ in. A staff is held vertically at a point the level of which is 80 ft. AD. The telescope is inclined at 9° to the horizontal and the readings taken on the staff are 6·65 ft., 5·14 ft., and 3·63 ft.

Find the horizontal distance of the point from the telescope and the level at the telescope. The height of the trunnion axis of the telescope is 4 ft. 6 in.

9. Equations (44) and (45), p. 266, may be transformed into the following expressions, which are more suitable for logarithmic computations :

$$v = s \cos \alpha \sin \beta \operatorname{cosec} (\alpha - \beta) \quad . \quad . \quad . \quad (44a)$$

$$h = s \cos \alpha \cos \beta \operatorname{cosec} (\alpha - \beta) \quad . \quad . \quad . \quad (45a)$$

Deduce these formulae.

10. A plane-table alidade with a tacheometric telescope is provided with a Beaman's stadia arc, which has engraved on it two special scales, auxiliary to the vertical circle scale of degrees. The scale V gives values of $\sin \alpha \cos \alpha$, the scale H gives $100 (1 - \cos^2 \alpha)$. The zero or horizontal reading of the former is 50.

If the staff intercept is 4·62, the axial web reading 3·14, and the instrument, fitted with an anallatic lens, has a constant of 100, find the vertical distances of the staff stations above or below the instrument axis when the V scale readings are 66 and 34 respectively. What are the corresponding readings on the H scale, and what are the horizontal distances to the staff ?

CHAPTER IX

PLANE TABLE SURVEYING

THE plane table consists of a flat board, varying in size from 16 in. × 12 in. to 30 in. × 24 in., fixed to the head of a light tripod by a wing nut or other suitable means. The head of the tripod is sometimes—but not always—provided with a light metal frame carrying three milled-headed screws for the purpose of levelling the table.

The other accessories may include :

(1) A brass, gun-metal, or box-wood straight-edge or alidade, fitted with two hair-line sights similar to those of a dial, described on p. 81. These hair-lines are so placed as to lie directly over or parallel to the " fiducial " or working edge of the rule, and this edge may with advantage be graduated to serve as a scale with which to plot any tacheometric or other distances that are directly observed in the field.

(2) A trough compass similar to that of a theodolite, generally detached, though sometimes a circular box compass, is let into the table near one edge as a permanent fixture.

(3) A small spirit level—generally, though not always, detached from the table.

(4) A plumb-bob.

(5) A ⊃ frame which may be used for large-scale plans to ensure that the instrument is correctly centered. It is placed horizontally round the edge of the table so that the extremity of the one leg coincides with the point on the paper, while exactly below it, from the extremity of the other leg, is suspended the plumb-bob, which should coincide with the corresponding point on the ground.

(6) Rollers on the underside of the frame which enable a continuous sheet of paper to be employed, in lieu of a number of separate sheets.

(7) A telescope, sometimes fixed to the " sight rule " in lieu of the hair-line sights, and preferably fitted with a stadia diaphragm.

It is sometimes permanently fixed in a horizontal position parallel to the " straight-edge ", but on the more elaborate and better-class instruments (see Fig. 9.1 by Stanley) it is capable of motion in a vertical plane, and provided with a graduated vertical circle read by means of verniers as in a transit theodolite.

As in the case of this latter instrument, the bubble tube or tubes attached to the rule, the horizontal axis of rotation of the telescope, and the line of collimation must be truly adjusted, though the same degree of accuracy is not acquired here.

The index error—if any—must also be observed, unless adjusting screws are provided.

The drawing-paper is attached to the board either by means of drawing-pins or special clips ; or it is stretched tightly through slits in the table from one roller to the other ; or it is secured along all the four edges if the outer portion of the frame is rebated to fit over the central panel, and held in place and tightened by cross battens fitting into slots on the underside.

As the work is actually drawn upon the paper in the field, the plane table is obviously not suitable for a damp or rainy climate, and for this reason it has not been greatly used in this country. On the Continent, however, and in the tropics it has been very extensively



FIG. 9.1.—Plane Table with Accessories

used both for topographical maps and for the filling in of detail on larger scale or cadastral maps, the chief points of which have previously been located by triangulation.

Plane table surveying is particularly suitable for the preparation of small-scale maps—though its use is by no means confined to these. It has the advantages over other surveying methods :

(1) That the plan is not prepared by one person—who has possibly never seen the tract of land—from the field notes of another Surveyor, but is drawn by the outdoor Surveyor himself while the country is before his eyes : after which it may or may not be reduced for the published plans.

(2) He can therefore see at a glance that the plan truly represents the tract of country surveyed and that none of the essential data has been overlooked.

(3) He can sketch on the plan, with a sufficient degree of accuracy, a great deal of detail that in any other method would entail a very

considerable amount of labour and expense ; and, as may be seen later, there are generally frequent opportunities for checking the accuracy of the plotted positions of important or governing points.

(4) Direct measurements may—unless it is more convenient to supplement the work with them—be almost entirely dispensed with, as the linear and angular dimensions are both to be obtained by graphical means.

Method of Procedure.—The following is the usual method of procedure adopted for a small survey with an ordinary instrument.

A base line AB (Fig. 9.2) is measured with a chain or tape upon a suitable piece of ground and its length ascertained. The table is then set up and levelled over one extremity A of the line, and a point *a* chosen on the paper to represent A. From *a* a line *ab* is drawn on the plan to represent AB to scale : the magnitude of the scale and the direction of *ab* on the paper being so chosen that the finished drawing of the tract of ground to be surveyed shall occupy the desired position and area on the sheet available.

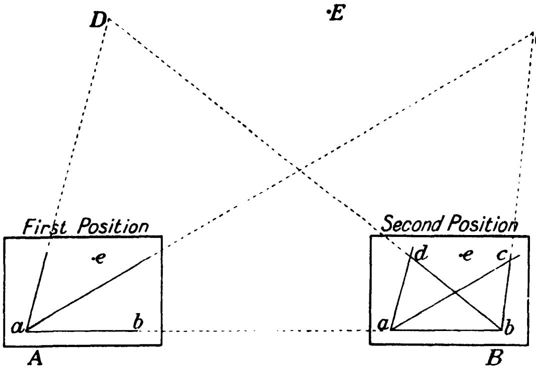


FIG. 9.2.—Principle of Plane Tabling

Theoretically the point *a* should be immediately above the corresponding station-point A, and for large-scale maps this is arranged by means of the ⊃ frame and plumb-bob previously mentioned ; but for small-scale maps it is sufficiently accurate to set up the table with its centre approximately over A, as any error introduced by not accurately centering *a* over A is negligible ; as is also any error due to the line of collimation of the telescope or the alidade sights not being precisely over the edge of the alidade, but parallel to it.

The alidade is laid along the line *ab* ; the wing nut, which fixes the table proper to the frame carrying the levelling screws and attached to the tripod head, is loosened ; and the table rotated until the telescope or the hair-line sights are directed towards B, when the instrument is said to be oriented. The wing nut is then clamped and the trough compass laid upon the paper, and turned until the needle point coincides with the central division of the scale, when, unless there is local attraction, the direction of the magnetic meridian may be drawn on the paper along the edge of the trough.

Or, as an alternative, the line representing the magnetic meridian may first of all be drawn on the paper—either parallel to one of the sides, or inclined to one of the sides, which then represents the true meridian, at an angle equal to the magnetic declination of the place.

Thus, if the trough compass is so placed that its edge lies along this line, the table may be rotated until the point of the needle coincides with the zero of its scale, when the table is oriented. Then the position of the point *a* having been chosen, the line of sight of the rule is directed towards B, while at the same time the fiducial edge passes through *a*, a ray is drawn from *a* towards B, and a length *ab* marked off to represent AB to the desired scale.

When the points *a* and *b* have been plotted, and the instrument has been set up over the station A and correctly oriented, rays are drawn through *a* towards any features which it is desired to locate, the directions being ascertained by placing the working edge of the alidade over *a* and turning about this point until the line of sight intersects the specified objects. Two such rays *a*D and *a*C are shown in Fig. 9.2, though on an actual plan it is not necessary to draw anything but small portions of such rays (say about 1 in. in length), near the positions in which it is judged the plotted point will fall on the plan. These rays should be drawn in lightly with a fine-pointed pencil, and, when there is any risk of confusion, a pencil note may be made on the paper to indicate to which station each line refers.

The instrument is then carried and set up at the next station B : the rule is laid along the line *ba*, the wing nut loosened, and the table rotated until the line of sight is directed to A. The wing nut is again clamped, so that the table is then fixed in the same position relative to the meridian that it occupied at A, *i.e.* it is oriented. Theoretically the point *b* should now be exactly over the station-point B, but unless the scale is large this refinement is unnecessary.

If some other point, *e.g.* E, in addition to A and B, has been previously plotted upon the paper at *e*, then a check may be made upon the orientation of the table at B by ascertaining that the ray *be* produced intersects E.

From *b* rays are now drawn towards the same points to which rays were drawn from *a*, as in Fig. 9.2 ; and the intersections of the corresponding rays fix the plotted positions *c*, *d* . . . of the observed stations C, D. . . .

It will readily be seen from the figure that *adb* is a similar triangle to ADB, and therefore that the distances AD and BD are represented by *ad* and *bd* to the same scale that *ab* represents AB, so that *d*, and similarly other points, *c* . . . etc., are located in their true positions on the plan.

Using these plotted points as guides, any minor details which it is desired to delineate may now be sketched with a considerable degree of accuracy upon the plan.

To extend the plan, when the whole field of view cannot conveniently be obtained from the two stations A and B, the instrument is set up and levelled at another station such as D. This point may be one of the positions plotted by intersection from A and B, a trigonometrical or traverse station, or a "resection" point found by the three-point or two-point problems explained later.

To orient the table the compass as a rule need not be referred to, but may be used if desired as a first approximation. The alidade is laid on the paper with its edge along the line *da*, and the wing nut being loosened, the table is rotated until *da* is directed to A. The wing nut is then clamped, and the orientation should be complete, but it is advisable to check, by noting that other rays such as *db*, *dc*, etc. intersect their corresponding stations B, C. . . .

Rays are then drawn from *d* as before, to intersect rays from other stations at which the instrument has been or may be set up.

If a plane table is being employed for the filling in of the details of a large trigonometrical or traverse survey, the work probably extends over a large number of sheets, on each of which a number of points such as *a* and *b* are plotted by co-ordinates. The plane-table work then connects and is controlled by these points, so that errors which might accumulate over long distances are eliminated.

In such a case the instrument may be set up and levelled over one of these primary points, and oriented by resection on to a second; or if more convenient, the instrument may be set up at any suitable commanding position, say O, and located from the primary points by means of the three-point problem method described below.

The ordinary process of location by intersection may with advantage be supplemented by stadia measurements if the rule is fitted with a tacheometric telescope. A very much larger number of points may be obtained from each instrument station, without confusion of the drawing, by sighting to a staff held in various positions by the staff man, who may often be mounted. The stations thus located are plotted by scaling the observed distances along the directions on the paper indicated by the alidade edge (*i.e.* by the method of polar co-ordinates); and additional checks are provided for the more important stations to which intersecting rays are taken.

The Three-Point Problem.—The "three-point problem" consists in the location on the plan of the position of the observer, by means of observations to three well-defined points, the positions of which are already delineated upon the drawing.

Thus let A, B, C be three station-points on the ground, represented by three points *a*, *b*, *c* on the plan, and let O be the position of the instrument in the field: the problem is then to locate on the plan the point *o*, which shall represent O, by means of observations from O to A, B, and C.

In Chapter XII are given a number of solutions to the general problem, and below are a few methods as applied to plane table surveying.

(1) **Triangle of Error Method.**—This solution is that which is most frequently used.

The table is oriented as correctly as possible with the trough or other compass (or if this is not available, by general observation), and the wing nut clamped to fix the table. The straight edge is then laid on the paper so as to intersect *a*, the line of sight directed to A, and a faint ray drawn near the position at which *o* is judged to be

situated. Similarly, two other rays are drawn, one through b in the direction bB and one through c in the direction cC. If the table is correctly oriented these rays should be concurrent—the point of intersection being the required point o.

If the table is not correctly oriented, then except when ABCO are concyclic a "triangle of error" is formed by the three rays Aa, Bb, Cc. and from the rules described below the direction in which the table is required to be rotated can be judged. The process is repeated until the triangle of error is eliminated. When ABCO are concyclic, the rays Aa, Bb, Cc are concurrent whether the table is correctly oriented or not, and it becomes impossible to locate o by this method—*i.e.* "failure of fix" results.

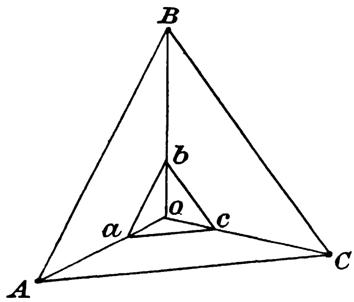


FIG. 9.3

Fig. 9.3 illustrates the case where the table is correctly oriented—and since abc and ABC are similar figures, and the sides of the one are parallel to the sides of the other, it can readily be shown that the rays Aa, Bb, Cc are concurrent, and that o represents on the plan the position O on the ground immediately below it.

Now suppose that the table is rotated through a small angle ϕ in a counter-clockwise direction, and that the axis of the table is at z (Fig. 9.4). Then o will move relative to the ground a distance $oo' = \phi.zo$ at right angles to zo. Similarly, a, b, c, will move to a_1, b_1, c_1 at right angles to oa, ob, oc respectively and through distances of ϕoa, ϕob, and ϕoc.

This movement is equivalent to a linear movement of the whole table through a distance oo_1 and a rotation of the table about o_1 through an angle ϕ in the counter-clockwise direction. The resultant effect will be to return z to its original position, and to give a, b, c the same displacements as before, since the movement of a, for instance, relative to z = movement of o relative to z + movement of a relative to o.

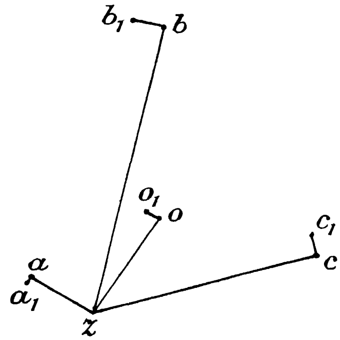


FIG. 9.4

It has already been explained that, for small-scale plans, it is not essential that the point o on the plan should be exactly above o on the ground, as the error is too small to show. Consequently if the table is not rotated, but moved linearly through a distance of oo, the rays from ABC are not appreciably altered in direction and are still concurrent at the same point on the paper if the table is correctly oriented—or form the same triangle of error if incorrectly oriented.

It will suffice, therefore, if we imagine the table rotated through an angle ϕ about o. The case where o is within the triangle abc is illustrated in Fig. 9.5. It will be seen that the displacements of a, b, c due to counter-clockwise rotation of ϕ, are ϕoa, ϕob, and ϕoc respectively. The rays drawn to ABC will be practically parallel to the original rays, but will now not be concurrent but form a triangle of error $o_1o_2o_3$. The true position of o is ϕoa from the ray aA, ϕob from bB, ϕoc from cC—*i.e.* Rule 1 is that the distances from the three rays are proportional to the lengths of the rays—*i.e.* proportional to OA, OB, OC, or to oa, ob, oc.

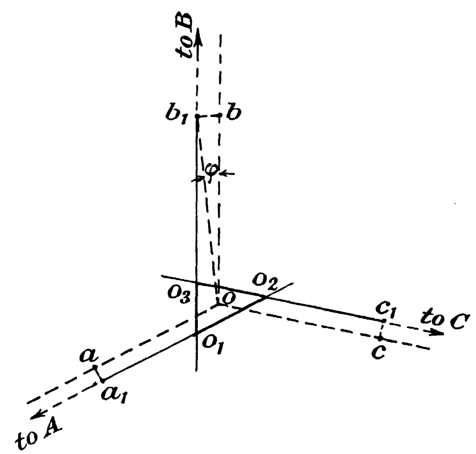


FIG. 9.5

It follows, therefore, that if the table is set up in the field, and the rays form a triangle of error, the true position of o can be roughly estimated, and the direction in which the table requires to be turned can be judged. Thus in Fig. 9.5 to make the ray b_1B pass through o the table must be turned clockwise.

It also follows, as can readily be seen on a sketch, that if o lies outside the triangle ABC, then o will lie outside the triangle of error ; if o lies inside the triangle ABC, then o lies inside the triangle of error. This is Rule 2. Rule 3 is that when o lies outside the triangle abc, it is situated in one of the two sectors where the rays aA, bB, cC are either all on the right or all on the left from o, when facing A, B, C in turn.

Two cases are illustrated in Figs. 9.6 and 9.7, B being the most distant station and A the nearest.

Fig. 9.6 : o is inside the triangle ABC, hence o lies within the triangle of error, nearer the ray oa. The table needs to be rotated in a counter-clockwise direction.

Fig. 9.7 : o is outside the triangle ABC, hence o lies outside the triangle $o_1o_2o_3$, and either in the area co_3o_1 when aA, bB, cC are all to the left of a point in it ; or in the area ao_2e where aA, bB, cC are all on the right of a point in it. Further inspection or trial shows o to be in the area co_3o_1 and the table needs to be rotated in a clockwise direction.

Other cases can be considered by the student, *e.g.* AOC in line : ABC in line : B on same side of AC as o : ABCO concyclic, etc.

In each case first draw a diagram imagining the table correctly oriented. Then imagine the table to be rotated either clockwise or counter-clockwise and scale off displacements of the rays amounts

proportional to the distances *oa*, *ob*, *oc*, as in Fig. 9.5. Draw the triangle of error. Now trace off on tracing paper only the positions of the points *a*, *b*, *c* (which are at a_1, b_1, and c_1) and triangle of error o_1, o_2, o_3, and apply the rules given above to verify that the position of *o* can be roughly estimated and the direction in which the table needs to be rotated can be judged.

(2) **Tracing-Paper Method.**—The table is set up and oriented as nearly as possible with the eye, or by means of a compass, and a piece of tracing paper is laid over the plan.

A point *o′* is chosen on the tracing paper over a position on the

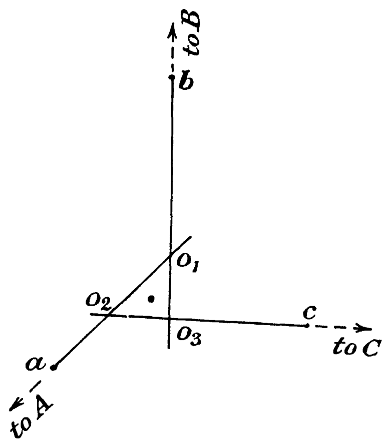


FIG. 9.6

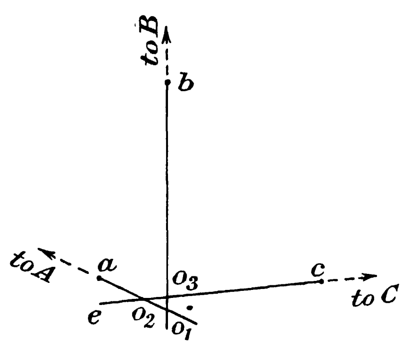


FIG. 9.7

plan which is judged to be approximately that of *o*, and rays are drawn from *o′* towards A, B and C. The tracing paper is then moved over the plan until the rays pass through the three points *a*, *b*, and *c* respectively, when the point *o′* may be pricked through on to the plan, thus determining the point *o*.

The alidade is then laid along the line *oa*, and the instrument turned if necessary until *oa* is directed towards A, when the table should be oriented.

As a check the rays B*b* and C*c* should now intersect at *o*. If not, a small triangle of error will result, and this may be corrected as already explained, or the correct position of *o* judged in relationship to it.

(3) **Bessel's Method** (Figs. 9.8 and 9.9) may be used either when the meridian is not marked upon the plan, or when a compass is not available or is useless owing to local magnetic attraction.

Theoretically the point *b* is placed over the station-point O, the straight-edge laid along the line *ba*, and the table rotated until *ba* is directed towards A ; the table is clamped and a ray *bd* drawn on the paper through *b* in the direction C*b* produced. The point *a* is next placed over the point O, *ab* directed towards B, and the ray *ad*

drawn in the direction Ca produced, intersecting the ray previously drawn through b in d. dc is joined, and the table rotated until dc and C are in the same line.

The table is now oriented and o can be found as in Method 1 by drawing rays Aa, Bb, Cc and producing these to intersect. It will be seen later that $adbo$ are concyclic. Actually the table is not moved as a whole in order that b and then a shall be in turn over the station O, but it is merely rotated on its axis, so that at first ba is directed to A, then ab to B, and finally dc to C—the error thus introduced being inappreciable. The construction may be proved as follows:

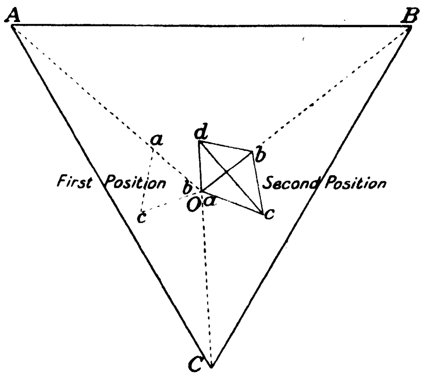


FIG. 9.8.—Bessel's Method

Let the angles at A be α_1 and α_2, at B, β_1 and β_2, and at C, γ_1 and γ_2 as in Fig. 9.9.

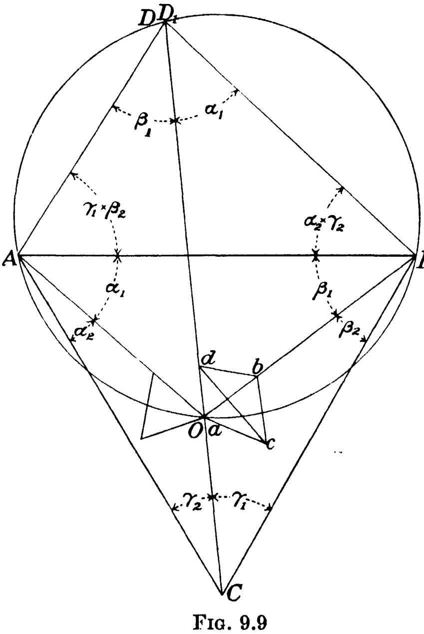


FIG. 9.9

Then the angle dab, being the exterior angle of the triangle aBC $= \gamma_1 + \beta_2$, and similarly the angle dba is equal to the angle daA $= \alpha_2 + \gamma_2$.

At A let AD be drawn on the opposite side of AB to C (if d is on the opposite side of ab to c as in this particular case), making an angle $(\gamma_1 + \beta_2)$ with AB and meeting Ca produced in D.

Similarly let BD_1 be drawn from B, making an angle $\alpha_2 + \gamma_2$ with BA and meeting Ca produced in D_1.

Then because the sum of the three angles of a triangle = 180° and

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 180°,$$

therefore the angle ADC $= \beta_1$ and the angle BD′C $= \alpha_1$.

But the angle ABa $= \beta_1 =$ ADa, therefore ADBa are concyclic (III. 21 Converse).

Similarly as each of the angles aAB and $aD_1B = \alpha_1$, therefore AD_1Ba are concyclic and D and D_1 must coincide.

The two figures ACBD and $acbd$ are therefore similar, because acb

is the plan of ACB to scale, and the angles DAB and DBA were made equal to the angles *dab* and *dba* respectively.

Therefore the angles *dca* and *dcb* are equal to γ_2 and γ_1 respectively.

Therefore when *acbd* is swung round *d* until *dc*C are in line, *ca* will be parallel to CA and *cb* parallel to CB, *i.e.* the table is oriented.

Actually the table is rotated on its axis—not round the point *d*—but the error resulting is negligible.

In Fig. 9.9 the point O is within the triangle ABC, but the same construction holds for other cases except when ABCO are concyclic, under which conditions C and D coincide and the point O becomes indeterminate.

(4) **Method 4.**[1]—Referring to Fig. 9.10 (*a*), *ae* is drawn at right angles to *ab*, and by fixing the fiducial edge of the rule along this line

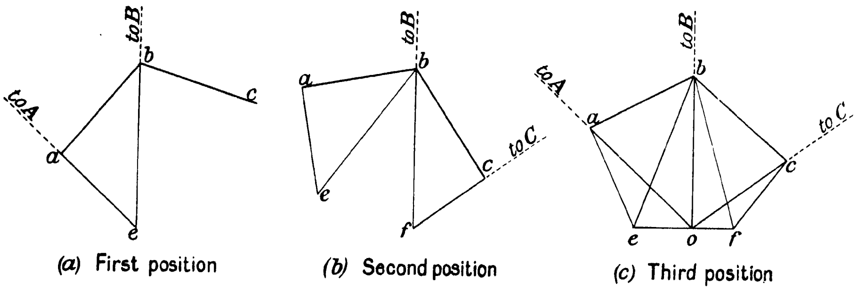


FIG. 9.10

the table rotated until *ea* is directed to A: a ray *be* is then drawn through *b* in the direction B*b* to intersect *ae* in *e*.

Similarly *fc*, drawn at right angles to *bc*, is directed to C, and *bf* is drawn in the direction B*b* to intersect *cf* in *f*.

ef is joined and *bo* drawn with a set square at right angles to *ef*. *o* then represents on the plan the position O of the table on the ground, and the table may be oriented by directing *ob* to B. As a check the rays A*a* and C*c* should now intersect at *o* on the line *ef*.

Failure of fix results as in Chapter XII, when *abco* are concyclic and *e* and *f* become coincident.

The angle subtended by AB at O is θ_2 say, so that as *e* is very close to *o* in comparison with the lengths *o*A, *o*B, and AB, therefore the angle A*e*B is approximately equal to the angle A*o*B,

i.e. $$\angle aeb = \theta_2.$$

Similarly if the angle BOC $= \theta_1$, then

$$\angle cfb = \theta_1.$$

By the construction, as the angles at *a* and *o* are each 90°, *baeo* are concyclic: and so also are *bcfo*. It follows therefore that $\angle aob = \theta_2$ and $\angle cob = \theta_1$; and Fig. 9.10 (*c*) represents the table correctly oriented, with *o* the required station position.

[1] *Engineering*, vol. xcvii., Jan. 16, 1914.

Two-Point Problem.—Let A and B (Fig. 9.11) be the actual stations on the ground, already plotted at *a* and *b* upon the plan : and let O be the position at which the instrument is to be fixed, while *o* is the corresponding point which it is desired to locate on the paper. A and B are assumed inaccessible, or unsuitable for instrument stations, otherwise the table may be set up at either of these points. and the usual method of procedure adopted as in Fig. 9.2.

Method 1.—The most simple though hardly reliable method, even when the station is entirely free from local magnetic influences, depends upon the orientation of the table by the compass. The instrument having been set up and levelled, the trough compass is placed with its edge parallel to the magnetic meridian as marked on the paper, and the table rotated until the needle floats freely with its point coincident with the zero of its scale. Rays are then drawn through *a* in the direction A*a*, and through *b* in the direction B*b*, when the point of intersection locates approximately the required position of *o*.

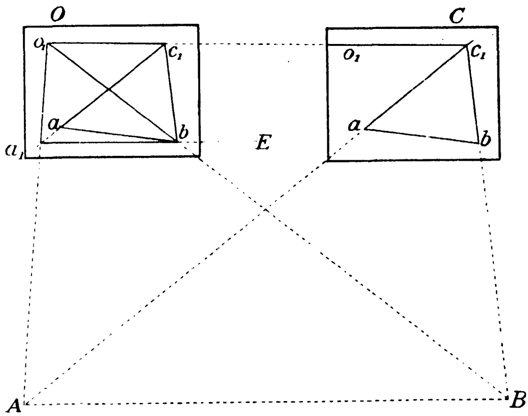


FIG. 9.11.—Two-Point Problem

Method 2.—The instrument is set up and levelled at any convenient point, say C, Fig. 9.11, which will provide good intersections from O, A, and B, and is oriented as nearly as possible by means of the compass, or by judging *ab* to be parallel with AB.

The alidade is laid on the paper and a ray drawn through *a* towards A ; similarly another ray is drawn through *b* towards B, intersecting the ray A*a* in c_1 as in Method 1.

Through c_1 is drawn a ray c_1o_1 in the direction c_1O.

The instrument is then carried to the station O, levelled and oriented parallel to its late position at C by placing the rule along the ray o_1c_1 and rotating the table until this is directed towards C. The table is then clamped and a ray drawn through *b* towards B and intersecting the line c_1o_1 in some point o_1. Next a ray is drawn from o_1 towards A, when, if the orientation of the table at C was correct, the point *a* on the paper should be bisected. If not, the intersection of o_1A with ac_1 fixes another point a_1 to represent A.

That is to say, $a_1bc_1o_1$ represent ABCO when working from the first assumed orientation at C.

But the true representation of AB is *ab*, so that the assumed orientation is inaccurate by the magnitude of the angle a_1ba provided that

this preliminary construction has been carefully executed: the scale will also be slightly in error. To adjust, the rule is laid along a_1b, and if there is no natural object already in alignment, a ranging rod E is placed at some considerable distance from the instrument in the line a_1b produced. The rule is then transferred to the line *ab*, and the table rotated through an angle a_1ba, *i.e.* until the line of sight again intersects the same object E, when *ab* should be parallel to AB and the table correctly oriented. The true position of *o* can then be determined by sighting through *a* to A and through *b* to B, and noting the intersection of the rays.

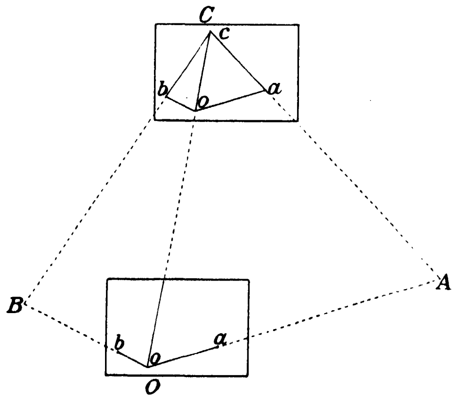


FIG. 9.12.—Two-Point Problem

Method 3.—If B, say, is accessible, but is unsuitable for, or cannot be occupied as, an instrument station, the following procedure may be followed. The instrument is set up and levelled at a point O near B (Fig. 9.12), and oriented as accurately as possible by means of the compass, or by noting that *ab* is approximately parallel with AB. (If the point O is chosen in the line AB or in the prolongation of AB, this orientation can be done very accurately.) A ray *bo* is then drawn through *b* in the direction of B and the point *o* plotted by measuring the distance BO on the ground and plotting *bo* to the correct scale upon the plan. Any small error in the orientation has no appreciable effect on the location of *o*, when the distance BO is small. When *o* has been located in this way, the orientation may be corrected if necessary, by rotating the table until the line *oa* is directed towards A.

Rays are then drawn from *o* to fix in the usual way any distant objects which it is required to survey, and a line *oc* is also drawn from *o* towards the next station-point C.

When the work at O has been completed the instrument is removed to C, levelled and oriented by adjusting the alidade edge to the line *co* and directing back to O.

A ray is then drawn through *a* towards A, and its intersection with *co* determines the point *c*. The orientation may be checked by means of the compass and the point *c* by resection from any located points which may now have become visible. Resection on to B will probably be of little value as it is so close to O.

From *c* rays are drawn to intersect those previously set out from O, and various other points are thus located.

It is important that points such as instrument stations found by resection should be checked if possible by one or more rays in addition

to those actually necessary for their location: *e.g.* if a station-point has been determined by a three-point problem method, its position should, if possible, be verified by a fourth ray from another triangulation, traverse or plotted position.

Traversing.—Unless the country to be surveyed is of a very open character, and furnished with a number of commanding positions suitable for plane table stations, and from which a good view of the surrounding country can be obtained, it is generally advisable, if not absolutely necessary, to supplement the methods mentioned above by plane table traverses.

Each of these traverses should, if possible, be "closed," either:

(1) By forming a complete circuit from a point already accurately located upon the plan, or

(2) By connecting two points, the relative positions of which have been previously determined by the primary and controlling

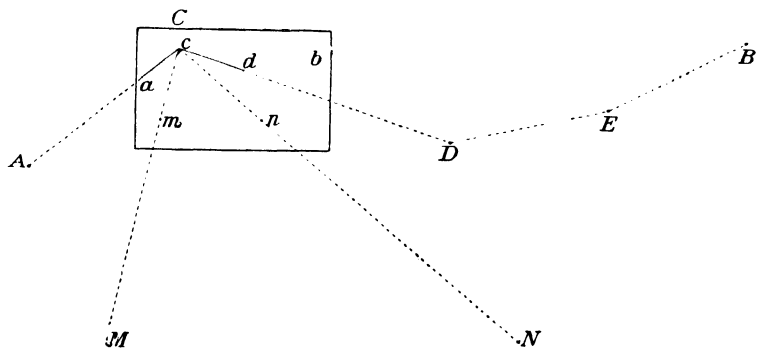


FIG. 9.13.—Traversing with the Plane Table

triangulation or traverse survey executed with more precise instruments, or

(3) By connecting two points determined by intersection or "resection" methods with the plane table.

Method 1.—The most accurate method of procedure is as follows:

Let A and B (Fig. 9.13) be the two points already plotted on the paper at *a* and *b*, and let it be required to connect them by means of a plane table traverse, either because the route is not commanded by positions suitable for instrument stations of the intersection method, or because the detail is such that otherwise a very large number of intersecting rays would be required, and the need of these and the possibility of confusion are obviated by a traverse.

The instrument is first set up and levelled at A, and oriented by fixing the edge of the alidade along the line *ab* and rotating the table until *ab* is directed to B.

The orientation is verified by noting that other rays such as *a*N and *a*M pass through the previously plotted and corresponding points

n and *m* on the paper (*e.g.* in Case 3 above M and N may be the positions from which *a* and *b* have been located).

A ray is drawn in the direction *a*C and the length AC ascertained either by stadia readings upon a staff at C, or by means of a chain or tape, or other method such as pacing. The length *ac* is then marked out on the paper from *a* to the correct scale and the point *c* consequently located.

The detail along the line AC is sketched upon the map : or it may be plotted directly to scale on the paper from chain and tape offsets taken in the usual manner : or preferably the data may be noted in a separate field book kept as explained in Chapter I.

The instrument is next moved to station C, oriented by sighting back along *ca* to A, and verifying upon any other points, such as B, M, and N, the plotted positions of which are known.

A ray *cd* is drawn in the direction CD, the length of CD ascertained, *cd* plotted to scale to fix the point *d*, and the detail along the line determined as before.

Similarly the instrument is set up in turn at each of the remaining traverse stations DE . . . orienting by a back sight to the last vacated station in each case.

The final line should eventually close at *b*, but if not, and the closing error is b_1b, a correction may be applied by the graphical method (3), p. 141. It is therefore advisable to put in the pencil work for the line *acde* . . . b_1 very lightly, and to keep detail such as offsets in a separate field book until the adjustment of the error of closure has been completed.

Method 2.—When the plane table sheets are to be considerably reduced to form the finished maps, so that the errors of closure may also be so reduced as possibly to be negligible on the smaller scale ; or when the detail to be surveyed is not of a very important character : or when several points in the route may be intersected and employed as control points ; the traverse may be very rapidly, though not so accurately, executed by a method similar to the loose needle method of dialling (p. 128).

To survey the route ACDE . . . B (Fig. 9.13) the instrument is set up and oriented at A (or possiblyat C) and a ray drawn in the direction *a*C. The length AC is measured by means of stadia readings from A, by direct chainage, or other convenient method ; the corresponding length *ac* is scaled on the paper and the point *c* fixed. The instrument is next moved to D—omitting C,—oriented by means of the compass, and a ray drawn through *c* in the direction of C. Along this line is scaled off a length *cd* to represent to scale the measured distance CD. From the point *d* so located the next ray is drawn towards E, and the point *e* located by scaling the length DE on the plan.

By proceeding in this way, setting up at alternate stations only, orienting by means of the compass, scaling the lengths of the various lines, and noting or sketching the detail, a very rapid traverse may be made, which should, if correct, close at B. There will, however,

most probably be an appreciable error of closure to be corrected—as in Method 1.

This method is, of course, only applicable when the various stations are practically free from any local magnetic attraction.

Contouring.—Contour lines may be determined by the following methods.

(1) For large-scale maps, the contours may be set out on the ground in the usual manner with a level or the plane table telescope, and the lines afterwards surveyed with the plane table by one of the methods described above.

(2) If the alidade is provided with a telescope, this may be fixed in a horizontal position, and the points on various contour lines determined as in Method 3 (p. 191), the distances from the instrument being deduced from the readings of the outer stadia wires.

(3) The contours may be ranged out by the ordinary alidade hair sights, by the use of a staff provided with a vane fixed at a definite height above the ground; the staff positions at which the vane is cut by the horizontal wire of the circular aperture being marked on the ground and surveyed later. *E.g.* if the altitude of the alidade sight is 80·60 ft. above datum, the vane will be fixed at 5·60 ft. from the bottom of the staff when locating the 75 contour.

(4) The altitudes of a number of governing points may be determined as the survey proceeds, by measuring the distances downwards from a sliding vane to the foot of the staff held at the various points, after the position of the vane has been fixed on the same level as the horizontal hair of the small circular aperture of the hair-line sight. This position is signalled by the instrument man to the staff man. The contours may afterwards be interpolated from the spot levels thus obtained.

(5) For small-scale maps, the distances to, and the altitudes of, a number of controlling points may be obtained either by the tangential or ordinary tacheometric methods, using inclined sights as explained in Chapter VIII. From these spot levels, the general contours may be interpolated and sketched in by the eye.

(6) The angle of elevation or depression (α) to a vane fixed on a staff at the same height as the observer's eye, or to the natural surface of the ground, may be ascertained from the reading of the vertical circle of the telescope; or from a specially graduated alidade sight such as that shown in Fig. 9.14.

The horizontal distance (d) is obtained by scaling from the map, or by means of a telemetric observation, and from this the altitude of the distant point may be calculated; *i.e.* the difference in level of the point sighted above or below the sighting aperture is $h = d \tan \alpha$.

The Indian Pattern Clinometer, as made by J. H. Steward, is shown in Fig. 9.14. The horizontal bubble arm is pivoted at one end, and fitted with a levelling screw at the other. The bubble tube is attached to the arm, and in some models is encased in it. At opposite ends of this arm are two vertical vanes, which can be folded

down when the instrument is not in use. In the one is a small circular sighting aperture, while in the other is a long vertical slit, along the sides of which are two graduated scales. One scale is in degrees subdivided to 20 minutes, and extending from $-23°$ to $+23°$, and the other is a corresponding scale of tangents, the zero in each case being opposite the sighting aperture.

To use the instrument, the plane-table is accurately levelled, the instrument directed towards the target, and the bubble brought to the centre of its run by means of the levelling screw seen in Fig. 9.14. The eye is then applied to the sighting hole and the reading on one (or both) of the opposite scales is noted, where it coincides with the target sighted. Then, if the horizontal distance to the target is known, the height may be deduced.

In some models, the long vane is fitted with a sliding frame which carries a horizontal cross-hair, and is operated by a rack and pinion device. The cross-hair can thus be adjusted to intersect the target, and the scale readings noted more easily.

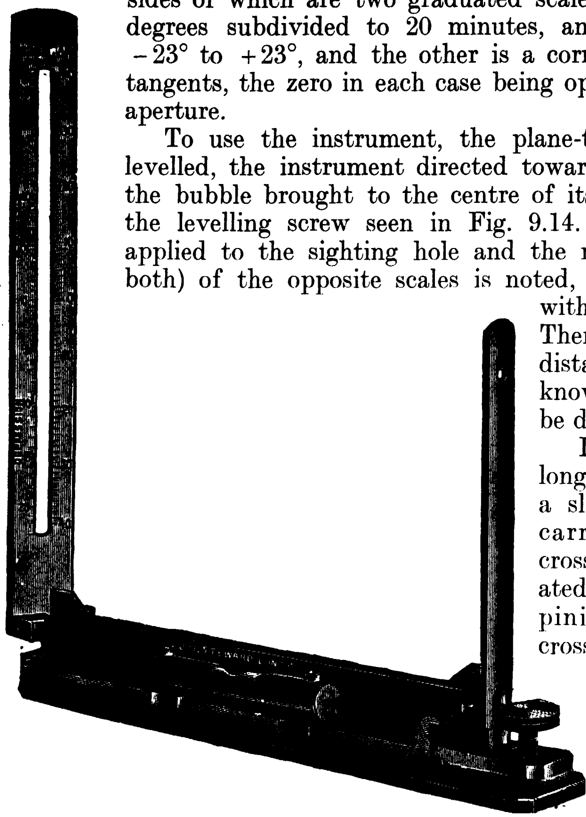


FIG. 9.14.—The Indian Pattern Clinometer

If the instrument is in adjustment, the bubble should be in the centre of its run, when the line joining the eye-hole and the zero of the scales is horizontal. To test this, reciprocal observations may be made between two stations, and the bubble adjusted if found necessary by means of the capstan-headed screws provided.

A further refinement in an Indian model is the provision of an extra sighting hole at the zero of the long vane, and a few degrees of graduated scale on the inner face of the short vane. Two readings can thus be taken from the one station, the second with the position of the vanes reversed. The average gives the true gradient to the target, and the bubble tube can, if necessary, be adjusted accordingly.

Further checks on the accuracy of the instrument can be made by direct measurements on the instrument itself, or by a comparison of results with those obtained by direct levelling or by using a theodolite.

Example.—The reduced level of an instrument station being 87·50 ft. and the height of the alidade above the ground 4·3 ft., find the reduced level of a station N, when an observation to this point gives a reading of ·033 on the tangent scale, while the distance to N is scaled from the plan as 5420 yds.

$$5420 \text{ yds.} = 16260 \text{ ft.},$$

so that the difference in height between N and the eye

$$= d \tan \alpha = 16260 \times \cdot 033 \text{ ft.} = 536 \text{ ft. nearly.}$$

The reduced level of N is therefore

$$87{\cdot}5 + 4{\cdot}3 + 536 = 628 \text{ ft. about.}$$

Had a vane at N 5 ft. above the ground been sighted, the reduced level of the ground would have been $628 - 5 = 623$ ft.

There is also a correction for curvature and refraction (p. 200) $= 0{\cdot}57 \times (3{\cdot}08)^2 = 5$ ft. approx., to be deducted.

It may be noted that the tangent reading ·033 covers a range ·0325 to ·0335, so that $d \tan \alpha = 536 \pm 8$ ft. nearly, *i.e.* there is a probable error from this cause alone of ± 4 ft.

Accuracy.—When filling in the topographical detail of a triangulation survey by straightforward plane table work, if care is taken to obtain good intersections, it is generally claimed that the accuracy is such that no point should be displaced an appreciable amount on the plan: *i.e.* that the accuracy is only restricted by the scale of the plan which is being prepared.

Unless great care is taken, however, large and appreciable errors may occur in large-scale plans, in plane table traverses, or in the determination of altitudes. The following are a few of the sources of error:

(i) The chief error in small-scale plans is due to the shrinkage and warping of the drawing paper, this being affected by the varying amount of moisture present in the atmosphere at different times.

(ii) The error due to any dislevelment of the table has been considered in Chapter IV, when dealing with the theodolite. The error is, however, likely to be much more considerable with a plane table, as the means of levelling are usually not so delicate as are those of a theodolite.

(iii) The error due to inaccurate centering of the table may be very appreciable for large-scale plans, when it is very necessary that the point a, representing the instrument station on the plan, shall be immediately above the corresponding point A on the ground.

Thus if d be the horizontal displacement of a from the vertical through A in a direction at right angles to AB, and if the length of the line $AB = L$, while that of its representation ab on the plan $= l$, then the error in the direction of ab is $\frac{d}{L}$ radians, and the displacement of b from its true position relative to a is $\frac{l \, . \, d}{L}$.

The maximum value of d is hardly likely to exceed 1 ft. at most, so that the displacement is negligible—*i.e.* less than, say, $\frac{1}{100}$th in.—unless $\frac{L}{l}$ is less than 1200, or unless the scale is less than about 100 ft. to an inch. Thus for scales of 100 ft. or more to the inch, it is not

necessary to centre the table accurately at each instrument station. (See Example 2, below.)

Similarly, an error due to the line of sight of the telescope or of the hair-line sights not being exactly over, but parallel to, the ruling edge of the alidade is negligible in most cases. Thus if the horizontal distance between the line of collimation and the alidade edge is 1 in., the displacement of b will be inappreciable unless the scale is 8 to 10 ft. or less to 1 in.

If the line of sight is not quite parallel to the edge of the rule, the accuracy will be unaffected.

(iv) If the instrument is provided with a telescope, the observations are liable to such errors as are mentioned in Chapter IV.

(v) Errors of bisection will cause errors in the direction of the rays on the paper—and these will in turn displace the intersected point from its true position on the plan, as explained in Chapter XVIII on Photographic Surveying.

(vi) If the tangent scale of the clinometer can be read to three significant figures, the deduced altitude will be correct only to about that number of significant figures. Altitudes are also subject to errors caused by the plotting and scaling of the horizontal distances from the plan.

(vii) An error may be introduced if the bubble is not in the centre of its run at the time of the observation, or if the zero line of the clinometer is not parallel to the bubble axis.

References.—*A Text-book of Topographical and Geographical Surveying*, Close and Winterbotham. *Organized Plane Tabling*, Ordnance Survey Training Series No. 1.

EXAMPLES

1. Determine the altitude of the instrument station O if the following observations were made from O to points A and B. Correct for curvature and refraction.

Angle of elevation to A = 4°-31′.	Scaled distance OA = 2570 ft.
,, ,, B = −2°-20′.	,, ,, OB = 4937 ft.
The reduced level of A = 1570·3.	Height of signal at A = 10·7 ft.
,, ,, B = 1151·2	,, ,, B = 26·5 ft.

Height of alidade above ground = 4·1 ft.

2. In setting up the plane table at a station A, the corresponding point a on the plan was not accurately centred above A.

If the displacement of a was 1 ft., in a direction at right angles to a ray AB, how much on the plan would be the consequent displacement of b from its true position if

(i) scale = 6 in. to 1 mile: distance AB = 9000 ft. ?
(ii) ,, = $\frac{1}{500}$th ,, = 120 ft. ?
(iii) ,, = 10 ft. to 1 in. ,, = 45 ft. ?

3. The reduced level of a plane table station being 100·00 ft., and the height of the alidade above the ground 4·2 ft., find the reduced level of a staff station A, when the reading upon the Indian clinometer scale is 0·027. The distance to A scaled from the plan is 7680 ft., and a vane 10 ft. above the ground at A was sighted.

CHAPTER X

CURVE RANGING

In the present chapter the chief methods employed in the setting out of railway and other curves—too large to be described with a radius rod or a string from a fixed centre—will be explained.

As a general rule a curve which is a circular arc is required to connect two straight lengths, and these must be tangential to the curve, in order that there shall be no abrupt break at the junctions.

Let BA and AC (Fig. 10.1) be the two straight lines intersecting at A, and let the curve touch them at the two tangent points T and T_1. If the chainage is being carried through the curve in the direction T to T_1, the point T is sometimes called the "Point of Curve" and T_1 the "Point of Tangent". The radii of the curve at T and T_1 will be at right angles to BA and AC respectively; and if these two perpendiculars be drawn, their intersection at O locates the centre of the circle of which the curve TT_1 is a part.

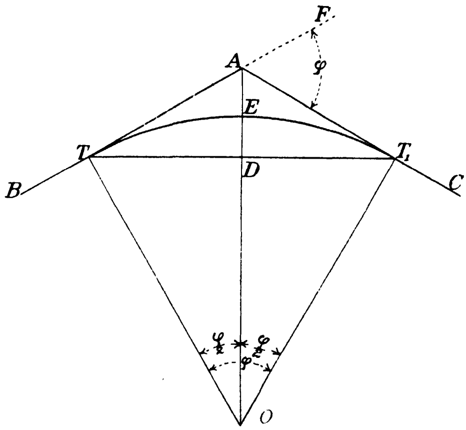


Fig. 10.1

The line joining T and T_1 is known as the "long chord" of the curve, and by the corollary to Euclid III. 17, the angle ATT_1[1] is equal to the angle AT_1T; also the tangent distance AT is equal to the tangent distance AT_1. If a line be drawn joining O to A, it will intersect the curve at E, the apex, crowning point, or summit of the curve.

The intercept DE on the line OA between E and the point of intersection D of OA and the long chord TT_1 is known as the "versed sine" of the curve.

The angle FAC $(=\phi)$ is known as the deflection angle of the curve,

[1] Sometimes the angle ATT_1 is known as the total deflection angle of the curve. See Rankine's *Method of Deflection Angles*, p. 314.

and is equal to the sum of the two interior and opposite angles of the triangle ATT_1, *i.e.*

$$\angle FAC = 2\angle ATT_1,$$

or

$$\angle ATT_1 = \angle AT_1T = \frac{\phi}{2}.$$

But the angle ATT_1, between the tangent AT and the chord TT_1, is equal to the angle subtended by TT_1 in the opposite segment of the circle (Euclid III. 32), and this is equal to half the angle TOT_1 subtended by TT_1 at the centre.

The angle ATT_1 is thus equal to $\frac{1}{2}\angle TOT_1$ so that $\angle TOT_1 = \phi$, *i.e.* the angle at the centre—TOT_1—is equal to the deflection angle—FAT_1—for the curve.

The angle TAT_1 is known as the intersection angle, and is equal to $180° - \phi$.

Generally the positions of the two straight lengths to be connected are fixed, and hence the deflection angle, ϕ, of the curve can be found directly from the plan or measured on the ground. Then as the angle $AOT = \frac{\phi}{2}$ and $\frac{AT}{OT} = \tan\frac{\phi}{2}$, we get

$$AT = R \,.\, \tan\frac{\phi}{2}, \qquad . \quad . \quad . \quad (1)$$

where R is the radius of the curve.

From this, if the tangent distances AT and AT_1 and ϕ are known, R can be calculated ; or if R and ϕ are known the positions of T and T_1 can be determined.

Again $\qquad \frac{R}{OA} = \cos\frac{\phi}{2}$ or $OA = R \,.\, \sec\frac{\phi}{2}$,

and as $OE = R$, $\qquad \therefore OA = R + EA = R \,.\, \sec\frac{\phi}{2}$,

i.e.

$$EA = R\left(\sec\frac{\phi}{2} - 1\right). \qquad . \quad . \quad . \quad (2)$$

From this, if the value of ϕ and the position of E are fixed, R can be found ; or if R and ϕ are given the position of E can be located.

In English practice, circular railway curves are generally defined by the radius, *e.g.* a 10-chain curve, a 20-chain curve, etc. While in America the curve is defined by the number of degrees at the centre subtended by a 100 ft. chord, *e.g.* a 5° curve is one in which a chord of 100 ft. subtends an angle of 5° at the centre.

In Fig. 10.2 let TN represent a chord of 100 ft. length and $\angle TON = \theta°$, *i.e.* θ is the degree of the curve.

Draw OM, bisecting TN at right angles, so that TM = 50 ft. and

$$\frac{TM}{OT} = \sin\frac{\theta}{2}, \text{ i.e. } 50 = R \,.\, \sin\frac{\theta}{2}. \qquad . \quad . \quad (3)$$

If θ be small, then $\sin\frac{\theta}{2}$ is approximately equal to $\frac{\theta}{2}$ in radians, *i.e.*

$$\sin\frac{\theta}{2}=\frac{\theta}{2}\cdot\frac{\pi}{180} \text{ where } \theta \text{ is in degrees.}$$

$$\therefore \mathrm{R}=\frac{50}{\sin\frac{\theta}{2}}=\frac{50\times 2\times 180}{\theta\,.\,\pi}=\frac{5730}{\theta} \text{ ft. approximately.} \qquad (4)$$

The length of the curve will approximately be $\frac{100\times\phi}{\theta}$, where ϕ is the deflection angle of the curve.

1. **Ordinates from the Long Chord.**—Small curves for street kerbs, etc., may be set out by means of offsets from the long chord, and a

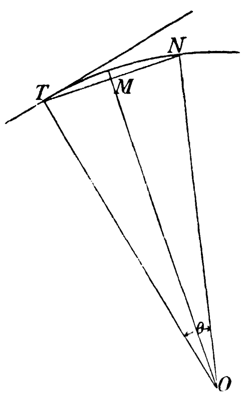


FIG. 10.2

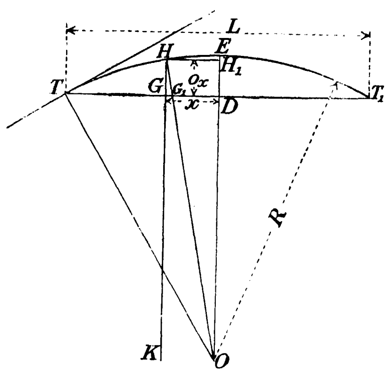


FIG. 10.3.—Offsets from Long Chord

formula for the ordinate at any distance x from the centre (Fig. 10.3) may be deduced as follows.

Let L be the length of the long chord, T and T_1 its extremities, DE the versed sine, and O the centre of the curve, and o_x the ordinate at a distance x from D.

By Euclid I. 47,

$$\mathrm{OT}^2=\mathrm{OD}^2+\mathrm{DT}^2,$$

i.e.

$$\mathrm{R}^2=(\mathrm{R}-\mathrm{DE})^2+\left(\frac{\mathrm{L}}{2}\right)^2,$$

i.e.

$$\mathrm{R}-\mathrm{DE}=\sqrt{\mathrm{R}^2-\left(\frac{\mathrm{L}}{2}\right)^2},$$

$$\therefore \mathrm{DE} \text{ or } o_0=\mathrm{R}-\sqrt{\mathrm{R}^2-\left(\frac{\mathrm{L}}{2}\right)^2}, \qquad (5)$$

from which o_0—the versed sine—can be calculated when R and L are known, or if o_0 and L are stated, R can be calculated.

Draw HGK (Fig. 10.3) parallel to EO, at a distance x along the chord from D, and draw HH_1 parallel to TT_1, cutting OE in H_1.

Then $$OH^2 = OH_1{}^2 + HH_1{}^2,$$

i.e. $$R^2 = (OD + o_x)^2 + x^2,$$

or $$OD + o_x = \sqrt{R^2 - x^2}$$

and $$o_x = \sqrt{R^2 - x^2} - OD,$$

or $$o_x = \sqrt{R^2 - x^2} - (R - o_o). \quad . \quad . \quad . \quad (6)$$

Example.—Calculate the ordinates at 25 ft. distances for a circular curve, having a long chord of 200 ft., and a versed sine of 10 ft.

By substitution in equation (5),

$$10 = R - \sqrt{R^2 - (100)^2}, \ \therefore R = 505 \text{ ft.}$$
$$R - o_0 = 495 \text{ ft.}$$
$$o_0 = 10 \text{ ft.}$$
$$o_{25} = \sqrt{505^2 - 25^2} - 495 = 9{\cdot}38 \text{ ft.}$$
$$o_{50} = \sqrt{505^2 - 50^2} - 495 = 7{\cdot}52 \text{ ft.}$$
$$o_{75} = \sqrt{505^2 - 75^2} - 495 = 4{\cdot}40 \text{ ft.}$$

If the radius of the curve is very large in comparison with the length of the chord TT_1, the ordinate at x, *i.e.* HG, is very nearly equal to the radial length HG_1, so that approximately, by III. 35,

$$TG \,.\, GT_1 = HG \,.\, 2R,$$

or $$HG = \frac{TG \,.\, GT_1}{2R}.$$

Applying this rule to the above example,

$$2R = \frac{10000}{10} = 1000 \text{ ft. nearly,}$$

and the ordinates are $$o_{25} = \frac{75 \times 125}{1000} = 9{\cdot}37 \text{ ft.}$$

$$o_{50} = \frac{50 \times 150}{1000} = 7{\cdot}50 \text{ ft.}$$

$$o_{75} = \frac{25 \times 175}{1000} = 4{\cdot}37 \text{ ft.,}$$

agreeing very approximately with those derived by the more accurate rule.

2. **Offsets from the Tangents.**—In this method the offsets are set out from the two tangents AT and AT_1. Let the offset aa_1 at a distance Ta from the tangent point be set out radially as in Fig. 10.4.

Then by Euclid. III36,

$$aT^2 = aa_1(aa_1 + 2R). \quad . \quad . \quad . \quad (7)$$

If aa_1 is negligible compared with 2R, as would usually be the case for a flat curve,

$$aT^2 = 2R \,.\, aa_1,$$

i.e.

$$aa_1 = \frac{aT^2}{2R},$$

or

$$o_x = \frac{x^2}{2R}, \qquad \ldots \qquad (8)$$

where o_x is the offset at a distance x from T.

For flat curves the centre O is often inaccessible and the ordinates are then set out, without any great error, at right angles to the tangents AT and AT_1, and at equal distances along these lines. This entails that the points so fixed on the curve are at unequal distances apart, which is occasionally an inconvenience, and the curve, though closely approximating to a circle, is, strictly speaking, parabolic.

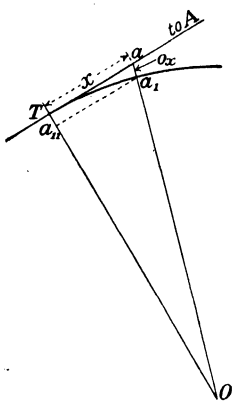


FIG. 10.4.—Offsets from Tangent

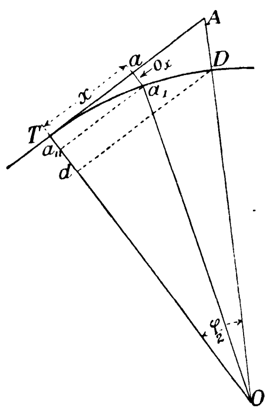


FIG. 10.5.—Offsets from Tangent

When aa_1 is not negligible compared with R, a more accurate expression for the radial ordinates may be deduced from (7). A more convenient form (equation 9) may be derived from (7) or directly as follows :

As T is the tangent point to the curve, $\angle$ATO is a right angle,

$$\therefore aO^2 = aT^2 + TO^2 \quad \ldots \quad (I.\ 47),$$

i.e.

$$(aa_1 + R)^2 = aT^2 + R^2,$$

or

$$aa_1 = \sqrt{R^2 + aT^2} - R,$$

or

$$o_x = \sqrt{R^2 + x^2} - R. \qquad \ldots \qquad (9)$$

An accurate expression for offsets to be set out perpendicular to the tangent TA may be derived as follows (Fig. 10.5).

Join Oa_1, and draw a_1a_{11} parallel to AT and perpendicular to OT.

Then $(Oa_1)^2 = Oa_{11}^2 + a_1a_{11}^2$,

i.e. $R^2 = (R - o_x)^2 + x^2$,

$R - o_x = \sqrt{R^2 - x^2}$,

and $o_x = R - \sqrt{R^2 - x^2}$. . . . (10)

The value of x required to locate the vertex D of the curve

$$= dD = OD \sin \frac{\phi}{2}. \quad . \quad . \quad . \quad (11)$$

Example.—Determine the offsets to be set out at ½-chain intervals along the tangents, to locate a 20-chain curve.

From equation (8), the 1st offset $o_{\cdot 5} = \frac{50^2}{2 \times 2000} = \cdot 625$ link,

the 2nd offset $o_1 = \frac{100^2}{2 \times 2000} = o_{\cdot 5} \times 2^2 = 2{\cdot}500$ links,

the 3rd offset $o_{1\cdot 5} = o_{\cdot 5} \times 3^2 = 5{\cdot}625$ links,

the 10th offset $o_5 = o_{\cdot 5} \times 10^2 = 62{\cdot}5$ links.

For comparison the offsets are shown in tabular form below: in column 3 are the results obtained by substitution in (9), and in column 5 those obtained from (10). Column 4 shows the errors resulting from the approximation assumed in equation (8), when the ordinates are set out in a radial direction; while column 6 gives the corresponding errors introduced when the ordinates are set out at right angles to the tangents.

TABLE

Distance x	Usual method Equation (8)	Accurate method for radial offsets Equation (9)	Difference	Accurate method for perpendicular offsets Equation (10)	Difference
Chains	Links				
·5	·62	·62	..	·62	..
1·0	2·50	2·50	..	2·50	..
1·5	5·62	5·62	..	5·62	..
2·0	10·00	9·98	·02	10·03	·03
2·5					
3	22·50	22·37	·13	22·63	·13
4	40·00	39·61	·39	40·41	·41
5	62·50	61·55	·95	63·51	1·01

An error is here introduced into the first decimal place of a link when x is about 3 chains.

Assuming this ordinate comes to about the centre point of the curve when set out at right angles to the tangent, the value of the deflection angle ϕ from equation (11) would be approximately 17°.

For values of ϕ larger than this, with a 20-chain curve, the errors

would therefore begin to be appreciable, but in such a case, as the offsets otherwise become inconveniently large (and are not negligible compared with R), the centre portion of the curve might be located from a supplementary tangent through D as in Fig. 10.6.

3. **Kröhnke's Tangential System.**—Here unequal abscissae are taken along the tangents TA and T_1A in such proportions that the points located on the curve by perpendicular ordinates are equidistant ; and tables are published to facilitate the work. Unless the chainage at the commencement of the curve, however, chances to be a whole number of chains, the points so determined are not more convenient than points found by Method 2.

4. **Jackson's**[1] **two methods** for setting out a circular curve from the tangents. In each method six points on the curve are fixed from TA by means of ordinates tabulated for different radii. A second tangent through the sixth point so determined is then set out, its

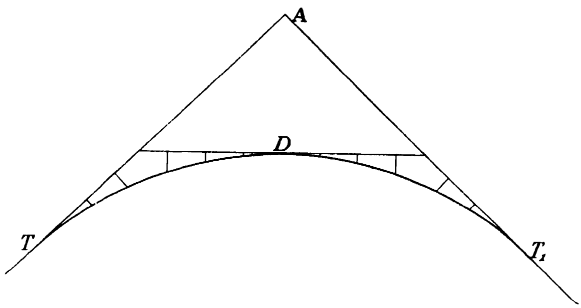


FIG. 10.6

direction being found from the third point by means of other tabulated data. A further six points are then located from this tangent, the same ordinates and abscissae being used as for the first six. A third tangent is then set out through the twelfth point, and the process repeated until the end of the curve is reached.

In one method equal abscissae are taken, giving points at unequal distances along the curve as in Method 2 ; in the other method unequal abscissae are taken, and equidistant points on the curve determined.

5. **Offsets from Chords.**—This method is that which is most usually employed when an angular instrument is not available and when it is necessary to resort to the " chain and tape ". The *modus operandi* is as follows :

It is generally desirable, in railway and other work, that the " chainage " shall be carried continuously throughout the whole line ; so that if the chainage at T is n chains $+ m$ links, the first chord length will be the remaining portion of the chain length, *i.e.* $(100 - m)$ links. The end of the mth link, which falls at T, is held there while the front end of the chain is swung round from a (Fig. 10.7), in line with T and A, through the calculated distance aa_1, thus fixing the point a_1, $(n+1)$

[1] Jackson, *Aids to Survey Practice.*

chains from the commencement of the work. The chain is next pulled in the direction of Ta_1 produced, until $a_1b = 1$ chain (or $\frac{1}{2}$ chain as the case may require), and, the rear end of the chain being held at a_1, it is swung round this point as centre, moving the fore end from b to b_1, through the second calculated offset distance bb_1. a_1b_1 is then prolonged to c, b_1c being the length of chord required, and the point c_1 found by swinging the chain round b_1 as centre, through a distance equal to the third calculated offset cc_1. This method probably gives better results than those by offsets from the tangents (Method 2), but is not nearly as accurate as those methods in which a theodolite is used. If the curve does not join the tangent AT_1 at the correct point T_1—and probably it will not do so at the first trial—the points a_1, b_1, c_1, etc., must be adjusted by repeating the work until a correct result is obtained. For ordinary railway curves of 20 or more chains radius

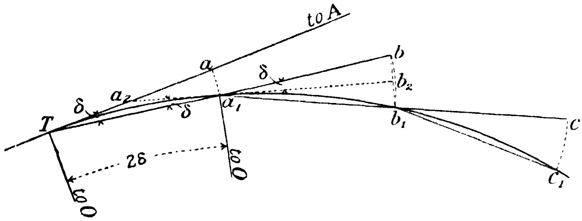


FIG. 10.7.—Offsets from Chords

a 1 chain length of chord is quite suitable, but for sharper curves a chord length of $\frac{1}{2}$ chain or less should be adopted.

Derivation of Formulae.—Let the angle aTa_1 (Fig. 10.7) between the tangent aT and the chord Ta_1 be δ radians. Then the angle subtended at the centre O by $Ta_1 = 2\delta$ (III. 20 and 32). But as the chord Ta_1 is nearly equal to the arc Ta_1,

$$Ta_1 = R \,.\, 2\delta$$

and

$$\delta = \frac{Ta_1}{2R}. \qquad . \quad . \quad . \quad . \quad (12)$$

Similarly, the chord aa_1 is nearly equal to the arc aa_1,

$$\therefore\ aa_1 = Ta_1 \,.\, \delta,$$

or by substitution from (12)

$$aa_1 = \frac{Ta_1^{\,2}}{2R},$$

or

$$o_1 = \frac{c_1^{\,2}}{2R}, \qquad . \quad . \quad . \quad (13)$$

where o_1 is the first offset and c_1 is the length of the first chord.

At a_1 draw a tangent to the curve, cutting Ta in a_2 and bb_1 in b_2.

Then $a_2T = a_2a_1$, because both are tangents to the circle from a_2, and $\angle a_2Ta_1 = \angle a_2a_1T$. Also $\angle a_2a_1T = \angle ba_1b_2$, which is vertically opposite (I. 15).

The triangles aTa_1 and ba_1b_2 are approximately similar as both are nearly isosceles,

$$\therefore \frac{bb_2}{a_1b} = \frac{aa_1}{Ta}, \; i.e. \; bb_2 = \frac{a_1b}{Ta} \cdot o_1 = \frac{c_2}{c_1} \cdot \frac{c_1^2}{2R} = \frac{c_2c_1}{2R}.$$

Also b_2b_1 being the offset from the tangent a_1b_2, may be written as

$$\frac{c_2^2}{2R}.$$

The second offset bb_1 is therefore equal to $bb_2 + b_2b_1$, *i.e.*

$$o_2 = \frac{c_2c_1}{2R} + \frac{c_2^2}{2R} = \frac{c_2(c_1+c_2)}{2R}. \qquad (14)$$

Similarly, $o_3 = \dfrac{c_3(c_2+c_3)}{2R}$, from which, when c_2 and c_3 are equal, as would usually be the case,

$$o_3 = \frac{c_3^2}{R}. \qquad (15)$$

The remaining offsets are all equal to this, when equal chords are employed, except the last one (which should finish at the tangent point T_1).

This, as the last chord length is probably an uneven number of links, is

$$o_n = \frac{c_n(c_{n-1}+c_n)}{2R}. \qquad (16)$$

Example (Lond. B.Sc.).—Two roads meet at an angle of 127°-30′. Calculate the necessary data for setting out a curve of 15 chains radius to connect the two straight portions of the road (*a*) if it is intended to set out the curve by chain and offsets only, (*b*) if a theodolite is available. Explain carefully how you would, in both cases, set out the curve in the field.

(*a*) The deflection angle ϕ (Fig. 10.1) is 180° − 127°-30′ = 52°-30′.

From equation (1) the tangent distances AT and AT_1, from which T and T_1 may be located, are each equal to R tan 26°-15′,

$$= 15 \times {\cdot}4931 = 7{\cdot}397 \text{ chains.}$$

The length of the curve is $\pi \times 15 \times \dfrac{52{\cdot}5}{180}$ chains

$$= 13{\cdot}744 \text{ chains.}$$

Taking 13 chords, each equal to 100 links, and one chord of 74·4 links, it is found by formula (13) that the first offset $o_1 = \dfrac{100^2}{2 \times 1500} = 3\frac{1}{3}$ links.

As the chords are all equal, with the exception of the 14th, the next 12 offsets —by formulae 14 or 15—are each equal to $\dfrac{100^2}{1500} = 6\frac{2}{3}$ links.

The last offset by formula (16) is

$$o_{14} = \frac{74{\cdot}4\,(100 + 74{\cdot}4)}{2 \times 1500} = 4\tfrac{1}{3} \text{ links.}$$

Similarly, if 50-link chords had been adopted,

$$o_1 = {\cdot}83 \text{ link,}$$
$$o_2 - o_{27} = 1{\cdot}66 \text{ links,}$$
$$o_{28} = {\cdot}61 \text{ link.}$$

(*b*) See solution after Method 7.

L

6. **Offsets from Chords.**—In those cases in which the ground outside the curve is not favourable for chaining, the curve may be set out by means of offsets from chords inside the circle.

Let the chords Tb_1, a_1c_1, b_1d_1, etc. (Fig. 10.8) be of equal length l. Then as the versed sine is given by equation (5) as $v = R - \sqrt{R^2 - (l/2)^2}$, the lengths a_1a_2, b_1b_2, c_1c_2, etc. can be calculated, when the radius and chord length are decided upon.

When R is large compared with l, the formula simplifies to $v = l^2/8R$.

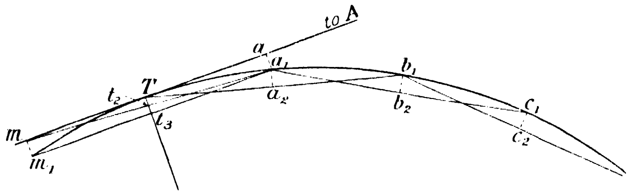


FIG. 10.8.—Offsets from Chords

In American practice, when a 100-ft. chord subtends a definite angle, *e.g.* $n°$, and the radius is not an even number of feet or chains, it may be more convenient to refer to tables for the values of the offsets, *e.g.* $v = R$ versin $n°$.

If Tm be set out along the straight length AT, and made equal to Ta, then ma_1 cuts the radial line Tt_3 at t_2 where $Tt_2 = \frac{1}{2}v$, *i.e.* $\frac{1}{2}$ a_1a_2.

To set out the curve, the method of procedure is to set out $Tt_2 = \frac{1}{2}v$ at right angles to TA, chain $Tm = l/2$, join mt_2 and produce to a_1, making $ma_1 = l$. At a_1 set out $a_1a_2 = v$ radially or approximately so, and produce Ta_2 to b_1, making $Tb_1 = l$. Incidentally, aa_1, the offset from the tangent to $a_1 = a_1a_2$ when R is large. Then set out $b_1b_2 = v$, $a_1b_2c_1 = l$, and so on until the end of the curve is reached.

7. **Rankine's Method of Tangential Angles**[1] (Fig. 10.9), which involves the use of one theodolite and a chain or tape, is the method which is most frequently adopted for setting out railway or other important curves. It is a very convenient operation, and the calculations involved are simple—particularly in American practice, where the "degree" of a curve is stated. It yields good practical results unless the chords are so long compared with the radius that the variation between the length of an arc and its chord becomes considerable.

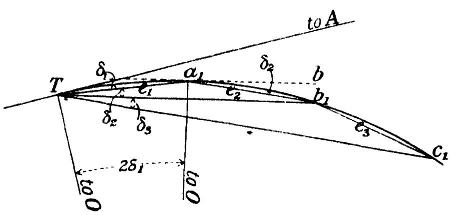


FIG. 10.9
Rankine's Method of Tangential Angles

[1] The term "Deflection" angle is often used instead of Tangential angle.

The procedure is as follows :

The theodolite is set up at the tangent point T (Fig. 10.9) with the vernier adjusted to 360°, and the telescope is directed to A and clamped. If the chainage at T is n chains $+m$ links, the length of the first chord will usually be $(100-m)$ links (or feet as the case may be). The first tangential angle $\text{AT}a_1$ is calculated from the formula derived below, and the vernier of the instrument fixed to read this angle, the telescope being then directed towards a_1. The end of the mth link is retained at T, and the remaining $(100-m)$ links swung round this point as a centre, while the person looking through the telescope indicates to the leader at a_1 the direction in which he must move in order to obtain exact coincidence with the cross-hairs.

a_1 having been located, and the instrument still remaining at T, the vernier is altered to read the second tangential angle $\text{AT}b_1$ and re-clamped. The telescope is consequently directed along the line $\text{T}b_1$, and the second chord a_1b_1 is set out by swinging the chain round a_1 as a centre until the image of b_1 coincides with the cross-hairs. The remaining chords are set out in a similar manner, *e.g.* the position of c_1 is fixed by swinging the chain round b_1 as a centre, and getting coincidence with the cross-hairs of the diaphragm when the telescope is directed along the line $\text{T}c_1$, and the vernier reads the third tangential angle $\text{AT}c_1$.

The formula for the angles will now be derived. The first tangential angle (δ_1) is the angle $\text{AT}a_1$, between the tangent TA and the chord $\text{T}a_1$, and is equal to the angle in the opposite segment. But this angle is half the angle subtended by $\text{T}a_1$ at the centre of the circle, so that $\angle \text{TO}a_1 = 2 \,.\, \delta_1$.

When the length of the chord is small compared with the radius of the curve, the chord $\text{T}a_1$ is approximately equal to the arc $\text{T}a_1$, and its length $l_1 = 2\pi\text{R} \,.\, \dfrac{2\delta_1}{360}$, where $2\pi\text{R}$ is the circumference of the complete circle,

$$\therefore\ \delta_1 = \frac{360}{2} \times \frac{l_1}{2\pi\text{R}} = \frac{90}{\pi} \,.\, \frac{l_1}{\text{R}} \text{ degrees}$$

$$= \frac{60 \times 90}{\pi} \,.\, \frac{l_1}{\text{R}} \text{ minutes,}$$

$$\textit{i.e. } \delta_1 = 1718{\cdot}9 \,.\, \frac{l_1}{\text{R}} \text{ minutes.} \qquad . \quad (17)$$

Similarly, the tangential angle δ_2 of a_1b_1 *from the tangent* at a_1, *i.e.*

$$\angle ba_1b_1 \text{ is } 1718{\cdot}9 \,.\, \frac{l_2}{\text{R}} \text{ minutes,}$$

and this angle, between the tangent ba_1 and the chord a_1b_1, is equal to the angle in the opposite segment, *i.e.*

$$\angle ba_1b_1 = \angle a_1\text{T}b_1,$$

therefore $\angle ATb_1$, the total tangential angle of b_1, *i.e.* Δ_2, is equal to $\delta_1 + \delta_2$.

Similarly, $\angle ATc_1$, the total tangential angle of c_1, *i.e.* Δ_3, is equal to $\delta_1 + \delta_2 + \delta_3$, where $\delta_3 = 1718{\cdot}9 \cdot \frac{l_3}{R}$ minutes.

The total tangential angle Δ_n for the last chord (*i.e.* $\delta_1 + \delta_2 + \delta_3 + \ldots + \delta_n$) should evidently be equal to $\angle ATT_1$.

Example.—Part (*b*) of the example given on p. 313 after Method 5. Here $\phi = 52°\text{-}30'$, R = 15 chains, and the length of curve = 13·744 chains.

There will, therefore, be 13 chords each of 1 chain length and one chord of 74·4 links.

$$\begin{aligned}\delta_1 = \delta_2 = \delta_3 \ldots = \delta_{13} &= 1718{\cdot}9 \,.\, \frac{1}{15}\\ &= 114{\cdot}6 \text{ minutes}\\ &= 1°\text{-}54'{\cdot}6,\\ \delta_{14} &= 1718{\cdot}9 \times \frac{{\cdot}744}{15}\\ &= 1°\text{-}25'{\cdot}3.\end{aligned}$$

From these data the tangential angles may be tabulated :

$$\begin{aligned}\Delta_1 &= \delta_1 & &= 1°\text{-}54'{\cdot}6\\ \Delta_2 &= \delta_1 + \delta_2 & &= 3°\text{-}49'{\cdot}2\\ \Delta_3 &= \delta_1 + \delta_2 + \delta_3 & &= 5°\text{-}43'{\cdot}8\\ &\vdots\\ \Delta_{13} &= \delta_1 + \ldots \delta_{13} = 13\delta_1 & &= 24°\text{-}49'{\cdot}8\\ \Delta_{14} &= \Delta_{13} + \delta_{14} & &= 26°\text{-}15'{\cdot}1.\end{aligned}$$

This value of Δ_{14} agrees with $\angle ATT_1 = \frac{\phi}{2} = 26°\text{-}15'$.

In the case of a curve deflecting towards the *left*, each of these values would be subtracted from 360° to give the required reading on the vernier of the instrument.

8. **The Two Theodolite Method.**—To set out a curve over rough ground, two theodolites may with advantage be employed (Fig. 10.10), and the use of a chain or a tape dispensed with ; but the method,

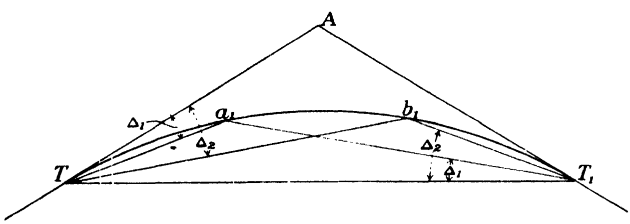


FIG. 10.10.—Two Theodolite Method

although simple and accurate, is not so often adopted as Rankine's Method 7, owing to the fact that it is seldom economical or convenient to employ two experienced surveyors with two instruments, upon a curve which can be satisfactorily located with one.

If the chainage at the tangent point T is n chains + m links,

the first point a_1 on the curve will be $(100-m)$ links from T, and the tangential angle ATa_1, *i.e.* δ_1, is calculated as in Method 7, *i.e.* $\delta_1 = 1718{\cdot}9 \times \frac{100-m}{R}$ minutes, where R is expressed in links.

This angle ATa_1 is equal to the angle in the opposite segment, *i.e.* $\angle TT_1a_1$: and similarly the second total tangential angle $\triangle_2$, *i.e.* $\angle ATb_1$, is equal to TT_1b_1.

Hence it is easily seen from Fig. 10.10 that if a theodolite is set up at T with its vernier reading 360°, and sighted to A, while another instrument, with its vernier reading 360° is fixed at T_1, and sighted to T, then by fixing the same reading $\triangle$ on each instrument, the intersection of the two lines of collimation gives a point on the curve. By using the tangential angles calculated by formula (17), the correct points a_1b_1, etc., can be fixed, so that the chainage is continued from the straight without interruption round the curve. Any particular point is quickly found if each of the instrument men at T and T_1 in turn waves the staff man into coincidence with his cross wires. A very few trials will usually suffice to locate the point.

In the case of a curve deflecting towards the *left*, each of the values $\triangle_1$, $\triangle_2$, $\triangle_3$, etc., would be subtracted from 360°, to give the required readings on the verniers of both instruments, the zeros being as already stated.

To set out the curve mentioned in the Example on p. 313, the table of tangential angles is exactly similar to that calculated on p. 316.

9. **Tacheometric Curve Ranging.**—Chaining may be dispensed with by the use of a tacheometer, but the method is, of course, much less exact than Method 7. If the instrument is set up at T (Fig. 10.11), each point on the curve is fixed by the tangential angle from TA as before, and by its distance measured tacheometrically *from* T—instead of directly, from the preceding point, as in Rankine's Method.

If $\triangle$ is the tangential angle (calculated from formula (17)), the angle subtended at the centre of the circle is $2\triangle$, and the length of the whole chord from T is $2R \sin \triangle$ (*cf.* formula (3)). The various lengths are therefore $2R \sin \triangle_1$, $2R \sin \triangle_2$, etc., $2R \sin \triangle_n$, where $\triangle_n = \delta_1 + \delta_2 + \ldots + \delta_n$, as in Method 7.

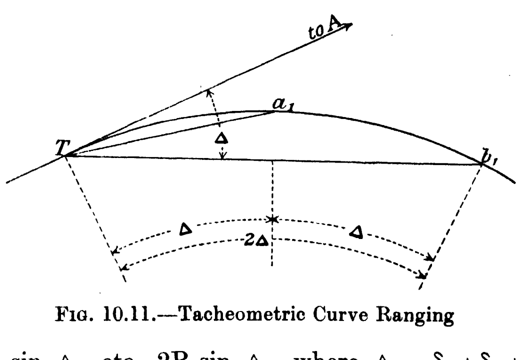


FIG. 10.11.—Tacheometric Curve Ranging

These may be deduced for any particular value of R and δ_1, δ_2, etc., by means either of ordinary sine or log sine tables, or of special tables. If the ground is moderately level, the required intercepts on the staff can be calculated by formula (4), p. 244, otherwise; if inclined sights (p. 250) are needed, the method would be very cumbersome.

Example.—Make the necessary calculations for setting out tacheometrically the curve of which the data are given on p. 313, the engineer's chain being referred to. The results may be expressed in tabular form.

TABLE

Tangential Angle, Δ	Length (D) of Chord from T. $D=2R \sin \Delta$	Intercept on staff with telescope approximately horizontal
	Feet	
$\Delta_1 = 1°\text{-}54'\cdot6$	100	·99
$\Delta_2 = 3°\text{-}49'\cdot2$	199·8	1·99
$\Delta_3 = 5°\text{-}43'\cdot8$	299·5	2·98
$\Delta_4 = 7°\text{-}38'\cdot4$	398·8	3·98
$\Delta_5 = 9°\text{-}33'$	497·7	4·97
⋮ ..	..	..
$\Delta_{13} = 24°\text{-}49'\cdot8$	1259·8	..
$\Delta_{14} = 26°\text{-}15'$	1326·8	..

Column (1) is calculated, as already explained, on p. 316, Method 7.
Column (2) is calculated from the formula $D=2R \sin \Delta$.

Column (3) is calculated from the formula $D=s\,.\,\frac{f}{i}+f+d$ ((4), p. 244), where $\frac{f}{i}=100$ and $(f+d)=1$.

It will be seen that, beyond the fifth point, the distances become too large to be clearly distinguished from T, so that then the instrument would need to be moved to point 5 and the tangential angles set out from there as explained later.

Points 6, 7, 8, 9, 10 would be fixed with the same intercepts on the staff as points 1, 2, 3, 4, 5 (col. 3): after which the instrument would be moved to 10, and the same intercepts used for 11, 12, 13, and 14 as for 1, 2, 3, 4, while the final intercept 15 would be 4·71—to finish at the tangent point T_1.

A few miscellaneous problems in connection with the ranging of circular curves will now be considered.

(*a*) To find the positions of the two tangent points T and T_1 for a curve of radius R, when the directions of the two tangents BA and AC have been decided upon, but the point of intersection A is inaccessible.

On the tangents BA and AC respectively, fix any two points E and F (Fig. 10.12). Set up the theodolite at E and F in turn, and measure the angles BEF and EFC. By subtracting each of these values from 180°, the angles AEF and AFE in the triangle AEF are found, and the deflection angle ϕ is equal to AEF + AFE.

The length EF is next measured and the distances EA and AF calculated, *i.e.*

$$AF=\frac{EF\,.\,\sin AEF}{\sin EAF} \text{ and } EA=\frac{EF \sin EFA}{\sin EAF}.$$

The radius of the curve R and the deflection angle ϕ being known, the tangent distances AT and AT_1 are calculated from formula (1), *i.e.*

$$AT=AT_1=R\,.\,\tan\frac{\phi}{2}.$$

To fix the point T on the ground, a length equal to (AT − AE) is chained off from E, and similarly T_1 is found by chaining a distance (AT_1 − AF) from F, after which the curve may be set out by one of the methods already described.

Example.—To find the positions of T and T_1 when the following values have been determined by direct measurement and R = 20 chains: BEF = 165°-36′, EFC = 168°-44′, EF = 3·40 chains.

The angle AEF = 180° − 165°-36′ = 14°-24′.
The angle AFE = 180° − 168°-44′ = 11°-16′.

$$\therefore \phi = 25°\text{-}40'.$$

$$AT = AT_1 = 20 \times \tan 12°\text{-}50' = 4{\cdot}556 \text{ chains.}$$

$$AE = \frac{3{\cdot}40 \times \sin 11°\text{-}16'}{\sin 25°\text{-}40'} = \frac{3{\cdot}40 \times {\cdot}1954}{{\cdot}4331} = 1{\cdot}534 \text{ chains.}$$

$$AF = \frac{3{\cdot}40 \times \sin 14°\text{-}24'}{\sin 25°\text{-}40'} = \frac{3{\cdot}40 \times {\cdot}2487}{{\cdot}4331} = 1{\cdot}952 \text{ chains.}$$

The distance of T from E = AT − AE = 3·022 chains.
The distance of T_1 from F = AT_1 − AF = 2·604 chains.

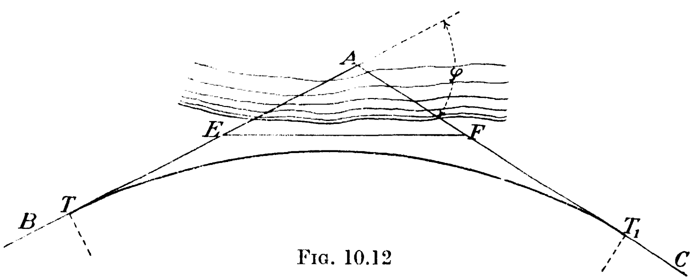


FIG. 10.12

(*b*) If it is not possible, on account of buildings or other objects intervening in the line of sight, to set out all the points on a curve by means of tangential angles (Method 7) from one position of the instrument T, the instrument may be moved and the work proceeded with as follows.

Suppose c_1 (Fig. 10.13) is the last point which can be seen from T, and that the tangential angle is $\triangle_c$. While the instrument is still stationed at T, and directed towards c_1, set out another point c_3 in the same line Tc_1 produced. Now move the instrument to c_1, fix the vernier at 360°, and direct the cross-hairs on to c_3.

If nc_1c_4 be the tangent at c_1, $nc_1 = n\mathrm{T}$, and the angle $nc_1\mathrm{T} = \triangle_c$; consequently the vertically opposite angle $c_3c_1c_4$ must be equal to $\triangle_c$, so that if the vernier of the theodolite is again set at $\triangle_c$ the telescope will point along the tangent c_1c_4.

The angle $c_4c_1d_1$ between this tangent and the next chord c_1d_1 is equal to the angle $c_1\mathrm{T}d_1$ in the opposite segment (*i.e.* δ_d, Method 7): hence by adding $\triangle_c$ to each, $c_3c_1d_1$ is seen to equal $\triangle_d$.

Similarly, the angle $e_1c_1d_1 = \angle e_1\mathrm{T}d_1$ and consequently $\angle c_3c_1e_1 = \triangle_e$, and $c_3c_1f_1 = \triangle_f$, etc.

From this it may be seen that the instrument can be moved from the tangent point to any other point on the curve, and the staking out proceeded with from that point, using the *same* calculated angles as before.

Instead of sighting to an arbitrary point such as c_3, when the theodolite is moved to c_1, the telescope may be sighted back to T with the vernier reading 360°, and then turned vertically through 180° (*i.e.* transitted); or it may be turned through 180° in azimuth, and

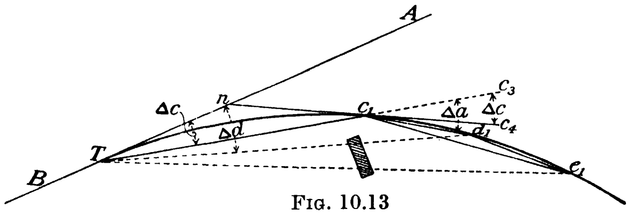


FIG. 10.13

the opposite vernier taken for the new points on the curve. If practicable, however, a point such as c_3 should be fixed—to eliminate possible errors due to the line of collimation of the instrument not being in adjustment.

(*c*) To determine a compound curve which will be tangential to the two straight lengths BA and AC (Fig. 10.14), and also touch at a point P, a third line MN intersecting these.

The curve will be made up of two circular arcs, each of which will be tangential to MN at P; *i.e.* MP is the tangent distance (=MT) of

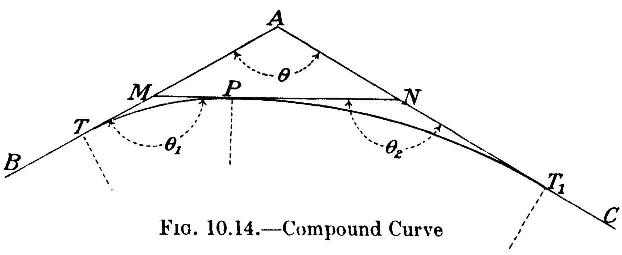


FIG. 10.14.—Compound Curve

the one, and NP ($=NT_1$) that of the other, so that T and T_1 can be easily located.

Measure the angles BMP and PNC $=\theta_1$ and θ_2, say. Then the radius R_1 of the curve between T and P is equal to MP . tan $\frac{\theta_1}{2}$; and R_2 the radius of the curve P to T_1 is equal to NP tan $\frac{\theta_2}{2}$, by formula (1), p. 306.

(*d*) To determine a simple curve which will be tangential to three lines BA, MN, and AC.

Let the angles at A, M, and N be θ, θ_1, and θ_2 respectively as in

Fig. 10.14, and let the curve touch MN in P, where P is at present not located.

By considering the main tangents AT and AT_1 by formula (1),

$$R = AT \cdot \tan \frac{\theta}{2}, \qquad (18)$$

while, by considering the tangents TM and MP,

$$R = MT \tan \frac{\theta_1}{2} = (AT - AM) \tan \frac{\theta_1}{2}. \qquad (19)$$

Similarly, $$R = (AT_1 - AN) \tan \frac{\theta_2}{2}. \qquad (20)$$

By substituting the value of R from (18) in (19),

$$AT \tan \frac{\theta}{2} = (AT - AM) \tan \frac{\theta_1}{2},$$

or $$AT \left(\tan \frac{\theta_1}{2} - \tan \frac{\theta}{2}\right) = AM \tan \frac{\theta_1}{2},$$

$$\therefore AT = \frac{AM \cdot \tan \frac{\theta_1}{2}}{\tan \frac{\theta_1}{2} - \tan \frac{\theta}{2}}, \qquad (21)$$

and $$R = \frac{AM \cdot \tan \frac{\theta_1}{2} \cdot \tan \frac{\theta}{2}}{\tan \frac{\theta_1}{2} - \tan \frac{\theta}{2}}, \qquad (22)$$

from which equations if the length AM is measured T can be located, and the curve staked out.

Example.—If $\theta = 160°$, $\theta_1 = 171°\text{-}44'$, $\theta_2 = 168°\text{-}16'$, and AM = 3 chains. Then by (21)

$$AT = \frac{3 \times \tan 85°\text{-}52'}{\tan 85°\text{-}52' - \tan 80°}$$
$$= \frac{3 \times 13{\cdot}8378}{13{\cdot}8378 - 5{\cdot}6713} = 5{\cdot}083 \text{ chains,}$$

and $$MT = 2{\cdot}083 \text{ chains,}$$
$$R = AT \tan 80°$$
$$= 5{\cdot}083 \times 5{\cdot}6713$$
$$= \underline{\underline{28{\cdot}83}} \text{ chains.}$$

(*e*) To determine a curve which shall pass through a given point P and be tangential to two straight lines.

Let the angle of intersection be θ, and let the co-ordinates of P from A be $x \cdot y$ as in Fig. 10.15. If measurements have been taken x_1 and y_1 from AT, x and y can be calculated.

Then $$\frac{OT}{OA}=\sin\frac{\theta}{2}.$$

But $$ON=\sqrt{OP^2-PN^2}=\sqrt{R^2-x^2},$$

$$\therefore\ OA=ON+NA=\sqrt{R^2-x^2}+y,$$

$$\therefore\ \frac{OT}{OA}=\frac{R}{\sqrt{R^2-x^2}+y}=\sin\frac{\theta}{2}, \qquad (23)$$

$$\therefore\ R\operatorname{cosec}\frac{\theta}{2}-y=\sqrt{R^2-x^2},$$

from which $$R=\frac{y\operatorname{cosec}\frac{\theta}{2}+\sqrt{y^2-x^2\left(\operatorname{cosec}^2\frac{\theta}{2}-1\right)}}{\operatorname{cosec}^2\frac{\theta}{2}-1}. \qquad (24)$$

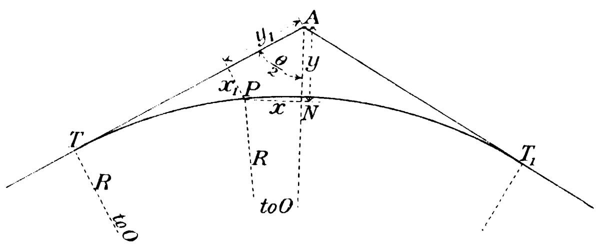


Fig. 10.15.

From this, if x, y, and $\frac{\theta}{2}$ are known, R can be calculated, the + sign of the quadratic being taken for a curve on the side of P remote from A.

If $x=0$, *i.e.* P is the apex of the curve,

$$R=\frac{y\left(\operatorname{cosec}\frac{\theta}{2}+1\right)}{\operatorname{cosec}^2\frac{\theta}{2}-1}=\frac{y}{\operatorname{cosec}\frac{\theta}{2}-1},$$

or $$R=\frac{y\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}. \qquad (25)$$

Example (Lond. B.Sc.).—The tangents to a railway curve meet at an angle of 147°. Owing to the position of a building a curve is to be chosen which will pass near a point A, 50 feet from the point of intersection of the tangents on the bisector of the angle 147°. Find to the nearest half degree, the degree of a suitable curve, and then calculate the tangent distances and show how near this curve passes to A.

Here $$\frac{R}{R+50} = \sin\frac{147^\circ}{2},$$

$$\therefore R = (R+50)\cdot 9588,$$

$$\therefore R = \frac{50 \times \cdot 9588}{\cdot 0412} = 1163 \text{ ft. approximately.}$$

Therefore by equation (4) $\theta = \frac{5730}{1163} = 4\cdot 9$ or 5° nearly.

If θ is made 5°, $R = \frac{50}{\sin 2^\circ\text{-}30'} = 1146\cdot 3$ ft.

But from (25) $$y = 1146\cdot 3\,(\text{cosec } 73^\circ\text{-}30' - 1)$$
$$= 1146\cdot 3 \times \cdot 0429$$
$$= 49\cdot 2 \text{ ft.}$$

Therefore curve passes about ·8 ft. from A.

Transition Curves.—The natural tendency of a body moving at a uniform speed is for it to continue its motion in a straight line (Newton's First Law), and if it is desired to prevent this, some external force must be applied.

In the case of a train moving round a curve, the force is applied by the action of the rails on the wheels, and, action and reaction being equal, the train consequently exerts an equal and opposite force on the rails.

This reaction is known as centrifugal force, and its magnitude is $\frac{Wv^2}{gr}$ lbs. where

W is the mass of the body in lbs.,
v is the velocity in feet per sec.,
r is the radius of curvature in feet,
$g = 32\cdot 2$ ft. per sec. per sec.

The train also exerts a downward force of W lbs. wt., so that the resultant thrust on the rails is as shown in Fig. 10.16, and as any tangential force tends to carry the train off the lines and to push the rails outwards off the sleepers, the track should be banked until it is at right angles to the resultant thrust. The horizontal component, however, varies as v varies, so that this can only be done to suit one particular speed—usually the greatest probable—on any given curve.

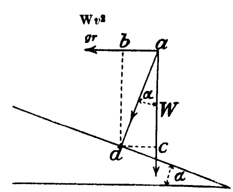


FIG. 10.16
Centrifugal Force

If $a = \tan^{-1}\frac{v^2}{rg}$ is the angle at which the resultant thrust is inclined to the vertical, then the plane of the rails should be inclined at a to the horizontal, and to effect this the outer rail is raised above the level of the inner.

The difference in level between the two rails is known as the cant or super-elevation and must be applied gradually at a rate not greater than, say, 1 in 300; and though theoretically a larger amount may sometimes be required, 5″ or 6″ is generally the maximum allowance.

In the case of a simple circular curve, because the curvature is the same at all points, the full allowance is required immediately the curve is entered upon, so that unless the first lengths of curve are to be insufficiently banked, the cant must be gradually applied along the *straight* length. This is unpleasant and evidently unsatisfactory, and to avoid it some form of Transition Curve or Curve of Adjustment should be employed to connect the straight portions to the main circular arcs.

The gauge, though slightly increased on a curve, is practically constant, so that if h be the cant, and G the distance between the centres of the rail heads,

$$\frac{h}{G} = \sin \alpha = \tan \alpha \text{ (nearly) as } \alpha \text{ is generally small.}$$

but
$$\tan \alpha = \frac{v^2}{gr},$$

$$\therefore \frac{h}{G} = \frac{v^2}{gr}. \qquad (26)$$

Therefore for any given gauge and maximum speed v, $h \propto 1/r$, so that if h is to be applied at a uniform rate, an ideal form for a transition curve would be one in which the curvature $(1/r)$ is zero at the tangent point on the straight, and increases uniformly in the same ratio as the distance along the curve until it joins the main curve (radius $=$ R) with a curvature equal to 1/R.

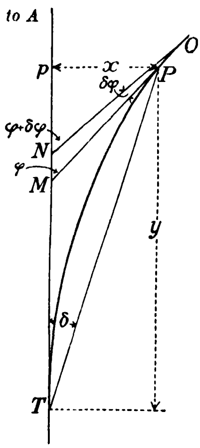


FIG. 10.17
Transition Curve

Let P and Q (Fig. 10.17) be any two points at a distance δl apart on a transition curve TPQ, T being the tangent point on the straight. Let the tangents at P and Q cut TA in M and N respectively, making $\angle$PMA $= \phi$, say, and $\angle$QNA $= \phi + \delta\phi$.

Then the average curvature of the strip PQ (which is equal to $1/r$ where r is the mean radius of PQ) is $\delta\phi/\delta l$.

If the curvature is to be proportional to the length l of the curve from T, then $\delta\phi/\delta l = c_1 l$, where c_1 is a constant, *i.e.*

$$\delta\phi = c_1 l \, \delta l,$$

from which by integration $\phi = c_1 l^2/2$,

or by writing $k = \sqrt{2/c_1}$, $\qquad l = k\sqrt{\phi} \qquad (27)$

The **clothoid spiral** represented by equation (27) is thus suitable for a transition[1] curve, the value of k being obtained from the value of ϕ, say ϕ_1, at the end of the transition curve (*i.e.* at its junction with the circular arc) when $l =$ L.

That this curve $l = k\sqrt{\phi}$ is very closely represented by the **cubic**

[1] "Transition Curves for Railways", *Proc. I.C.E.* (cxl.), James Glover; and "Setting out a Transition Curve", *Proc. I.C.E.* (cxcvii.), W. Hewson.

parabola $x = cy^3$, where x is the ordinate to the curve at a distance y along the tangent TA from T, may be shown as follows.

From Fig. 10.17 it will be seen that

$$dy = dl \,.\, \cos \phi$$

$$= dl\left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \text{etc.}\right).$$

Substituting $\left(\frac{l}{k}\right)^2$ for ϕ from equation (27),

$$dy = dl\left(1 - \frac{l^4}{2k^4} + \ldots\right),$$

and on integrating $$y = l - \frac{l^5}{10k^4} + \ldots$$

But as ϕ (which is expressed in radians) is a very small angle, k is very large, and the second and following terms are negligible for all ordinary cases, consequently

$$y = l \text{ nearly.} \qquad \ldots \qquad (28)$$

Similarly, $$dx = dl \sin \phi$$

$$= dl\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \ldots \text{etc.}\right)$$

$$= dl \,.\, \left(\frac{l^2}{k^2} - \frac{l^6}{6k^6} + \ldots\right), \qquad \ldots \qquad (29)$$

and on integration $$x = \frac{l^3}{3k^2} - \frac{l^7}{42k^6} + \ldots$$

$$= \frac{l^3}{3k^2} \text{ nearly,}$$

or, as $l = y$, $$x = \frac{1}{3k^2} \,.\, y^3, \qquad \ldots \qquad (30)$$

or $$x = c \,.\, y^3, \qquad \ldots \qquad (31)$$

where $$c = \frac{1}{3k^2}.$$

To find the radius of curvature r at any point, the usual approximation may be used, *i.e.*

$$\frac{1}{r} = \frac{d^2x}{dy^2}.$$

From equation (31)

$$\frac{dx}{dy} = 3cy^2,$$

$$\frac{d^2x}{dy^2} = 6cy,$$

or as $$y = l \text{ nearly, and } c = 1/3k^2,$$

$$1/r = 2l/k^2, \qquad \ldots \qquad (32)$$

i.e. the curvature is proportional to the length l.

It now needs to be shown how the transition curve can be fitted between the tangent BA and the circular curve, so that the transition curve shall join the straight tangentially, and also that where the transition curve joins the circular curve both curves shall have the same tangent and the same radius of curvature.

If the main curve is to have a radius of R feet, this curve will need to occupy some position such as that shown in Fig. 10.18 to allow the transition curve to be accommodated between the arc $T_{11}E$ and the straight TA.

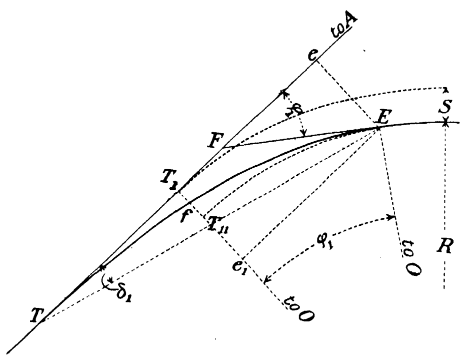


FIG. 10.18

Transition Curve

This displacement $T_1T_{11} = S$ is known as the shift, and T_1 is the tangent point for an imaginary circle of radius $OT_1 = R + S$ having the same centre O as the main curve. Before T_1 can be located S must be determined, and an expression for S is derived below:

Let E be the junction point of the transition and the circular curve; then each curve must have the same ordinate eE, the same tangent, and the same curvature at this point. The curvature of the circle is $1/R$, and the tangent EF is at right angles to the radius OE.

Then as OE and OT_1 are perpendicular to EF and BA respectively, the angle $EOT_1 = \phi_1$.

T_1T_{11} is therefore equal to $eE - e_1T_{11}$ where e and e_1 are the projections of E on BA and OT_1 respectively, *i.e.* the shift

$$S = T_1T_{11} = eE - R\ (1 - \cos\phi_1)$$

$$= eE - R\left\{1 - \left(1 - \frac{\phi_1^2}{2!} + \ldots\right)\right\}$$

$$= eE - \frac{R\phi_1^2}{2} \text{ nearly.}$$

But from equation (30)

$$eE = \frac{(Te)^3}{3k^2} = \frac{L^3}{3k^2},$$

because L, the length of the transition curve TE, is by equation (28) nearly equal to the abscissa $Te = y$.

Also from equation (32) $R = \frac{k^2}{2L}$, because the radius of curvature at E is equal to that of the circular curve which it joins; and from equation (27) $\phi_1^2 = \frac{L^4}{k^4}$.

Therefore by substitution $S = \frac{L^3}{3k^2} - \frac{k^2}{2L} \cdot \frac{L^4}{2k^4}$

$$= \frac{1}{12} \frac{L^3}{k^2},$$

or

$$S = \frac{L^2}{24R}, \qquad (33)$$

the form of expression employed in Froude's Curve of Adjustment.

If the radius R of the main curve has been decided upon, and if the angle of intersection between the two straight lengths BA and AC is known, the position of T_1 relatively to the point of intersection A may be found by equation (1), as for a circular curve of radius (R + S).

Also, if the amount of super-elevation (*i.e.* h feet) to be applied to the circular portion of the curve of radius R has been computed, the length L of the transition curve may be determined by one of several methods :

(*a*) It may be of such a length that the cant is applied at a uniform rate of, say, 1 in 300, when $L = 300 . h .$ feet, in which case if h is limited to 6 inches, the maximum value of L is 150 feet ;

(*b*) It may be of such a length that the *rate* at which the super-elevation h is to be applied shall not exceed, say, 2 inches per second ;[1]

(*c*) It may be of a constant length, say 150 ft., a figure which is fixed empirically.

Or (*d*) it may be of such a length that the rate of change of radial acceleration shall not exceed, say, 1 ft. per sec.^{3}, whence $L = v^3 \div R$.

But before the setting out of the curve can be commenced, it is necessary to locate T, the point of tangency of the transition curve, and it may be shown that T_1T is equal to $\frac{L}{2}$.

Thus

$$\begin{aligned} TT_1 &= Te - Ee_1 \\ &= Te - R . \sin \phi_1 \\ &= Te - R . \left(\phi_1 - \frac{\phi_1^{\,3}}{3!} + . . .\right) \\ &= Te - R\phi_1 \text{ nearly} \\ &= L - \frac{k^2}{2L} \cdot \frac{L^2}{k^2} \\ &= \tfrac{1}{2} . L. \qquad (34) \end{aligned}$$

The curves may now be set out by one of several methods :

(*a*) The central circular curve of radius R may be set out from T_{11} or from E by any of the usual methods already described in the present chapter.

(*b*) The transition curve may be set out—

(1) By means of offsets from TA.

(2) By means of offsets from TT_1 and from the arc $T_{11}E$.

(3) By means of deflection angles from T.

[1] Some American railways specify that the rate shall not exceed $1\frac{1}{8}$ inches per second.

(1) Offsets from TA.

From equation (30) $x=\frac{y^3}{3k^2}$,

and from equation (32) $k^2=2lr=2LR$,

i.e.
$$x=\frac{y^3}{6LR}$$
$$=y^3\cdot\frac{4S}{L^3}. \qquad . \quad . \quad . \quad (35)$$

The ordinate eE is therefore 4S, and the ordinate T_1f is therefore $\frac{1}{2}$S, *i.e.* the transition curve bisects and is bisected by T_1T_{11}.

(2) Offsets from TT_1 and from the arc $T_{11}E$.

At n feet from T the offset from the tangent TA

$$=\frac{n^3}{L^3}\cdot\frac{L^2}{6R}\left(i.e.\ \frac{n^3}{L^3}\cdot 4S\right).$$

At $L-n$ feet from T the offset from the tangent TA

$$=\frac{(L-n)^3}{L^3}\cdot\frac{L^2}{6R}.$$

At $L-n$ feet from T the offset from the tangent TA to the circular curve

$$=\frac{L^2}{24R}+R\ \sqrt{R^2-\left(\frac{L}{2}-n\right)^2}.$$

The intercept between the transition curve and the circular arc at n feet from E, where n is less than $\frac{1}{2}$L, is therefore

$$\left\{\frac{L^2}{24R}+R-\sqrt{R^2-\left(\frac{L}{2}-n\right)^2}\right\}-\frac{(L-n)^3}{L^3}\cdot\frac{L^2}{6R},$$

i.e.
$$\frac{L^2}{24R}+R-R\left\{1-\left(\frac{L-2n}{2R}\right)^2\right\}^{\frac{1}{2}}-\frac{(L-n)^3}{6RL},$$

which reduces to
$$\frac{n^3}{L^3}\cdot\frac{L^2}{6R}.$$

That is, the offset from the circular curve to the transition curve at a distance n from E is equal to the offset from the tangent to the transition curve at a distance n from T, hence half the curve may be set out by offsets from the tangent TT_1, and the remaining half by equal offsets from the circular curve $T_{11}E$.

It also follows from the above result that the rate at which the curve approaches any osculating circle is constant, and equal to the rate at which it approaches the tangent TA.

(3) Deflection angles from T.

By the differentiation of equation (30), *i.e.* of $x=\frac{1}{3k^2}\cdot y^3$,

$$\frac{dx}{dy}=\frac{y^2}{k^2},$$

also from Fig. 10.17 $$\frac{dx}{dy}=\tan\phi=\frac{pP}{pM}=\frac{x}{pM}=\frac{y^3}{3k^2}\cdot\frac{1}{pM},$$

$$\therefore \frac{y^2}{k^2}=\frac{y^3}{3k^2}\cdot\frac{1}{pM},$$

or $$pM=\frac{y}{3}. \qquad (36)$$

Hence if the deflection angle ATP $=\delta$,

$$\tan\delta=\frac{x}{y},$$

and $$\tan\phi=\frac{3x}{y},$$

i.e. $$\tan\delta=\tfrac{1}{3}\tan\phi,$$

or, since δ and ϕ are very small angles,

$$\tan\delta=\tan\tfrac{1}{3}\phi,$$

and $$\delta=\tfrac{1}{3}\phi.$$

But $$\phi=\frac{l^2}{k^2}\text{ radians},$$

hence $$\delta=\tfrac{1}{3}\cdot\frac{l^2}{k^2}\text{ radians},$$

$$=\frac{1}{6RL}\cdot l^2\text{ radians},$$

$$=\frac{1800}{\pi RL}\cdot l^2\text{ minutes}. \qquad (37)$$

If the circular curve is defined by the number of degrees subtended at the centre by a chord of 100 ft., *i.e.* if the circular curve is a D° curve, then

$$\phi_1=\frac{L}{2}\cdot\frac{D}{100}\text{ degrees},$$

and $$\delta_1=\frac{LD}{600}\text{ degrees}=\frac{LD}{10}\text{ minutes},$$

where δ_1 is the total deflection angle ATE,

and $$\delta=\frac{l^2}{L}\cdot\frac{D}{10}\text{ minutes}. \qquad (38)$$

Numerous tables are published to facilitate the calculations of the deflection angles for different spiral curves.

In order to set out the circular curve from E, the direction of the tangent at E may be found by sighting to T, and turning through an angle TEF, *i.e.* $\phi_1-\delta_1$, *i.e.* $\tfrac{2}{3}\phi_1$, or $2\delta_1$

$$=\frac{LD}{5}\text{ minutes},$$

or, as the point F is $\frac{2}{3}$L from T, the line FE may be set out by sighting to F.

Compound Curves.—If it is required to insert a transition curve between two circular curves of radii R_1 and R_2 respectively, the amount of shift S_1 and S_2 required for each is calculated by equation (33).

The points T_1, P and T'_1 (Fig. 10.19), at which the curves of radii (R_1+S_1) and (R_2+S_2) would touch the three tangents, are then computed as explained on p. 320, and in Fig. 10.14.

The transition curves at T_1 and T_1' are located as already explained. At P, when the two curves have been shifted the proper amounts S_1

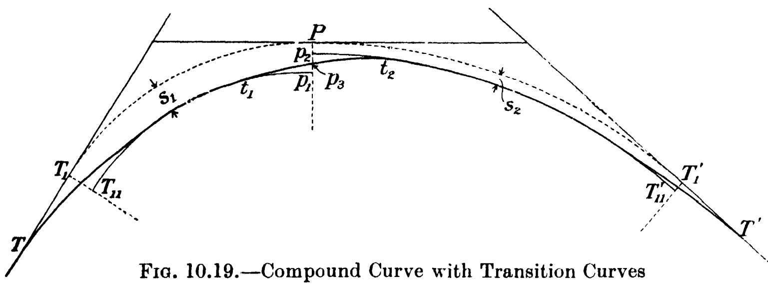


FIG. 10.19.—Compound Curve with Transition Curves

and S_2, the distance between the tangents of the shifted curves is S_1-S_2, *i.e.*

$$p_1p_2=S_1-S_2.$$

Also, if the cants on the two curves are h_1 and h_2 respectively, and if f is the maximum rate at which the change of cant is to be applied, then the length L of transition curve required at P is

$$L=f(h_1-h_2).$$

Otherwise the length L may be fixed empirically or by considering the rate of change of cant per second.

To proceed to locate this length L of transition curve, use may be made of the fact that the rate at which the transition curve approaches either of the osculating circular curves is constant, and equal to the rate at which it approaches the tangent at its origin. Hence the transition curve is bisected at a point (p_3) midway between the two curves, so that, knowing p_1p_2, *i.e.* S_1-S_2, and the length L, the offsets may be set out from the circular arcs t_1p_1 and t_2p_2.

Thus, from p_3 set out $\frac{1}{2}$L in each direction to meet the circular curves in t_1 and t_2 respectively. Then the ordinate x from the circular curve to the transition curve at any distance y from t_1 or from t_2 is

$$x=\frac{y^3}{\left(\frac{L}{2}\right)^3}\frac{1}{2}(S_1-S_2)$$

$$=\frac{4 \cdot y^3 (S_1-S_2)}{L^3}. \qquad (39)$$

If the two curves are reverse, a similar procedure is usually adopted, the distance between the shifted curves being $S_1 + S_2$, and the length of transition curve $L = f(h_1 + h_2)$. The ordinate x from the circular curve to the transition curve (which bisects p_1p_2 in p_3), at a distance y from t_1 or from t_2, is

$$x = \frac{4y^3}{L^3}(S_1 + S_2) \quad . \quad . \quad . \quad (40)$$

The two portions of the transition curve would then have a common tangent at p_3, the middle point of p_1p_2, but by this method, when the cant is changed at a uniform rate, the cant on the transition curve near the centre point p_3 may be in the opposite direction to the curvature.

An alternative method would be to set out two separate transition curves from the point of tangency of the two circles of radii $R_1 + S_1$ and $R_2 + S_2$, *i.e.* from P. One curve of length $L_1 = fh_1$ from P to t_1, would be located by means of offsets from the circular curve t_1p_1, and the other of length $L_2 = fh_2$, from the arc p_2t_2.

The offset x at a distance y from $t_1 = \frac{y^3}{L_1^3} . S_1$, and the offset x at a distance y from $t_2 = \frac{y^3}{L_2^3} . S_2$, so that the distance p_1p_2 between the circular curves would be divided in the ratio of $S_1 : S_2$ at P.

In this case the cant on the transition curve would become zero when the curvature is nil, as the curve passed through P, but the two portions of the transition curve would not be quite tangential to the same line at their junction as is the case in the usual method.

Example.—Calculate the data required to locate transition curves for a compound curve composed of two arcs of 40 chains and 15 chains respectively.

Assuming a speed of 40 miles per hour, *i.e.* 58·7 ft. per sec., h_1 for the 40-chain curve, by equation (26),

$$= \frac{4 \cdot 9 \times (58 \cdot 7)^2}{32 \cdot 2 \times 40 \times 66} \text{ ft.}$$
$$= \cdot 2 \text{ ft. nearly,}$$

where 4·9 ft. is the distance from centre to centre of the rails.

Similarly, h_2 for the 15-chain curve $= \cdot 5$ feet (maximum). Applying the cant at a rate of 1 in 300, $L_1 = 60$ ft., $L_2 = 150$ ft., or $L_1 = 1 \cdot 00$ chain and $L_2 = 2 \cdot 50$ chains say.

Then $$S_1 = \frac{(100)^2}{24 \times 4000} \text{ links} = 1 \cdot 04 \text{ links,}$$

and $$S_2 = \frac{(250)^2}{24 \times 1500} \text{ links} = 1 \cdot 74 \text{ links.}$$

The ordinates for setting out L_1 are

$$\frac{y^3}{(100)^3} . 4 \times 1 \cdot 04 \text{ links,}$$

so that for pegs at 50-link intervals the offsets are ·52 and 4·16 links.

Similarly, for locating L_2 the ordinates from the equation $x = \frac{y^3}{(250)^3} \times 4 \times 1 \cdot 7$ are ·06, ·44, 1·50, 3·56, and 6·96 links.

At the junction of the two curves L = 1·50 chains and the offsets are calculated from the formula (39),

$$x = \frac{4y^3}{150^3} \times \cdot 70 \text{ link},$$

i.e. at 50 links from t_1 or t_2 $\quad x = \cdot 10$ link.

Vertical Curves.—In American practice the longitudinal inclination or gradient of a railway line is usually expressed as a percentage, so that an inclination of x in 100 is an x% grade.

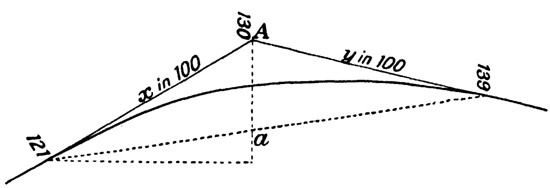


FIG. 10.20.—Vertical Curves

When two grades meet, it is advisable to insert a curve in a vertical plane to round off the angle that would otherwise be formed (Fig. 10.20). Such a curve is known as a Vertical Curve; its form is generally parabolic, and its length is determined by the rate at which it is decided to change the gradient of the line.

The American Railway Engineering Association recommend that for first-class railway work the change of grade shall be about ·1% per 100 ft. chain at summits, and ·05% per chain at sags, though these rates are often exceeded.

Thus, if an x% up grade meets a y% down grade, the total change of grade is $(x+y)$%, and the length of vertical curve required to connect the two tangents will be

$$\frac{x+y}{\cdot 1} = 10\,(x+y) \text{ chains (Engineer's).}$$

The nearest even number L of complete chains would be adopted. The vertical curve would then commence at $\frac{L}{2}$ chains from the apex, and finish at $\frac{L}{2}$ chains beyond the apex.

The elevations of the beginning and end of the curve, and of other stations along the tangents, may then be computed if the elevation of the apex A is known; and by assuming the curve to be parabolic the distances that this is below the tangent at each point can be calculated, and hence the elevations of the points on the curve can be found, as in the following example.

Example.—If the up grade is ·8% and the down grade ·9% the change of grade is 1·7% and the length of the vertical curve is 17 chains. This would be taken as 18 chains to give 9 chains on each side of the vertex. If the elevation at the apex A (chainage = 130·00 say) is 80·40 ft., the elevation of the beginning of the curve at a chainage of 121·00 is 80·40 − (9 × ·8) = 73·20 ft.; and at the end of the curve—chainage 139·00—the elevation is 80·40 − (9 × ·9) = 72·30 ft.

The elevation of the point a (Fig. 10.20) is therefore $\frac{73 \cdot 20 + 72 \cdot 30}{2} = 72 \cdot 75$ ft.

Also, as the parabola will bisect Aa, the elevation of the parabola at A $=\dfrac{80{\cdot}40+72{\cdot}75}{2}=76{\cdot}57$ ft.

The elevation at a chainage of 122·00 on the curve may be found as follows:
Elevation of station 122·00 on tangent $=80{\cdot}40-(8\times{\cdot}8)=74{\cdot}00$ ft.

Distance of curve below tangent at this point $=\left(\dfrac{1}{9}\right)^2\times(80{\cdot}40-76{\cdot}57)$

$$=\frac{1}{81}\times3{\cdot}83=0{\cdot}05 \text{ ft.}$$

The elevation of the point 122 on the curve is therefore $74{\cdot}00-0{\cdot}05=73{\cdot}95$ ft.
Similarly, the elevation of other points on the curve may be calculated.

Other examples are given on p. 334 to be solved by the student.

If the inclinations are expressed as fractions instead of percentages, the same principles are applicable, though the calculations may be more laborious.

EXAMPLES

(1) Calculate the ordinates at 25-ft. distances for a circular curve having a long chord of 350 ft. and a versed sine of 15 ft.

(2) (U. of L.) On a railway survey the azimuths of two intersecting courses are 275°-30′ and 289°-00′ respectively. It is decided to connect these courses by a 4° curve and the chainage of the intersection is 132 chains +24·5 (the 100-ft. chain being used).

Fill in the following table for the curve:

Curve No. 39

Azimuths			Deflection		Degree	Rad.	Tang. Dist.	Length	Chainage		
1st Tang.	Long Chord	2nd Tang.	R.	L.					Curve begins	Apex	Curve ends

(3) Work out the necessary quantities for setting out the curve in Question 2 by

(1) The method of offsets from chords (usual method).
(2) ,, ,, ,, (method 6).

(4) (U. of L.) The chainage at the point of intersection of the tangents to a railway curve is 3876 links, and the angle between them is 124°.

Find the chainage at the beginning and end of the curve if it is 40 chains radius, and calculate out the angles which are required in order to set out this curve (*a*) with a theodolite, (*b*) with a chain and tape only.

(5) In Fig. 10.14 BA and AC are intersected by a straight length MN and the angles $\theta_1=130°$ and $\theta_2=150°$. Calculate the lengths AT and AT_1 if the curve TP is 15 chains and PT_1 20 chains radius. Find also the position of P, M, and N, and the length MN.

(6) In Fig. 10.14 if $\theta=130°$, AM $=7{\cdot}00$ chains, AN $=6{\cdot}00$ chains, and MP $=4{\cdot}50$

chains, find the radius of a curve to pass through P and to be tangential to AB and AC. If the chainage of A = 120·56 chains, what would be the chainage at T, P, and T_1.

(7) (U. of L.) A light railway is to be carried round the shoulder of a hill, and its centre line is to be tangent to each of the three lines AB, BC, and CD as follows:

Line	Bearing	Length
AB	North 30° E.	..
BC	East	600 ft.
CD	South	..

Calculate the radius of the curve and the lengths required for setting out the tangent points.

(8) (U. of L.) The whole circle bearings of two straight portions AB, BC of a railway line are 60° and 90° respectively. The chainage of the point B is 85·18 chains, the 100-ft. chain being used.

The two lines are to be connected by two transition curves, and a circular curve of 12 chains radius.

The length of each transition curve is to be 150 ft. Calculate the chainage for the beginning and end of the three curves. Also prepare a field book from which the transition curves and the circular curve could be set out with a chain and theodolite.

(9) Two straight lengths, BA and AC, are intersected by a third line which makes the angle BMN = 130° and the angle MNC = 150°.

A compound curve, composed of two curves the radii of which are 20 and 25 chains respectively, is to be employed to join BA and AC as in Fig. 10.19. Make the necessary calculations for the location of these curves if transition curves are to be inserted between them and at the junctions with the tangents BA and AC.

The chainage at A = 144 chains + 54 links, and the maximum velocity to be allowed for is 45 miles per hour.

(10) Find the reduced level of the various station pegs on a vertical curve connecting two uniform gradients of

(*a*) +0·7% and −0·9% respectively.
(*b*) +0·7% and +0·9% ,,
(*c*) −0·7% and −0·9% ,,

Take the reduced level of the intersection point as 160·60 ft. in each case, and the maximum rate of change of grade as 0·2% at summits and 0·1% at sags.

The + sign denotes an ascending and the − sign a descending grade.

(11) (U. of W.) A symmetrical parabolic curve is inserted at a summit where an up grade of 1 in 30 changes to a down grade of 1 in 60. The chainage at the intersection point is 1550 feet and the reduced level 140.00. Tabulate the data necessary to set out the curve, if the sight line is 600 feet, the eye being 3·75 feet above ground.

CHAPTER XI

EARTHWORK CALCULATIONS

BEFORE proceeding to find an expression for the volume of a cutting or embankment it may be advantageous to consider the mensuration of one or two geometrical figures.

(1) To find the volume of a regular or irregular cone or pyramid (Fig. 11.1). Let a_0 be the base area, and d_0 the perpendicular distance of the vertex O from the base; and let a be the area of any section parallel to the base at a perpendicular distance d from the vertex.

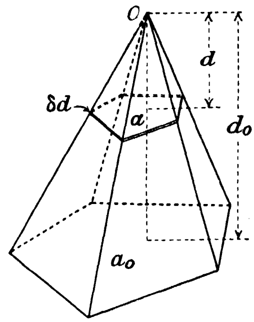


FIG. 11.1

Then $\frac{a}{a_0} = \frac{d^2}{d_0^2}$ as the *linear* dimensions of the areas are proportional to d and d_0 respectively.

The volume of an elementary strip of a thickness δd at a distance d from O is therefore

$$a \cdot \delta d = a_0 \cdot \frac{d^2}{d_0^2} \cdot \delta d,$$

so that the volume of the solid is the integral of this expression between the limits $d=0$ and $d=d_0$, *i.e.*

$$V = \frac{a_0}{d_0^2}\int_0^{d_0} d^2 \cdot dd = \frac{a_0}{3} \cdot \frac{d_0^3}{d_0^2} = \frac{a_0 d_0}{3},$$

or if a_c is the central section

$$\frac{a_c}{a_0} = \frac{\left(\frac{d_0}{2}\right)^2}{d_0^2} \text{ or } a_0 = 4 \cdot a_c,$$

$$\therefore V = \frac{a_0 d_0}{3} = \frac{2 \cdot a_0 d_0}{6} = (a_0 + 4a_c)\frac{d_0}{6}. \quad . \quad . \quad (1)$$

(2) In the case of a truncated pyramid or cone (Fig. 11.1) the limits of integration in the above expression are d_2 and d_0, *i.e.* the perpendicular distances from the vertex of the completed cone on to the top and base areas, a_2 and a_0 respectively, so that

$$V = \frac{a_0}{d_0^2}\int_{d_2}^{d_0} d^2 \, . \, dd$$

$$= \frac{a_0}{3 \, . \, d_0^2}(d_0^3 - d_2^3)$$

$$= \frac{a_0}{3 \, . \, d_0^2}(d_0 - d_2)(d_0^2 + d_0 d_2 + d_2^2)$$

$$= \frac{d_0 - d_2}{3}\left(a_0 + \frac{a_0 d_2}{d_0} + \frac{a_0 d_2^2}{d_0^2}\right). \quad . \quad . \quad (2)$$

But $$a_2 : a_0 : : d_2^2 : d_0^2,$$

$$\therefore \frac{a_0 d_2^2}{d_0^2} = a_2,$$

so that equation (2) may be written

$$V = \frac{d_0 - d_2}{3}(a_0 + \sqrt{a_0 a_2} + a_2). \quad . \quad . \quad (3)$$

The central area $$a_c : a_0 : : \left(\frac{d_0 + d_2}{2}\right)^2 : d_0^2,$$

i.e. $$a_c = a_0 \, . \, \frac{(d_0 + d_2)^2}{4 \, . \, d_0^2},$$

i.e. $$4a_c = a_0\left(1 + \frac{2d_2}{d_0} + \frac{d_2^2}{d_0^2}\right)$$

$$= a_0 + \frac{2a_0 d_2}{d_0} + a_2,$$

$$\therefore \frac{a_0 d_2}{d_0} = \tfrac{1}{2}(-a_0 + 4a_c - a_2). \, . \quad . \quad . \quad (4)$$

Substituting in (2)

$$V = \frac{d_0 - d_2}{6}(a_0 + 4a_c + a_2). \quad (5)$$

This equation is obviously true in the special case when $a_0 = a_c = a_2$, *i.e.* is applicable to such solids as cylinders and parallelepipeds which may be considered as frusta of cones and pyramids of an infinitely great altitude.

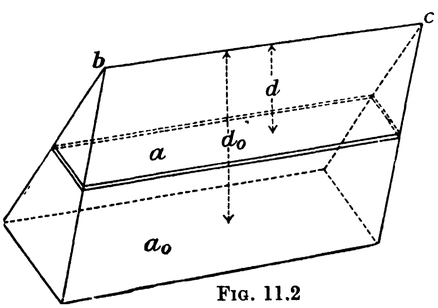


Fig. 11.2

(3) The volume of a wedge or prism (Fig. 11.2) having its two end faces parallel may be derived in a similar manner.

The linear dimensions parallel to the intersection line or vertex *bc*

remain constant for all sections, while dimensions at right angles to this are proportional to d.

Therefore, using the same notation as before, $\frac{a}{a_0} = \frac{d}{d_0}\left(\text{not } \frac{d^2}{{d_0}^2}\right)$.

The volume of a thin strip parallel to bc, and of thickness δd, is therefore

$$a \,.\, \delta d = \frac{a_0 \,.\, d}{d_0} \,.\, \delta d,$$

and the total volume

$$\mathrm{V} = \frac{a_0}{d_0}\int_{0}^{d_0} d \,.\, dd$$

$$= \frac{a_0 \,.\, {d_0}^2}{2d_0} = \frac{a_0 d_0}{2}, \qquad (6)$$

or for a truncated prism

$$\mathrm{V} = \frac{a_0}{2d_0}({d_0}^2 - {d_2}^2),$$

or as

$$a_2 = \frac{a_0 d_2}{d_0} \therefore \mathrm{V} = \frac{d_0 - d_2}{2}(a_0 + a_2). \qquad (7)$$

$$a_c = a_0 \frac{\frac{d_0 + d_2}{2}}{d_0} = \frac{a_0(d_0 + d_2)}{2d_0} = \tfrac{1}{2}(a_0 + a_2),$$

or

$$4 \,.\, a_c = 2\,(a_0 + a_2),$$

so that if V is expressed in terms of a_0, a_c, and a_2, from (7)

$$\mathrm{V} = \frac{d_0 - d_2}{2}(a_0 + a_2)$$

$$= \frac{d_0 - d_2}{6}(3a_0 + 3a_2), \text{ (multiplying numerator and denominator by 3)}$$

$$= \frac{d_0 - d_2}{6}(a_0 + 2(a_0 + a_2) + a_2)$$

$$= \left(\frac{d_0 - d_2}{6}\right)(a_0 + 4a_c + a_2). \qquad (8)$$

A parallelepiped may also be considered as a special case, *i.e.* as the frustum of a prism of an infinitely great altitude when a_0, a_c, and a_2 are all equal.

The Prismoidal Formula.—From equations (1), (5), and (8) it is evident that the volume of a whole or truncated cone, pyramid, or prism may be expressed by the same rule, viz. that the volume is equal to the sum of the areas of the two parallel end sections + four times the area of the central section, multiplied by $\frac{1}{6}$th of the perpendicular distance between the end sections.

A prismoid is a solid figure having two parallel plane end areas,

A_0 and A_2 say, which are not necessarily similar, but may be of any shape whatever provided that the surfaces joining their perimeters are capable of being generated by straight lines continuous from the one perimeter to the other.

Let the perpendicular distance between the two parallel end faces be 2D, and let the central section have an area A_1.

Then the figure is evidently composed of a number of regular or irregular cones, pyramids, and prisms, of a uniform height 2D, so that the above-stated general formula is applicable, *i.e.*, as

$$A_0 = \Sigma a_0,$$
$$A_1 = \Sigma a_c,$$
$$A_2 = \Sigma a_2,$$

and

$$2D = 2\,(d_0 - d_2),$$
$$V = \frac{\Sigma(d_0 - d_2)}{6}\,(a_0 + 4a_c + a_2)\ .$$
$$= \frac{D}{3}\,(A_0 + 4A_1 + A_2). \quad . \quad . \quad . \qquad (9)$$

If the dimensions of the end areas are measured, those of the central section may frequently be ascertained by calculation and the formula (9) applied, or, if desired, all the required dimensions may be obtained by direct measurement.

Example.—Find the volume of a tank which is excavated in level ground to a depth of 8′ - 6″. The top, which is rectangular in shape, has an area of 50 ft. × 20 ft., while the bottom is 33 ft. × 3 ft.

Applying formula (9) we have

$$D = 4{\cdot}25 \text{ ft.},$$
$$A_0 = 50 \times 20 = 1000 \text{ sq. ft.},$$
$$A_2 = 33 \times 3 = 99 \text{ sq. ft.},$$
$$A_1 \text{ (taking a mean of the linear dimensions)} = 41{\cdot}5 \times 11{\cdot}5,$$

and

$$4A_1 = 41{\cdot}5 \times 11{\cdot}5 \times 4 = 1909 \text{ sq. ft.},$$
$$\therefore\ V = \frac{4{\cdot}25}{3}\,(1000 + 1909 + 99) = 4261\tfrac{1}{3} \text{ cub. ft.}$$

An incorrect result would be obtained were the volume assumed to be either the mean of the top and bottom areas × 8·5 ft. or as the area of the central section × 8·5 ft.

For the purposes of comparison the values obtained by these methods are given

$$(a)\ V_1 = \frac{1000 + 99}{2} \times 8{\cdot}5 = 4670\tfrac{3}{4} \text{ cub. ft.},$$
$$(b)\ V_1 = (41{\cdot}5 \times 11{\cdot}5) \times 8{\cdot}5 = 4056\tfrac{5}{8} \text{ cub. ft.}$$

The discrepancy between the results would have been much less had the areas been more nearly equal.

In order to compute the volume of a cutting or embankment, cross-sections are taken at sufficiently close intervals to split up the total length into what is virtually a series of prismoids.

If the ground is of fairly uniform slope, the sections may be equidistant, when the calculations are simplified.

Thus if A_0, A_1, A_2, $A_3 \ldots A_n$ are the areas of the different cross-sections at a distance D apart, then as long as D is sufficiently small to ensure that the solids between A_0 and A_2, A_2 and A_4, A_4 and A_6, etc., are approximately prismoidal, *i.e.* as long as the shape of the surfaces can be approximately traced by straight lines extending over these distances, the volume enclosed is

$$\frac{D}{3}(A_0+4A_1+A_2)+\frac{D}{3}(A_2+4A_3+A_4)+\ldots$$

The volume, where there are an odd number of areas (*i.e.* n is even), is then given by a formula of the same form as Simpson's rule for areas in Chapter I, *i.e.*

$$V=\frac{D}{3}(A_0+4A_1+2A_2+4A_3+\ldots 2A_{n-2}+4A_{n-1}+A_n). \quad . \qquad (10)$$

When there is an even number of cross-sections, one of the end divisions must be treated separately by any convenient method, while the Prismoidal Formula (10) is applied to the remainder.

If the value of D cannot be kept constant throughout, the volume must be calculated in separate lengths over which one value of D is constant, and the results added. (Cf. the application of Simpson's rule to the calculation of areas, p. 33).

The cross-sections may be taken along horizontal instead of vertical planes, if desired, and this method is often very convenient, *e.g.* for the determination of the capacity of reservoirs, or the amount of filling in hollows or of excavation in hills. If the contour lines are delineated, the enclosed areas give the values of A_0, A_1, A_2, etc., while the vertical interval corresponds to D in the above formula.

Example.—Calculate the cubic contents of the length of embankment of which the cross-sectional areas at 50-ft. intervals are as follows :

Distance.	0	50	100	150	200	250	300 ft.
Area.	425	640	726	1590	1790	2600	1130 sq. ft.

$$V=\frac{50}{3}\{(425+1130)+4(640+1590+2600)+2(726+1790)\}$$

$$=\frac{50}{3}\{1555+19{,}320+5032\}\text{ cub. ft.}=15{,}992\text{ cub. yds.}$$

The Trapezoidal Formula.—A more simple rule than the prismoidal rule for volumes is

$$V=\frac{D}{2}(A_0+2A_1+2A_2+\ldots 2A_{n-1}+A_n). \quad . \qquad . \quad (11)$$

This is mathematically correct, whether n is even or odd, if the volume between successive sections can be considered as built up of a number of prisms and parallelepipeds or cylinders, each of a height

D (*i.e.* with the omission of pyramids or cones), as by equation (7) the volumes of such figures are given by the formula

$$V_1 = \frac{D}{2} (A_0 + A_1).$$

The total volume V is then

$$V = \frac{D}{2} (A_0 + A_1) + \frac{D}{2} (A_1 + A_2) + \ . \ . \ . \ \frac{D}{2} (A_{n-1} + A_n),$$

from which equation (11) is easily deduced.

Example 1.—The volume of the tank in the example on p. 338 was found by the Trapezoidal rule to be $4670\frac{3}{4}$ cub. ft., whereas the correct value obtained by the Prismoidal formula was $4261\frac{1}{3}$ cub. ft.

In this example, however, it is evident that the rule is not strictly applicable, as the solid is composed of a central parallelepiped, 4 prisms at the edges and 4 pyramids at the angles.

The more closely together the cross-sections are taken, the more accurate is the result, *e.g.* if the value of D is taken as 4·25, and the central area is considered, the discrepancy occurs only in the 8 much smaller pyramids at the angles, instead of in the 4 large ones, and the derived result is

$$V = \frac{4{\cdot}25}{2}\{1000 + (2 \times 477{\cdot}25) + 99\} = 4363{\cdot}7 \text{ cub. ft. nearly.}$$

Example 2.—Calculate the volume of the embankment in the example on p. 339 by the application of the Trapezoidal Rule.

$$V = \frac{50}{2}\{(425 + 1130) + 2(640 + 726 + 1590 + 1790 + 2600)\} \text{ cub. ft.}$$

$$= 15{,}044 \text{ cub. yds.,}$$

a result which is nearly 6% less than that derived by the Prismoidal Rule.

Average Area.—A still more approximate method which corresponds to Rule 5 for areas, p. 32, is

$$V = \frac{nD\ (A_0 + A_1 + A_2 + \ . \ . \ . + A_n)}{n + 1}. \qquad . \quad (12)$$

Example.—Apply this rule to determine the contents of the embankment in the example on p. 339, and that above.

$$V = \frac{300\ (425 + 640 + 726 + 1590 + 1790 + 2600 + 1130)}{7 \times 27} \text{ cub. yds.}$$

$$= 14{,}129 \text{ cub. yds.}$$

This result is about $11\frac{1}{2}$% less than that given by the Prismoidal Rule, and about 6% less than that given by the Trapezoidal Rule.

Cross-Sectional Areas.—The areas of the various cross-sections taken for the calculation of quantities may be determined by any of the rules given in Chapter I, which may be suitable for the particular case, *e.g.* when the capacity of a reservoir is to be estimated, the areas enclosed by the different contour lines may be determined conveniently by means of a planimeter, or by means of a computing scale, or by give and take lines.

In certain cases, however, it may be advantageous to employ a special formula, or to resort to specially compiled earthwork tables

or diagrams—for instance, in the computation of the volumes of cuttings or embankments for road or railway work.

After the route of the proposed work has been finally decided upon, a longitudinal section is taken along the centre line, and, at suitable intervals, cross-sections at right angles to this are determined as explained in Chapter VI.

The "formation" level is then laid down upon the longitudinal section. The exact position of this is a matter for the judgment of the engineer, who must consider such points as

(1) The ruling gradient which it is inadvisable to exceed.

(2) The relative and actual quantities of cutting and embankment, so that it may be unnecessary either to haul excavated material a great distance to an available tip, or to bring from any remote points filling-in material for an embankment.

(3) The levels of existing roads and waterways which may need to be crossed or joined, and which cannot be unduly interfered with.

(4) The number and extent of bridges, viaducts, aqueducts, tunnels, etc., which the choice may entail.

From the longitudinal section the height of the formation line above datum at the various cross-sections is ascertained either by calculation or by scaled measurements, and when this is known the profile of the cutting or embankment may be set out on each section.

Thus in Figs. 11.3, 11.4, and 11.5 let

b = the width of a cutting at formation level TS.

h = the depth of cutting on the centre line QN.

s_1 and s_2 = the horizontal distances ER and DP from the centre to the intersection of the side slopes with the original ground level at R and P respectively. These values are known as side-widths, side-distances, or half-breadths.

n horizontal to 1 vertical = the inclination of the side slopes.

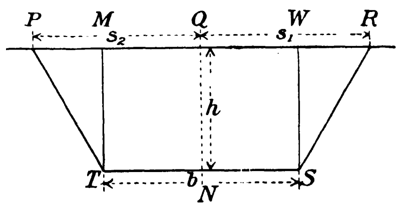


FIG. 11.3

m horizontal to 1 vertical = the transverse or "sidelong" inclination of the ground.

h_1 = the vertical height FR of the point R above the formation level.

h_2 = the vertical height GP of the point P above the formation level.

Case 1. When $m=\infty$, *i.e.* when the ground is level transversely (Fig. 11.3).

The area
$$A=\Big(b+(s_1+s_2)\Big)\frac{h}{2}$$
$$=2\ (b+nh)\frac{h}{2}$$
$$=(b+nh)h. \quad . \quad . \quad . \quad . \quad (13)$$

Case 2. When the slope of the ground does not cut the formation level (Fig. 11.4).

Let O be the point on the centre line at which the side slopes would intersect, *i.e.*

$$NS : ON :: n : 1,$$

or

$$ON = \frac{b}{2n}.$$

The area

$$PQRST = \text{the } \triangle\, QRO + \triangle\, QPO - \triangle\, TSO$$

$$= \tfrac{1}{2}\left\{\left(h + \frac{b}{2n}\right)s_1 + \left(h + \frac{b}{2n}\right)s_2 - \frac{b^2}{2n}\right\}$$

$$= \tfrac{1}{2}\left\{(s_1 + s_2)\left(h + \frac{b}{2n}\right) - \frac{b^2}{2n}\right\}. \qquad (14)$$

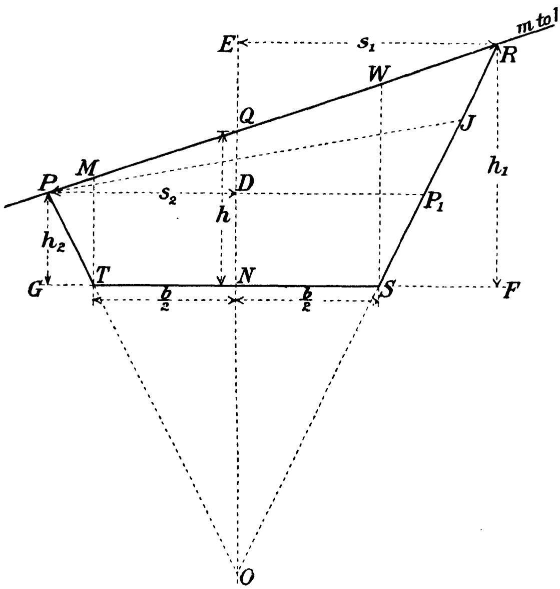


FIG. 11.4

This formula is equally true whether QR and QP are of the same slope or otherwise, provided that the values of s_1 and s_2 are known. If the heights of P and R above TS are known, the formula may be expressed in terms of these distances, *i.e.* the area

PQRST = the sum of the four triangles PTN + RSN + PQN + RQN

$$= \tfrac{1}{2}\left\{\frac{b}{2} \cdot h_2 + \frac{b}{2} \cdot h_1 + hs_2 + hs_1\right\}$$

$$= \tfrac{1}{2}\left\{\frac{b}{2}(h_1 + h_2) + h\,(s_1 + s_2)\right\}. \qquad (15)$$

Case 3 (Fig. 11.5). When the slope of the ground cuts the forma-

tion level, so that one portion of the area is cutting and the other portion embankment.

The distance XN from the point of intersection to the centre of the formation level is mh.

As in Case 2 $$O_1N = O_2N = \frac{b}{2n},$$

and the triangle

$$\begin{aligned} XRS &= \text{the } \triangle\, QRO_1 + \text{the } \triangle\, QXO_1 - \text{the } \triangle\, XSO_1 \\ &= \tfrac{1}{2}\left\{\left(\frac{b}{2n}+h\right)s_1 + \left(\frac{b}{2n}+h\right)mh - \left(\frac{b}{2}+mh\right)\frac{b}{2n}\right\} \\ &= \tfrac{1}{2}\left\{\left(\frac{b}{2n}+h\right)(s_1+mh) - \frac{b}{2n}\left(\frac{b}{2}+mh\right)\right\}. \quad (16) \end{aligned}$$

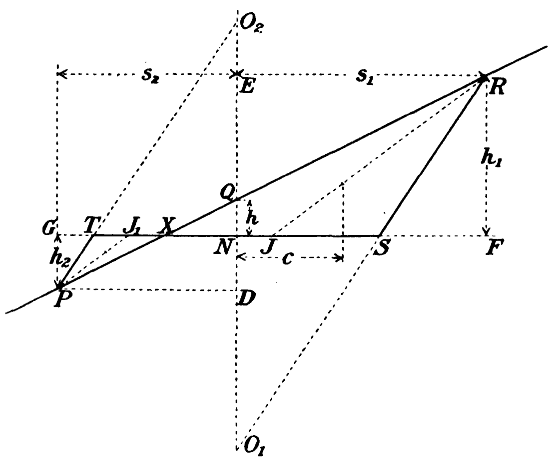


FIG. 11.5

Similarly, the triangle

$$\begin{aligned} PTX &= \text{the } \triangle\, QPO_2 + \text{the } \triangle\, QXN - \text{the } \triangle\, O_2TN \\ &= \tfrac{1}{2}\left\{\left(\frac{b}{2n}-h\right)s_2 + h\,.\,mh - \frac{b}{2}\cdot\frac{b}{2n}\right\} \\ &= \tfrac{1}{2}\left\{\left(\frac{b}{2n}-h\right)(s_2-mh) - \frac{b}{2n}\left(\frac{b}{2}-mh\right)\right\}. \quad (17) \end{aligned}$$

Or, if the heights h_1 and h_2 are known,

the area $$XRS = XS \times \tfrac{1}{2}h_1 = \tfrac{1}{2}\left(\frac{b}{2}+mh\right)h_1, \quad (18)$$

while the area $$XPT = TX \times \tfrac{1}{2}h_2 = \tfrac{1}{2}\left(\frac{b}{2}-mh\right)h_2. \quad (19)$$

The formulae (13) to (19) deduced above apply equally to embankments and to cuttings, as may be seen by inverting Figs. 11.3, 11.4, and

11.5, and numerous tables have been devised from these and other similar equations.

Sometimes the quantities are tabulated in two tables. One table furnishes the volume V_c of that portion immediately above or below the formation level, while the other table furnishes the quantity V_s in the two portions above or below the slope batters.

The central portion MWST at successive sections has areas $a = bh$ and $a' = bh'$, where h and h' are the respective central heights of the ground above or below the centre of the formation level.

The volume between successive cross-sections may be considered as approximately the frustum of a prism, so that the cubic contents are obtainable from equation (7), *i.e.*

$$V_c = \frac{D}{2}(a + a') = \frac{D}{2} \cdot b \cdot (h + h').$$

A correction is made when the sidelong slope is considerable or when the ratio of h to h' differs very much from unity.

Similarly, each of the side portions may be considered as the frustum of a pyramid, and the volume is obtained by equation (3), *i.e.*

$$V_s = \frac{D}{3}\{(a_1 + \sqrt{a_1 a_1'} + a_1') + (a_2 + \sqrt{a_2 a_2'} + a_2')\},$$

where a_1 and a_2 are the cross-sectional areas of the triangles WSR and PMT at one section, and a_1' and a_2' those corresponding at the following section.

Quantities on Curves.—In the formulae derived above, it has been assumed that the longitudinal centre line of the cutting or embankment is straight, and now the case when the centre line is curved on plan will be considered.

If the shape of a solid is such that it may be considered as the space swept through by a constant area rotating about a fixed axis, then by the well-known rule of Guldinus the volume is equal to the area multiplied by the length of the path traced by the centre of gravity of the area.

Thus, if an area A is rotated through an angle θ, about an axis at O, the path of any small element of area δA has a length of $r \cdot \theta$, where r is the distance of the element from O, and the volume swept through is $\delta A \cdot r \cdot \theta$.

The volume swept through by the whole area $= \int r \cdot \theta \cdot dA$, but $\int r \cdot dA$ is by definition equal to $A \times R$, where R is the distance of the c.g. of the area from O.

Hence the total volume is $A \cdot R \cdot \theta$, and, as $R\theta$ is the distance traced by the c.g., the total volume is equal to the area of the cross-section multiplied by the length of the path of the c.g.

For example, a right cone may be considered as formed by the complete revolution of a right-angled triangle about one of its shorter sides.

E.g. in Fig. 11.6 AB $=h$, the height of the cone.

BC $=r$, the radius of the base.

The area of the triangle ABC $=\frac{1}{2}\,.\,r\,.\,h$, and the c.g. G sweeps through a circle of radius GF, where GF is perpendicular to AB – the axis of revolution.

G lies on the median CE, at one-third of the distance from E,

$$\therefore\ GF=\tfrac{1}{3}\,.\,BC=\tfrac{1}{3}r,$$

and the path of G $=2\pi\,.\,\frac{r}{3}$.

The volume of the cone by Guldinus' rule is thus

$$\tfrac{1}{2}rh\times\tfrac{2}{3}\pi r=\frac{\pi r^2h}{3},$$

or $\frac{1}{3}$ base area × perpendicular height, a result which agrees with the equation on p. 335.

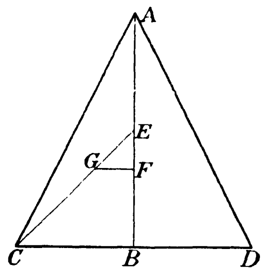


FIG. 11.6

Applying this rule to a cutting or embankment, it will be seen that if the shape and area A of the cross-section is constant, the volume will be the product of the area and the length between the sections measured along the curve through the c.gs.

For variable sections the previously derived formulae may be considered as satisfactory when the length along the path traced by the c.gs. is substituted for the length along the longitudinal centre line of the formation level.

It is, however, more convenient to apply the correction to the *areas* instead of to the *distance* D between the sections, because the eccentricity of the c.g. varies for successive sections, unless these are of uniform shape and size, and consequently the actual length of the mean path would be awkward to determine.

Let c be the eccentricity of the c.g. of a particular cross-section of area A. Then the volume swept over in a short length subtending θ radians at the centre of the curve is approximately $A(R\pm c)\theta$.

While, if the eccentricity is neglected, the usual rules would give $V=AR\theta$. The resulting error in a length $R\theta$ on the centre line $=\pm Ac\theta$, or

$$\pm\frac{Ac}{R}\text{ per unit length.}$$

If, therefore, to each area is added or subtracted an amount $\frac{Ac}{R}$ according to whether the c.g. falls on the side of the longitudinal axis remote from the centre of the curve or otherwise, and the usual rules then applied to the corrected areas, a sufficiently near approximation to the true volume is obtained. In fact, it is generally considered sufficiently accurate to neglect these corrections altogether, except in

exceptional circumstances, such as a deep cutting or embankment on a sharp curve round a hillside (Fig. 11.5).

The values of c for the various conditions may be derived as follows :

Case 1. When the surface of the ground is level and the area is symmetrical about the centre line, then $c=0$, and no correction is necessary.

Case 2 (Fig. 11.4). Let PP_1 be drawn parallel to TS.

Then the area may be considered as composed of the two figures RPP_1 and PP_1ST.

The area
$$\begin{aligned} RPP_1 &= \tfrac{1}{2}PP_1 \times (h_1 - h_2) \\ &= s_2 (h_1 - h_2) \text{ since } PP_1 = 2s_2 \\ &= s_2 \frac{(s_1+s_2)}{m}. \qquad (20) \end{aligned}$$

The distance of the c.g. of the triangle RPP_1 from P is $\tfrac{2}{3}$rds of the horizontal projection of the median PJ

$$= \tfrac{2}{3}\left(\frac{PP_1 + (s_1+s_2)}{2}\right) = \tfrac{1}{3}(2s_2 + s_1 + s_2) = \frac{s_1 + 3s_2}{3},$$

and the distance from the centre line QN

$$= \frac{s_1+3s_2}{3} - s_2 = \frac{s_1}{3}. \qquad (21)$$

The distance of the c.g. of the whole section, to the right of QN, *i.e.*

$$c = \frac{\left(\text{area } RPP_1 \times \frac{s_1}{3}\right) + \text{area } PP_1ST \times 0,}{\text{total area}},$$

or from (20), (21), and (14),

$$c = \frac{s_2(s_1+s_2)}{m} \cdot \frac{s_1}{3} \Big/ \tfrac{1}{2}\left\{(s_1+s_2)\left(h + \frac{b}{2n}\right) - \frac{b^2}{2n}\right\}$$

$$= \frac{2s_1s_2(s_1+s_2)}{3m\left\{(s_1+s_2)\left(h+\frac{b}{2n}\right) - \frac{b^2}{2n}\right\}}, \qquad (22)$$

or from (15)
$$c = \frac{2s_1s_2(h_1-h_2)}{3\left\{\frac{b}{2}(h_1+h_2) + h\,(s_1+s_2)\right\}}. \qquad (23)$$

Case 3 (Fig. 11.5). When the ground surface intersects the formation level.

(*a*) For the right-hand section RXS, the c.g. lies on the median RJ of the triangle RXS.

$$\begin{aligned} NJ &= (XJ - XN) \\ &= \tfrac{1}{2}(XN + NS) - XN \\ &= \frac{NS - XN}{2} = \tfrac{1}{2}\left(\frac{b}{2} - mh\right), \end{aligned}$$

and the horizontal distance

$$c = \text{NJ} + \tfrac{1}{3}(s_1 - \text{NJ})$$
$$= \frac{s_1}{3} + \tfrac{2}{3} \cdot \tfrac{1}{2}\left(\frac{b}{2} - mh\right)$$
$$= \tfrac{1}{3}\left(s_1 + \frac{b}{2} - mh\right). \quad . \quad . \quad . \quad (24)$$

(*b*) For the left-hand figure TPX, the c.g. lies on the median PJ_1

and $$\text{NJ}_1 = \tfrac{1}{2}\left(\frac{b}{2} + mh\right),$$

and the horizontal distance

$$c_1 = \text{NJ}_1 + \tfrac{1}{3}(s_2 - \text{NJ}_1)$$
$$= \frac{s_2}{3} + \tfrac{2}{3} \cdot \tfrac{1}{2}\left(\frac{b}{2} + mh\right)$$
$$= \tfrac{1}{3}\left(s_2 + \frac{b}{2} + mh\right). \quad . \quad . \quad . \quad (25)$$

Sidewidths.—Expressions for the sidewiths s_1 and s_2, and for the heights h_1 and h_2, may be deduced as follows :

Case 1 (Fig. 11.3). When $1/m = o$, *i.e.* the surface of the ground has no " sidelong " slope.

Here evidently $$s_1 = s_2 = \frac{b}{2} + nh. \quad . \quad . \quad . \quad (26)$$

Case 2 (Fig. 11.4).

s_1 = the longer sidewidth—up the bank in the case of a cutting and down the bank in the case of an embankment

$$= \frac{b}{2} + \text{SF}.$$

But $$\text{SF} = nh_1$$
$$= n(h + \text{EQ})$$
$$= n\left\{h + \frac{1}{m}\left(\frac{b}{2} + \text{SF}\right)\right\},$$
$$\therefore \text{SF}\left(1 - \frac{n}{m}\right) = nh + \frac{n \cdot b}{2m}$$

and $$\text{SF} = \frac{mn}{m-n} \cdot h + \frac{n \cdot b}{2(m-n)},$$

$$\therefore s_1 = \text{SF} + \frac{b}{2} = \frac{b}{2} + \left(h + \frac{b}{2m}\right)\frac{mn}{m-n}. \quad . \quad . \quad (27)$$

Similarly, $$\text{GT} = nh_2$$
$$= n\left\{h - \frac{1}{m}\left(\frac{b}{2} + \text{GT}\right)\right\},$$

and
$$\mathrm{GT}\left(1+\frac{n}{m}\right)=nh-\frac{n}{m}\cdot\frac{b}{2},$$

$$\therefore \mathrm{GT}=\frac{nm}{m+n}\left(h-\frac{b}{2m}\right)$$

and
$$s_2=\frac{b}{2}+\left(h-\frac{b}{2m}\right)\frac{mn}{m+n}. \qquad (28)$$

Case 3 (Fig. 11.5). The expression for s_1 is similar to equation (27), but towards the side on which the formation level is intersected by the surface of the ground :

$$\begin{aligned}\mathrm{GT}&=nh_2\\&=n(\mathrm{QD}-h)\\&=n\left\{\frac{1}{m}\left(\frac{b}{2}+\mathrm{GT}\right)-h\right\},\end{aligned}$$

and
$$\mathrm{GT}\left(1-\frac{n}{m}\right)=\frac{n}{m}\cdot\frac{b}{2}-nh,$$

$$\therefore \mathrm{GT}=\frac{mn}{m-n}\left(\frac{b}{2m}-h\right)$$

and
$$s_2=\frac{b}{2}+\left(\frac{b}{2m}-h\right)\frac{mn}{m-n}. \qquad (29)$$

The values of h_1 and h_2 are

$\left(h+\frac{s_1}{m}\right)$ and $\left(h-\frac{s_2}{m}\right)$ respectively in Case 2,

and $\left(h+\frac{s_1}{m}\right)$ and $\left(h-\frac{s_2}{m}\right)$ respectively in Case 3,

the last expression for h_2 being negative, *i.e.* in the opposite direction to h and h_1.

Setting out Sidewidths.—In addition to their application in the calculations for quantities, the values of s_1 and s_2 so obtained are employed in setting out the limits of the slopes on the ground, and fixing the positions of the boundary or fence pegs, say 6 ft. more remote from the centre line, though sometimes for this purpose it is considered sufficiently accurate to scale the distance s_1 and s_2 from the cross-sections.

An alternative method of locating with a level the tops of cuttings or the bottoms of embankments, particularly on steep or uneven ground, is as follows :

The levelling staff is held at Q on the centre line of the work and a reading is taken. The staff is then held at a point K say, which is judged to be approximately the correct position of the top of the slope.

As a guide to this approximation, the values of s_1 and s_2 may be scaled from the cross-section, or taken from the calculated values, or the sidewidth value for level ground $\left(i.e.\ \frac{b}{2}+nh\right)$ may be computed,

and as a first trial an allowance made with the eye to compensate for the "sidelong" slope, *i.e.* the staff is held at a selected point rather more distant from the central position on the upper side of a cutting or the lower side of an embankment, or rather less distant on the reverse sides.

If the altitude of this point K is found to be say h' ft. above formation level, then K will be the correct position and lie at the junction of the batter of the proposed work with the surface of the ground, if the measured distance from Q is equal to the calculated distance $\frac{b}{2}+nh'=s'$ say.

If not, a further trial is made, and the staff held at a little greater or a smaller distance than s' ft. from the centre line, according as the calculated distance s' or the measured distance to K is the greater. A new reading is then taken, giving the height above formation level as, say, h''.

The calculated distance, $\frac{b}{2}+nh''=s''$, say, will now generally be found to agree sufficiently closely with the measured distance, but, if not, the process may be continued, and any required degree of accuracy obtained.

Example.—An embankment is made on ground which has a transverse slope of 1 in 10. The width of the bank at formation level is 30 ft. and the side slopes of the embankment are 2 horizontal to 1 vertical.

The heights of the bank at the centre of the formation level are 10, 14, and 16 ft., at three consecutive sections spaced 50 ft. apart.

Find (1) The sidewidths.
(2) The cross-sectional areas.
(3) The volume of this length of embankment when the centre line is straight.
(4) The volume of this length when the centre line is a circle of 500 ft. radius.

(1) By equation (27), *i.e.* $s_1=\frac{b}{2}+\left(h+\frac{b}{2m}\right)\frac{mn}{m-n}$, the sidewidths on the downhill side of the embankment are

$$\text{Section (i)}\quad 15+\left(10+\frac{15}{10}\right)\frac{10\times 2}{10-2}=43{\cdot}75',$$

$$\text{,, (ii)}\quad 15+\left(14+\frac{15}{10}\right)\frac{10\times 2}{8}=53{\cdot}75',$$

$$\text{,, (iii)}\quad 15+\left(16+\frac{15}{10}\right)\frac{10\times 2}{8}=58{\cdot}75',$$

while on the uphill side the sidewidths by equation (28), *i.e.* $s_2=\frac{b}{2}+\left(h-\frac{b}{2m}\right)\frac{mn}{m+n}$, are

$$\text{Section (i)}\quad 15+(10-1{\cdot}5)\frac{20}{12}=29{\cdot}17\text{ ft.},$$

$$\text{,, (ii)}\quad 15+(14-1{\cdot}5)\frac{20}{12}=35{\cdot}83\text{ ft.},$$

$$\text{,, (iii)}\quad 15+(16-1{\cdot}5)\frac{20}{12}=39{\cdot}17\text{ ft.}$$

(2) The cross-sectional areas by equation (14), *i.e.* $A=\frac{1}{2}\left\{(s_1+s_2)\left(h+\frac{b}{2n}\right)-\frac{b^2}{2n}\right\}$. are

$$\text{Section (i)}\quad \tfrac{1}{2}\left\{(43\cdot75+29\cdot17)\left(10+\frac{15}{2}\right)-\frac{30^2}{4}\right\}=525\cdot5 \text{ sq. ft.,}$$

$$\text{,,}\quad \text{(ii)}\quad \tfrac{1}{2}\left\{(53\cdot75+35\cdot83)\left(14+\frac{15}{2}\right)-\frac{30^2}{4}\right\}=850\cdot5 \text{ sq. ft.,}$$

$$\text{,,}\quad \text{(iii)}\quad \tfrac{1}{2}\left\{(58\cdot75+39\cdot17)\left(16+\frac{15}{2}\right)-\frac{30^2}{4}\right\}=1038 \text{ sq. ft.}$$

(3) The volume of the embankment by the prismoidal rule is

$$V=\frac{50}{3}\{525\cdot5+4\,(850\cdot5)+1038\}$$
$$=82758 \text{ cub. ft.}$$
$$=3065 \text{ cub. yds.,}$$

while by the trapezoidal rule

$$V=\frac{50}{2}\{525\cdot5+2\,(850\cdot5)+1038\}$$
$$=81612\cdot5 \text{ cub. ft.}$$
$$=3023 \text{ cub. yds. nearly,}$$

by average areas

$$V=2\times50\left\{\frac{525\cdot5+850\cdot5+1038}{3}\right\}$$
$$=80467 \text{ cub. ft.}$$
$$=2980 \text{ cub. yds.}$$

(4) The eccentricity from equation (22) is $c=\frac{s_1s_2(s_1+s_2)}{3m}\Big/\text{area.}$

therefore for section (i) $c=\frac{(43\cdot75\times29\cdot17)(43\cdot75+29\cdot17)}{30\times525\cdot5}$,

from which by logarithms, or a slide rule, $c=5\cdot90$ ft.

$$\text{For section (ii)}\quad c=\frac{(53\cdot75\times35\cdot83)(53\cdot75+35\cdot83)}{30\times850\cdot5}$$
$$=6\cdot76 \text{ ft.,}$$

$$\text{for section (iii)}\quad c=\frac{(58\cdot75\times39\cdot17)(58\cdot75+39\cdot17)}{30\times1038}$$
$$=7\cdot24 \text{ ft.}$$

The correction to be applied, using the prismoidal formula, is

$$\frac{50}{3}\left\{525\cdot5\times\frac{5\cdot90}{500}+4\,.\,\frac{850\cdot5\times6\cdot76}{500}+\frac{1038\times7\cdot24}{500}\right\}\text{cub. ft.}$$
$$=1120 \text{ cub. ft.}=41\cdot5 \text{ cub. yds. about.}$$

To the trapezoidal rule result, the correction is

$$\frac{50}{2}\left\{525\cdot5\times\frac{5\cdot90}{500}+\frac{2\times850\cdot5\times6\cdot76}{500}+\frac{1038\times7\cdot24}{500}\right\}\text{cub. ft.}$$
$$=1105\cdot7 \text{ cub. ft.}=41 \text{ cub. yds. nearly.}$$

The above correction is added if the centre of the curve lies on the uphill side of the embankment, and subtracted if the centre lies on the downhill side.

Example 2.—At a certain section the height of the centre of the embankment in Example 1 is 0·5 ft.

Find the cross-sectional areas of excavation and filling in at this point, and

the corrections to be applied per unit length when the radius of the centre line of the formation level is 500 ft.

The sidewidth for the embankment portion is by equation (27)

$$s_1 = \frac{b}{2} + \left(h + \frac{b}{2m}\right)\frac{mn}{m-n}$$
$$= 15 + (0{\cdot}5 + 1{\cdot}5)\frac{20}{8} = 20{\cdot}0 \text{ ft.},$$

while for the cutting portion by equation (29)

$$s_2 = \frac{b}{2} + \left(\frac{b}{2m} - h\right)\frac{mn}{m-n}$$
$$= 15 + (1{\cdot}5 - 0{\cdot}5)\frac{20}{8}$$
$$= 17{\cdot}5 \text{ ft.}$$

The area of embankment by equation (16)

$$= \tfrac{1}{2}\{(7{\cdot}5 + 0{\cdot}5)(20 + 5) - 7{\cdot}5(15 + 5)\}$$
$$= 25{\cdot}0 \text{ sq. ft.}$$

The area of cutting by equation (17)

$$= \tfrac{1}{2}\{(7{\cdot}5 - 0{\cdot}5)(17{\cdot}5 - 5) - 7{\cdot}5(15 - 5)\}$$
$$= 6{\cdot}25 \text{ sq. ft.}$$

The eccentricity of the c.g. for the embankment by equation (24)

$$= \tfrac{1}{3}(20 + 15 - 5) = 10{\cdot}0 \text{ ft.},$$

and the correction per foot run

$$= \frac{Ac_1}{500} = \frac{250}{500} = 0{\cdot}5 \text{ cub. ft.},$$

to be added if the curve is concave towards the uphill side of the slope, and subtracted if convex.

The eccentricity of the c.g. for the excavation from equation (25) is

$$\tfrac{1}{3}(17{\cdot}5 + 15 + 5) = 12{\cdot}5 \text{ ft.},$$

and the correction per foot run

$$= \frac{Ac_2}{500} = \frac{6{\cdot}25 \times 12{\cdot}5}{500} = 0{\cdot}156 \text{ cub. ft.},$$

to be added if the curve is convex towards the uphill side of the slope, and subtracted if concave.

EXAMPLES

1. (U. of L.) A railway cutting lies approximately east and west.

Beginning at the west end at chainage 32 + 00 the levels taken along the centre line are given in the accompanying page from the level book. The cutting is 30 ft. wide at formation level and the side slopes are $1\frac{1}{2}$ to 1. The sidelong slope of the ground is downwards from the north side of the railway, and the slope is shown at each station in the level book. Reduce the line of levels and plot the longitudinal section to the following scales :

Horizontal, 50 ft. to 1 in.
Vertical, 10 ft. to 1 in.

The formation level is 33·00 ft. above datum at station 32 and the grade falls 1 in 40, the centre line of the whole section being in cutting. Plot the twelve cross-sections given to the same scales as the horizontal section.

Plot the plan of the area required for the cutting, making an allowance of 4 ft.

on either side, from the top of the slope to the fence. Calculate the area between the fences, using Simpson's rule.

Mark down the area of each cross-section in square feet and determine the volume of the cutting in cubic yards.

N.B.—The 100 ft. chain is used.

Chainage	B.S.	Int.	F.S.	H.I.	Reduced Level	Remarks	Sidelong slope, 1 in.
	9·37				27·30	O.B.M. stone S.E. corner church	
	13·21		0·32			T.P.	
32		9·5					160
33		7·2					200
34		7·5					level
34 + 50		2·0					level
35		0·3					level
	8·82		0·36			T.P.	
36		5·5					200
37		15·7					75
	0·34		15·83			T.P.	
38		10·7					35
	0·21		15·53			T.P.	
39		1·8					10
40		8·2					4
41		14·0					4
42		14·7					3

2. (U. of L.) In a certain railway cutting the width at formation level is 20 ft., the sides of the cutting slope at $1\frac{1}{2}$ to 1, and the surface of the ground has a uniform side slope of 1 in 10. Find the volume (in cubic feet) of the excavation between two points 200 ft. apart on the centre line, the depth of cutting at the first point being 25 ft., and at the second point 30 ft., while at a point half-way between the depth is 26 ft.

3. If the points in Question 2 lie on a curve of 1000 ft. radius, what would be the correction to be applied?

4. (U. of L.) The base of a railway cutting is 32 ft. in width, the depth of formation level is 34 ft. below the centre line of the railway, the side slopes are $1\frac{3}{4}$ to 1, and the surface of the ground falls 1 in 8. Calculate the half-breadths for the cutting.

At a distance of 1 chain along the centre line the depth of formation level is 28 ft., and at a distance of 2 chains it is 20 ft. Find the volume of earthwork to be removed.

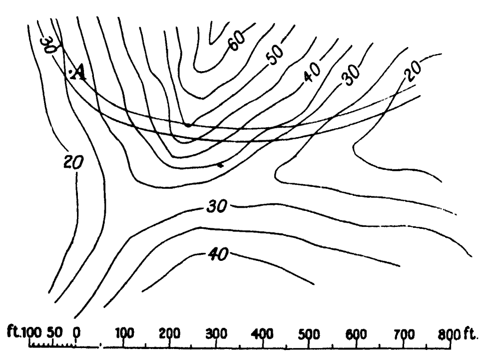


FIG. 11.7

5. (I.C.E.) In the sketch printed on this page, you are shown a contour map with a road 30 ft. wide marked on it. The road is to be made with a gradient falling 3 in 100 from the point A, whose level is to be made 40. If the sides of the banks and cuttings are to slope $1\frac{1}{2}$ to 1, mark the top and toe of the slope on each side

of the road, and write down the height of cut or bank at centre of the road every 100 ft. from A.

6. (U. of B.) At every hundred yards along a piece of ground, levels were taken; they are as follows:

Yards	Reduced Levels Feet
0	210·00
100	220·22
200	231·49
300	237·96
400	240·53
500	243·10
600	245·47
700	249·93
800	250·00

A cutting is to be made for a line of uniform gradient passing through the first and last points, namely, through the 210·00 ft. mark and the 250·00 ft. mark. What is the gradient?

Taking differences between the height of formation and the original height of surface to the nearest $\frac{1}{4}$ ft., calculate the volume of the cutting on the assumption that there is a natural side slope of 10 to 1.

Given breadth of formation = 36 ft.

s = slope of cutting on each side = $1\frac{1}{2}$ to 1.

7. Show that equations (27), (28) and (29) on pp. 347-348 may be written:

$$S_1 = \frac{m(b+2nh)}{2(m-n)} \quad . \quad . \quad . \quad . \quad . \quad (27a)$$

$$S_2 = \frac{m(b+2nh)}{2(m+n)} \quad . \quad . \quad . \quad . \quad . \quad (28a)$$

$$S_2 = \frac{m(b-2nh)}{2(m-n)} \quad . \quad . \quad . \quad . \quad . \quad (29a)$$

CHAPTER XII

HYDROGRAPHIC SURVEYING

HYDROGRAPHIC Surveying, as the name implies, is that branch of surveying which in one way or another is concerned with the measurement of a body of water. It includes such operations as the determination of contour lines under water, the cross - sections and discharges of streams, the location of high and low watermarks, and the boundaries of lakes, the " set " of tides and the directions of currents, etc.

Contour lines under water, or " Lines of Equal Depth ", may be referred for engineering purposes to a datum at the mean low-water level of ordinary spring tides, or to a datum at mean sea-level, or to any other arbitrary datum. In this country it is often convenient to refer to the Ordnance datum (O.D.).[1]

For nautical purposes the mean *low*-water level is generally referred to, but it should be borne in mind that this level may vary considerably along different parts of a coast; and although it is immaterial for nautical charts, the datum may not be suitable for engineering works unless these are of limited extent.

To determine these contour lines, measurements known as soundings are taken downwards from the surface of the water to the bed of the sea, river, or lake as the case may be, and from the spot-levels or cross-sections so obtained the contours are interpolated, as explained in Chapter VI.

For very shallow water and for the interval between high and low-water marks, it is generally preferable to take the levels directly with a dumpy or other level, and an ordinary levelling staff—the staff man being provided with " waders " if necessary.

For deeper water this method is not applicable, and the vertical downward distances from the surface are measured by means of a graduated rod (up to say 15 ft.), a brass or iron sounding chain subdivided into feet by tellers, or a sounding line of hemp having small indicating tapes inserted at every foot of its length, and with either fathoms[2] or 10-ft. lengths specially distinguished.

Whichever of these measuring appliances is used, it is heavily weighted with a leaden weight; and sometimes a device for bringing to the surface a specimen of the bed is attached. This device may be a cup-shaped scoop, or merely a piece of tallow inserted into a hollow in the bottom of the weight.

For nautical charts the depths are generally required in fathoms,

[1] See p. 178.

[2] 1 fathom = 6 ft.

while for engineering works the foot unit, decimally divided, is employed.

In a fast-flowing stream or current the boat from which the soundings are being taken is pulled in the direction of the current, and then while a reading is being observed, allowed to float freely with the stream. The boat is thus carried down stream at approximately the same velocity as the sounding line, so that any great deflection of this from the vertical is obviated, and a more correct reading is obtained than would otherwise be the case.

Tide Gauges.—As explained above, soundings are the vertical distances downwards from the surface of the water to its bed, and as the altitude of the surface of the sea or of a tidal river is not constant, owing to the rise and fall of the tides, it follows that the depth of a sounding at any particular point depends upon the state of the tide at the time. Similarly the surface levels of lakes, rivers, and other bodies of water vary as the climatic conditions change.

Consequently, to determine the absolute altitude, or the depth of a point on the bottom with reference to some fixed datum, the altitude of the surface at the time of observation of the sounding must be known. This information is obtained from a tide gauge, which in its simplest form consists of a vertical staff, graduated in feet and decimals of a foot (or in some other convenient notation), and placed in such a position that the surface level of the water may be read from the scale at any time.

The position of the zero of the scale relatively to the mean low-water level of ordinary spring tides is found by observation, or its altitude relatively to any other arbitrary datum may be found by accurate levelling from a bench mark.

Generally a rather more elaborate form of tide gauge is used, as it would be very difficult to obtain an accurate reading on a simple exposed scale on account of waves, etc.

In one form the water is admitted through a series of small openings near the foot, and well below low-water level—into a small cylindrical or rectangular tube, in which is a copper or other float. The oscillations due to the waves are by this means deadened, and a comparatively steady motion is communicated to the float as the mean sea-level rises or falls. Attached to the float is a vertical scale, the movements of which relatively to a fixed datum may be periodically noted. If desired, the vertical motion of the float may be communicated by means of a wheel and axle arrangement or other mechanical contrivance to a pencil, which is thereby caused to move through a distance proportional to the rise of the tide.

The pencil bears against a ruled diagram paper pasted on or otherwise attached to the circumference of a drum, which revolves by clockwork once in twenty-four hours, so that when the paper is removed a graph is obtained, of which the vertical ordinates are proportional to the height of the tide at definite times denoted by the horizontal ordinates to a scale of say 1″ per hour. The vertical scale depends upon the actual range of the tides.

By this means a record is obtained automatically, and it is unnecessary to have an observer stationed continuously at the gauge.

As each high tide occurs about 12 hours 25 minutes later than its predecessor, it is not necessary to replace the diagram every day, for the records of a week or more may be obtained upon one paper. The results consist very roughly of a series of harmonic curves, each day's curve being approximately 50 minutes on the horizontal scale in advance of the previous day's record, and the curves are consequently easily distinguishable.

If the gauge is not automatic, readings must be taken on the scale at frequent intervals, and the exact time of observation noted : the intervals being 5, 10, or 15 minutes or even more in some cases.

From these results, by interpolation, the surface level at any particular time may be determined, so that if, when a sounding is being taken, the time of observation is also noted, all the depths may be easily reduced to a common datum.

Location of Soundings.—The next difficulty is to locate the exact points at which particular soundings are taken, so that the data may be plotted upon a plan or chart.

(1) Perhaps the simplest method is one which is applicable only to a moderately narrow sheet of water such as a river across which a tape or a rope may be stretched.

If a general contoured survey of the river-bed over a considerable distance is required, a chain line (or a series of such lines forming a traverse or a portion of a theodolite or other survey) is set out along one bank, and at definite intervals along this line, and in suitable directions, sections are taken across the stream.

The distances along the cross-sections at which soundings are taken may be measured either by means of a tape stretched from bank to bank and self-supporting ; or a tape suspended by loops from a taut wire or other rope ; or by means of a rope stretched from bank to bank, and marked at intervals of 5 ft. (say) with linen or other tags.

The sounding party may take the soundings from a boat, which if necessary, as in the case of a swift-running stream, may be attached to a second rope, fixed near or at about a boat's length up-stream from the first or graduated rope. This enables the boat to be pulled quickly into position, and held there without danger.

The method is also particularly suitable when a single accurate cross-section of a river is required, and for determining the discharge by means of a current meter, pitot tubes, etc.

In still water, or in the case of a very slowly flowing stream, the second rope may be dispensed with, and the boat sculled into position each time.

(2) When a line of soundings is required to be taken completely, or only partly, across a river or other body of water, too wide to be conveniently spanned by a tape or rope as in method (1), the direction of the line may be set out on the banks by means of two posts or ranging rods A, B, Fig. 12.1 ; or if the nature of the banks precludes this, by means of two buoys, or of one post and one buoy.

The oarsmen are thus enabled to pull the boat into such a position that when A appears to be immediately behind B, a sounding may be taken, or a current meter reading may be obtained if required, on the correct line AB.

The position along the line AB may be determined :

(*a*) By a direct tape measurement from B.

(*b*) By stadia readings taken from an instrument at B on to a staff or rod held in the boat.

(*c*) By an angular measurement taken, when a signal from the boat is given, with a theodolite or other angular instrument stationed at some arbitrary position G, previously located with reference to the line AB.

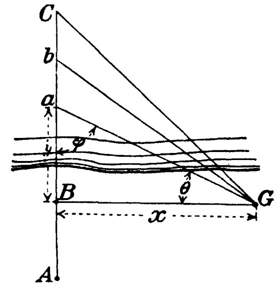


FIG. 12.1.
Location of Soundings

Thus if x be the perpendicular distance of G from some point B on AB, and if θ be the observed angle subtended at G by the distance from B to the boat, then this distance y to the position of the sounding $=x \tan \theta$.

From this formula the values of y for the various points can be quickly tabulated and plotted ; or if some other point such as A is referred to as a zero for the different angles, it may be more convenient to plot the positions by means of a protractor, and dispense with the calculations.

(*d*) By an angular measurement, ϕ, taken from the boat with a sextant at the time of sounding, to some point G already located on the map. Here $y=x \cot \phi$. For a short section G might be conveniently marked with a ranging rod.

For a long line of soundings, for instance off the sea-coast, G may be any well-defined object in the neighbourhood, such as a flagstaff, church spire, or even a buoy, and in this case the positions *a*, *b*, *c*, etc., might be plotted with a protractor.

The direction of the line AB is determined by any of the usual methods from the main chain line along the banks.

(3) A much more expeditious method of locating soundings for a chart is by means of simultaneous angular measurements taken from two instrument stations on shore ; a method, however, which requires two instrument men in addition to the sounding party in the boat.

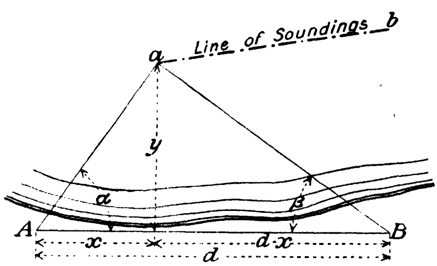


FIG. 12.2.—Location of Soundings

Thus if A and B are the two stations (Fig. 12.2) the positions of which on the map are known, a theodolite is set up at each ; or if these instruments are not available, sextants or other angular instruments may be used—plane tables would be practicable in some cases, if only a moderate number of points are to be located.

Each operator selects some definite referring object already plotted on the map (possibly the station at which the other instrument is fixed), and with the vernier of the horizontal scale clamped at zero, directs the cross-hairs of his telescope to this.

The lower axis is then clamped, and not again altered during the observations from this station unless occasionally it is found upon checking that the zero requires a little adjustment.

The telescope is then directed to some definite mark on the boat, the progress of which is followed until a signal is given by the sounding party, *e.g.* by means of a flag, when the angle subtended at each station is read and booked. The telescope is then redirected to the boat, and the next signal awaited, it being unnecessary again to adjust the zero to the referring object, unless there is ample time and a check is desired.

The position of each sounding may be observed independently in this way, but to do so would entail considerable labour, increase the risk of misunderstandings, and retard the progress of the work. Consequently it is a common practice to observe only the first and last positions on a line of soundings, the intermediate positions being approximately located by sounding at regular time intervals while rowing at a steady rate along the line. The time intervals need not be measured with a stop-watch, so long as the soundings are taken at a steady rate, as the total number of soundings taken over the distance being known, and the terminations of the line fixed by the angular observations, the intermediate positions may be found by interpolation. Almost invariably the points would be plotted by means of a protractor, but if desired the co-ordinates of each position could be calculated and plotted with a linear scale.

Thus let d be the distance AB, and α and β the observed angles at A and B—each measured from the other station as a referring object, and let the co-ordinates of the position of the boat (a) be x and y from A.

Then
$$y = x \tan \alpha = (d - x) \tan \beta,$$
$$\therefore x (\tan \alpha + \tan \beta) = d \tan \beta,$$
$$x = \frac{d \tan \beta}{\tan \alpha + \tan \beta},$$
and
$$y = \frac{d \tan \alpha \tan \beta}{\tan \alpha + \tan \beta}.$$

The two-theodolite method is also very useful in determining the direction of currents or the "set" of tides. Floats are liberated from a boat, or by other means, and points upon the paths followed by each are located by sets of two simultaneous angular measurements from the known station-points.

By noting the times of the observations, the rate of flow may be easily calculated, after the directions of the currents have been plotted and the lengths of their paths scaled from the chart.

(4) When the ground near the shore is of a moderately high altitude, the positions of soundings or of floats may be located by means of

observations from one theodolite, the polar co-ordinates of the points being thereby furnished.

The observations would include :

(*a*) A horizontal angle or bearing α from some referring object shown upon the map, and

(*b*) A vertical angle of depression.

Then, knowing the reduced level of the instrument axis, and from the tide gauge records, that of the surface of the water at the time of observation, the horizontal distance may be calculated, *e.g.* if the height of the instrument axis above the water surface be h feet, and the angle of depression be θ, the horizontal distance r from the instrument station to the sounding or float station is

$$r = h \cot \theta,$$

and if this is measured along a line upon the plan having a bearing of α from the referring object, the required point is located.

If preferred, the rectangular co-ordinates with reference to the instrument station as origin may be computed, *i.e.*

$$x = r \cos \alpha = h \cot \theta \cos \alpha,$$
$$y = r \sin \alpha = h \cot \theta \sin \alpha.$$

(5) When no survey is being carried out on shore, but all the operations are being completed from boats, the positions of the soundings may be located by means of two simultaneous angular observations to three well-defined points already marked upon the map.

If no prominent natural objects are available, poles may be erected, and their positions determined and plotted by ordinary surveying methods from recognizable points on the plan.

The position of each individual sounding may be located, or, as in the previous methods, merely the two end positions of each line of soundings. Unlike method (3), the two angles can generally be taken by one observer, though he may sometimes employ two instruments to enable him to measure the angles as nearly simultaneously as possible, and to read their values at his leisure afterwards.

The location of points by this method gives rise to the well-known " Three-Point Problem ", which is also of importance in plane table surveying, and in that connection has been dealt with in Chapter IX.

Three-Point Problem.—Let a, b, c be the three points upon the plan representing the three selected objects A, B, and C upon the shore to which observations are made ; and let o be that point on the plan which corresponds to the position O of the observer in the boat ; *i.e.* the point which it is required to locate on the plan from the observed values of the angles AOB $= \theta_2$ and BOC $= \theta_1$.

The following are a few of the methods by which this problem may be solved, though it should be noticed that when A, B, C, and O are concyclic the solution is indeterminate, a condition to which the expression " Failure of Fix " is applied.

Method 1 (Mechanical).—The most usual solution is obtained by

the use of a station-pointer (Fig. 12.3). This instrument consists of three arms, the fiducial edges of which radiate to a common centre: the middle arm is fixed, while the two outer are capable of rotation about the centre of the instrument, and these two are fitted with verniers reading to 1 minute, and with clamp and tangent screw arrangements for accurate adjustment.

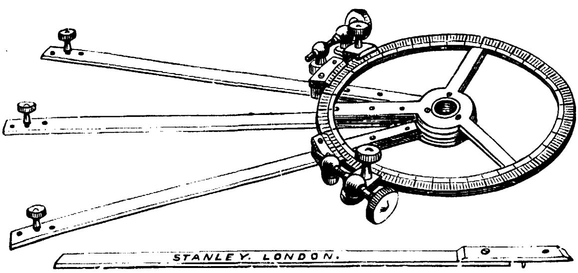


FIG. 12.3.—Station-pointer

To use the instrument the arms are so set by means of the verniers as to include the observed angles θ_1 and θ_2, and the instrument is moved over the paper until by trial each edge passes simultaneously through one of the three points a, b, c on the chart. The centre o is then marked with a hard pencil or pricker, or rays drawn along the edges of the arms are produced to intersect, and the point o thus obtained.

A similar method consists in using a sheet of tracing paper upon which three concurrent straight lines are drawn, to include the angles θ_1 and θ_2. This is moved over the chart as in the case of a station-pointer, and the position of the intersection point is pricked through on to the plan to give the point o.

When A, B, C, and O are concyclic, any number of points o will satisfy the conditions, so that the solution is indeterminate, as already stated in Chapter IX.

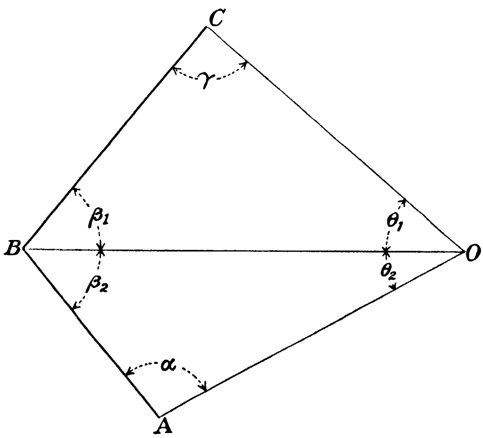


FIG. 12.4.—Three-point Problem

Method 2 (Mathematical) (Fig. 12.4).—Let the angle CBO $=\beta_1$, ABO $=\beta_2$, BCO $=\gamma$, and BAO $=\alpha$.

Then if $\beta_1+\beta_2=\beta$, say, this value can be obtained by measurement from the chart, and as the sum of all the angles in two triangles is equal to 360°,

$$\gamma+\alpha=360-\beta-\theta_1-\theta_2=\phi \text{ say.} \qquad (1)$$

We now want an expression for $\gamma-\alpha$ so that γ and α may be determined. Applying the sine formula to the two triangles in turn we have

$$\text{OB}=\text{BC}\sin\gamma/\sin\theta_1=\text{AB}\sin\alpha/\sin\theta_2, \qquad (2)$$

so that $\sin \gamma/\sin \alpha = \text{AB} \sin \theta_1/\text{BC} \sin \theta_2\}$. . (3)

$$= k \text{ say}$$

$$\tan \tfrac{1}{2} (\gamma - \alpha) = \frac{\sin \tfrac{1}{2} (\gamma - \alpha)}{\cos \tfrac{1}{2} (\gamma - \alpha)},$$

$$\tan \tfrac{1}{2} (\gamma + \alpha) = \frac{\sin \tfrac{1}{2} (\gamma + \alpha)}{\cos \tfrac{1}{2} (\gamma + \alpha)},$$

$$\frac{\tan \tfrac{1}{2} (\gamma - \alpha)}{\tan \tfrac{1}{2} (\gamma + \alpha)} = \frac{\sin \tfrac{1}{2} (y - \alpha) \cos \tfrac{1}{2} (\gamma + \alpha)}{\sin \tfrac{1}{2} (\gamma + \alpha) \cos \tfrac{1}{2} (\gamma - \alpha)}$$

$$= \frac{\tfrac{1}{2} (\sin \gamma - \sin \alpha)}{\tfrac{1}{2} (\sin \gamma + \sin \alpha)}$$

$$= \frac{\sin \gamma/\sin \alpha - 1}{\sin \gamma/\sin \alpha + 1} = \frac{k - 1}{k + 1}$$

$$\therefore \tan \tfrac{1}{2} (\gamma - \alpha) = \frac{k - 1}{k + 1} \cdot \tan \tfrac{1}{2} (\gamma + \alpha). \quad . \quad (4)$$

Now as AB, BC, θ_1 and θ_2 are known, k can be found from (3). $(\gamma + \alpha)$ is known from (1), and hence $(\gamma - \alpha)$ can be found from (4) and γ and α calculated. Then

$$\beta_1 = 180° - \gamma - \theta_1 \quad \text{and} \quad \beta_2 = 180° - \alpha - \theta_2.$$

The length OB can be calculated from equation (2), and, knowing β_1 and β_2, the point O is located. As a check OA and OC can be calculated from the sine formula, *i.e.* $\text{OA} = \text{OB} \sin \beta_2/\sin \alpha$. If the co-ordinates of A, B, C are known, or if the lengths and bearings of AB and BC are known, the co-ordinates of O can be determined by the methods described in Chapter V. If A, B, C, O are concyclic, $\beta + \theta_1 + \theta_2 = 180°$, and hence from equation (1)

$$\gamma + \alpha = 180°,$$

or $$\gamma = 180° - \alpha.$$

$$\therefore \sin \gamma = \sin (180° - \alpha)$$

$$= \sin \alpha,$$

and $$\sin \gamma/\sin \alpha = 1,$$

i.e. $$k = 1.$$

From equation (4) then

$$\tan \tfrac{1}{2} (\gamma - \alpha) = 0 \times \tan 90° = 0 \times \infty,$$

which is indeterminate, and satisfied for all values of $(\gamma - \alpha)$.

Method 3 (Geometrical).—From the points b and c (Fig. 12.5) on the chart, and on the opposite side of the line bc to the supposed position of o, draw lines cc_1 and bb_1, making the angles c_1cb and b_1bc each equal to θ_1 (*i.e.* the observed angle COB). Draw co_1 and bo_1 at right angles to c_1c and b_1b respectively, and let these intersect in o_1.

With o_1 as centre, and o_1b or o_1c as radius, describe a circle bco.

Similarly, on the opposite side of ab to o set off angles θ_2 and determine a point o_2; then with this as centre describe a circle abo, cutting the previous circle bco in o, which is the point required.

The centres o_1 and o_2 may also be found by other methods, as, for example, that shown for o_2 in Fig. 12.5. Here a_1ab was made equal to θ_2 and ao_2 drawn at right angles to a_1a to meet a_2o_2, the vertical bi-

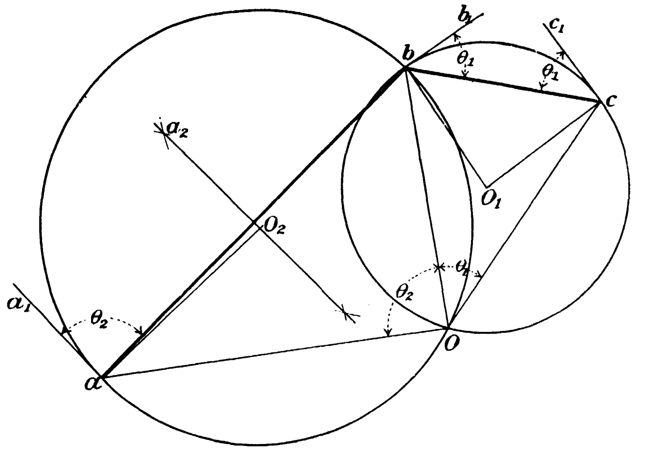


FIG. 12.5.—A Good Fix

sector of ab in o_2; or, again, the angles cbo_1 and bco_1 may be constructed, each equal to $90° - \theta_1$, and the angles bao_2 and abo_2 each equal to $90° - \theta_2$, and the points o_1 and o_2 fixed in this way.

Proof.—Join oa, ob, oc.

Then by Euclid III. 32, as c_1c is at right angles to o_1c—a radius of the circle bco—the angle c_1cb between the tangent c_1c and the chord cb is equal to the angle cob in the opposite segment, *i.e.* $cob = \theta_1$. Similarly, the angle $boa = \theta_2$, so that o is the point required.

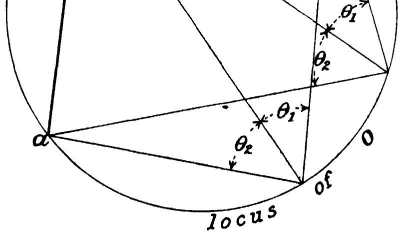


FIG. 12.6.—Failure of Fix

If the two circles intersect at an angle, such as that in Fig. 12.5, the point o may be fairly accurately determined and the solution is known as a "good fix". If, however, the two circles coincide (Fig. 12.6), *i.e.* if the four points a, b, c, o are concyclic, then the point o cannot be properly located and "failure of fix" ensues, as any position on the circle satisfies the conditions.

In intermediate cases, where the two circles nearly coincide, the point at which they intersect cannot be clearly distinguished, and a "bad fix" is obtained (Fig. 12.7).

As the sum of the two opposite angles of a quadrilateral inscribed

in a circle is equal to two right angles (III. 22), the four points *a*, *b*, *c*, *o* are concyclic if the sum of the angles β, θ_1 and θ_2 is equal to 180°, so that if the value of β is known it can be predicted, when the readings θ_1 and θ_2 are taken, whether the "fix" will be good or bad or whether it will fail altogether, and other points can then if necessary be observed too.

Or again, when the four points are concyclic, the angles BCA and BOA, being in the same segment, are equal, so that if θ_2 is found to be nearly equal to the angle BCA, a satisfactory fix will not be obtained, and other observation points should be chosen.

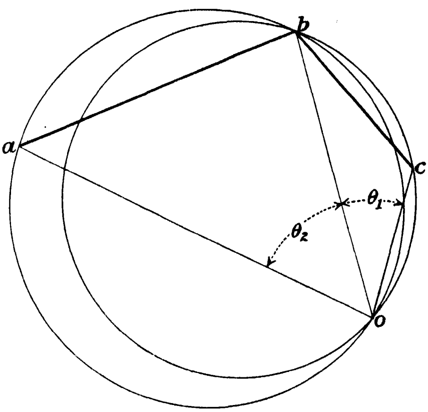


FIG. 12.7.—A Bad Fix

Method 4 (Geometrical) (Fig. 12.8).—Join *ca*. Draw *cd* and *ad* towards *b*, making angles θ_2 and θ_1 respectively with *ca*, and let these intersect in *d*.

Join *db*, and produce indefinitely to cut the circumscribing circle of the triangle *dca* in the point *o*. Join *oa*, *ob*, *oc*. Then the angle

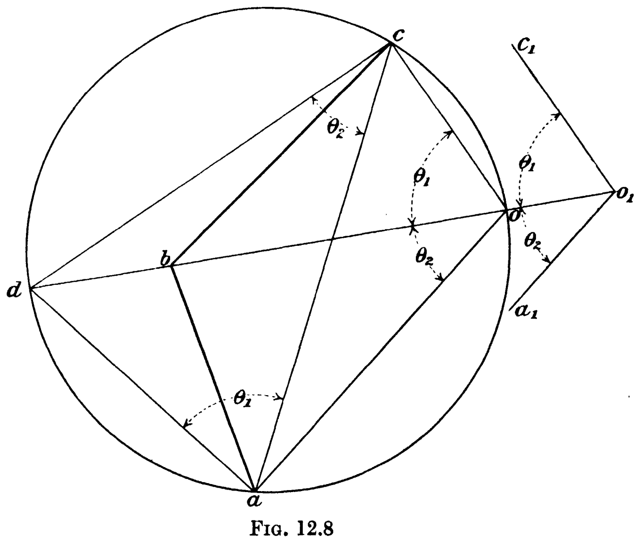


FIG. 12.8

aod = the angle *acd* (III. 21), because they are in the same segment of the circle *adco* ; *i.e.* the angle $aob = \theta_2$.

Similarly the angle $cob = \theta_1$, therefore *o* is the point required.

In the actual construction, instead of a circle being described through the three points *a*, *d*, *c*, the point *o* may be found by assuming any point o_1 on *db* produced and drawing o_1c_1 and o_1a_1 making angles θ_1 and θ_2 respectively, with o_1b towards *c* and *a*, and then drawing parallels through *c* and *a* to intersect on *db* in *o*.

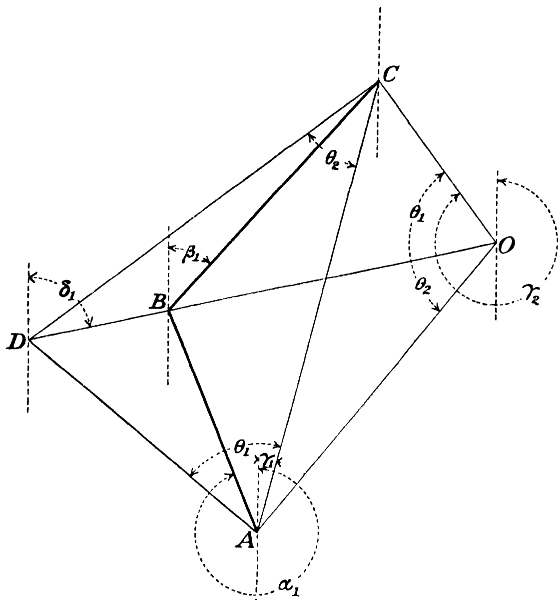


FIG. 12.9

In the case of "failure of fix" *d* and *b* would coincide, and consequently the direction of the line *dbo* would be unfixed, and the locus of *o* would be the circumscribing circle of the triangle *cba* as in Fig. 12.6. Other geometrical constructions[1] were described in previous editions of this book, but they are now omitted.

Method 5 (by Calculation) (Fig. 12.9).—When the co-ordinates of A, B, and C and the bearings of AB and BC are known as in the case of a traverse or triangulation survey. Let A_x and A_y, B_x and B_y, C_x and C_y be the co-ordinates of A, B, and C respectively, β_1 the bearing of BC, α_1 the bearing of AB, and γ_1 the bearing of AC.

Then from the triangle ADC, the length AC being known, and also the angles θ_1 and θ_2 (by construction as in Method 4),

DC is equal to $\dfrac{\text{AC} \sin \theta_1}{\sin (180 - \theta_1 - \theta_2)}$, and can therefore be calculated.

The bearing of CA is $180 + \gamma_1$, and that of CD is $180 + \gamma_1 + \theta_2$, and the co-ordinates of D referred to axes through C are

$D_x' = \text{CD} \sin (180 + \gamma_1 + \theta_2)$ and $D_y' = \text{CD} \cos (180 + \gamma_1 + \theta_2)$,

and the independent co-ordinates are

$$D_x = C_x + D_x',$$
$$D_y = C_y + D_y'.$$

Then, knowing the co-ordinates of B and D, the bearing δ_1 of DB can be calculated; *i.e.*

$$\delta_1 = \tan^{-1} \frac{B_x - D_x}{B_y - D_y}.$$

[1] See for example *Engineering*, vol. xcvii., January 16, 1914.

The method, of course, fails when B and D coincide, as $\delta_1 = \tan^{-1}\frac{0}{0}$, which is indeterminate.

$$\text{The bearing of CD} = 180 + \gamma_1 + \theta_2,$$
$$\therefore \text{ the bearing of DC} = \gamma_1 + \theta_2,$$
$$\text{and the angle CDO} = \delta_1 - \gamma_1 - \theta_2,$$
$$\text{and CO} = \frac{\text{DC} \sin(\delta_1 - \gamma_1 - \theta_2)}{\sin \theta_1}.$$

The bearing of DO $= \delta_1$ and that of OD $= 180 + \delta_1$, the bearing of OC $= \gamma_2 = 180 + \delta_1 + \theta_1$, therefore the co-ordinates of O referred to an axis through C are

$$\text{OC} \sin \gamma_2 \text{ and OC} \cos \gamma_2,$$

and the independent co-ordinates

$$O_x = C_x + \text{OC} \sin \gamma_2,$$
$$O_y = C_y + \text{OC} \cos \gamma_2.$$

Discharge of Streams.—Occasionally it falls within the duties of a surveyor to determine the discharge of a stream.[1]

1. **Current Meter.**—For a large river or stream, probably the most accurate result is obtained by means of a current meter (Fig. 12.10).

The meter is suspended from a boat, or from some other convenient position such as a bridge, by means of a chain or cord which is heavily weighted at the lower end to prevent any appreciable deflection in a down-stream direction by the current, since such deflection renders uncertain the position of the point at which the velocity is measured. Sometimes, instead of a chain or cord, an iron rod is used, and this is probably better for swift currents, since the rod can rest upon the bed of the river, and may be more easily held in position after it has been adjusted vertically by the aid, if necessary, of a wire or cord attached to its lower extremity.

The meter rests upon a collar clamped to the rod at any required height above the bottom, in such a manner that it is perfectly free to rotate in a horizontal plane, and thus, by the action of the rudder shown in the figure, it adjusts itself to meet the current.

If the suspension is by means of a chain or cord there is a certain freedom of motion in a vertical as well as in a horizontal plane; but the former motion is prevented if the suspension is by means of a rod. The meter consists either of a propeller-shaped screw mounted upon a horizontal spindle or of a small wheel mounted upon a vertical spindle and fitted with a number of cup-shaped or conical vanes as in Fig. 12.10, which shows the instrument as made by Gurley. In the latter case the action of the current causes the wheel to rotate in the opposite direction to that in which the concavity of the cups faces,

[1] See also the following reports by M. A. Hogan, H.M. Stationery Office:
(1) "River Gauging: A Report on Methods and Appliances suitable for use in Great Britain."
(2) "Current Meters for use in River Gauging."

and by means of an electrical arrangement the number of revolutions can be recorded on a dial placed in the boat or on shore. In one form of the instrument, after every 50 revolutions an electrical contact is made which causes a bell to ring and to continue ringing for 20 revolutions; *i.e.* the bell rings during 20 revolutions and is silent for 30 revolutions. If the time between one moment of contact and some following one be noted with a stop-watch, the number of revolutions per second (n) may be deduced; or the number of seconds (t) to complete 50 revolutions (*i.e.* the time interval from one contact to the next) may be calculated from readings taken over a more or less extended interval of time.

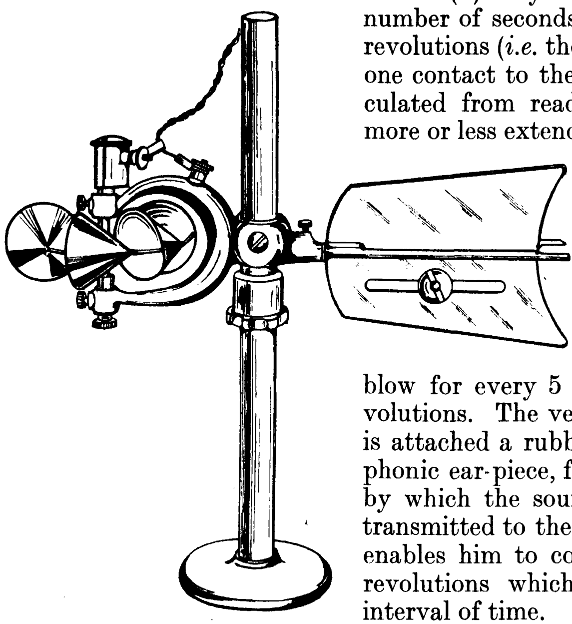


FIG. 12.10.—Current Meter

In another form of meter the revolutions of the bucket wheel are indicated by the sound of a hammer striking against a diaphragm — one blow for every 5 or for every 10 revolutions. The vertical rod, to which is attached a rubber tube and a telephonic ear-piece, forms the conductor by which the sound of the stroke is transmitted to the observer's ear, and enables him to count the number of revolutions which occur in a given interval of time.

From the equation of the meter the velocity of the stream at the point of observation may then be deduced. *To calibrate the meter* it is suspended from a projection jutting out in front of the bow of a boat, and immersed at least 2 ft. to eliminate any surface effects.

The boat is then towed or sculled over a known distance in still water at as constant a speed as possible, the number of revolutions recorded by the meter and the time taken being also noted. The boat, being in rear of the meter, does not disturb the water until the meter has passed through it, and consequently does not affect the reading.

From these data are calculated the relative velocities of the meter and the water, and also the number of revolutions per second, or the time taken to complete say 50 revolutions, and from a series of such experiments a law may be discovered connecting these quantities.

The equation, table, or curve so ascertained will be applicable to observations made in a flowing stream from a stationary boat or from a bridge, as it is the velocity relative to the meter which is measured in each case.

Example.—The following example shows some of the results obtained by towing a meter through still water in the manner described above:

TABLE

Dist.	No. of Revs. of Meter Wheel	Time in secs.	t = time for 50 revs.	n = revs. per sec.	v = vel. in ft. per sec.	log v.	log t.
162	200	57	14·25	3·51	2·84	·453	1·154
240	300	67	11·17	4·48	3·59	·555	1·048
273	350	70	10	5	3·9	·591	1
245	300	60	10	5	4·08	·611	1
281	350	62	8·86	5·75	4·53	·656	·947
240	300	46	7·67	6·53	5·22	·718	·885
240	300	41	6·83	7·32	5·85	·767	·834

Curve (1) (Fig. 12.11) shows the relationship between v and n; as this is a straight line the equation is of the form

$$v = cn + b, \qquad . \quad . \quad . \quad (1)$$

where c and b are constants.

Substituting values taken from the diagram such as $v=3$, $n=3{\cdot}77$, and $v=6$, $n=7{\cdot}54$, we get

$$3 = 3{\cdot}77c + b, \qquad . \quad . \quad . \quad (2)$$

$$6 = 7{\cdot}54c + b. \qquad . \quad . \quad . \quad (3)$$

Multiplying (2) by 2, and subtracting from (3),

$$b = 0,$$

and therefore from (2)

$$c = \frac{3}{3{\cdot}77} = {\cdot}796\,;$$

i.e. the equation is $\qquad v = {\cdot}796\, n. \qquad . \quad . \quad . \quad (4)$

Curve (2) shows the relationship between v and t, and Curve (3) that between log v and log t.

Assuming the equation to curve (2) is of the form $v = \frac{k}{t^p}$, where k and p are constants, then curve (3), which is a straight line, has the equation

$$\log v = \log k - p \log t. \qquad . \quad . \quad . \quad (5)$$

But when $\log t = o$, $\qquad \log v = \log k$,

and also $\qquad p = \dfrac{\log k - \log v}{\log t} = \tan\theta,$

where θ is the slope of the line (3) when the scales are equal.

The two logarithmic scales being equal, and θ being 45°, $p = \tan 45° = 1$, and curve (2) is a rectangular hyperbola. As the origin of the co-ordinates is not shown on the diagram, k may be found by

the substitution of say $\log v = \cdot 4$ and $\log t = 1{\cdot}2$ in equation (5), these values being obtained from curve (3) ; *i.e.*

$$\cdot 4 = \log k - 1{\cdot}2,$$
$$\log k = 1{\cdot}6,$$
$$k = 39{\cdot}81.$$

The equations of the meter over the range of velocities covered by the experiments are therefore

$$v = \cdot 796n, \qquad (6)$$

or

$$v = \frac{39{\cdot}81}{t}. \qquad (7)$$

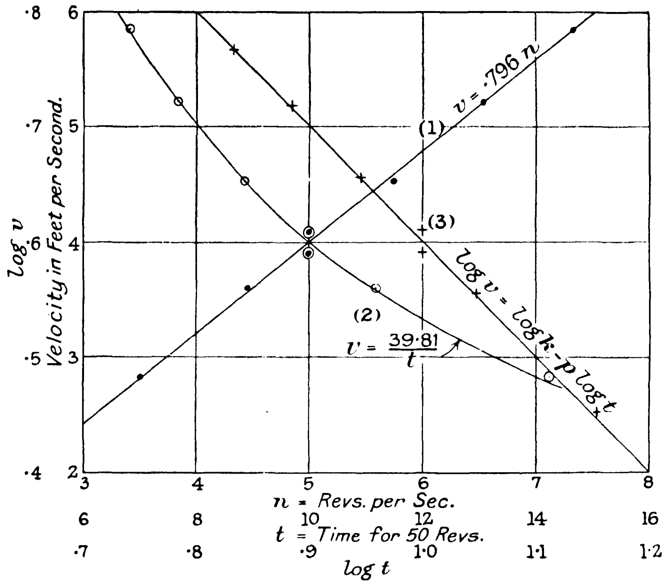


FIG. 12.11.—Determination of Current Meter Constants

After the calibration or rating of the meter has been ascertained the method of procedure is as follows.

The cross-section of the river is accurately determined by means of soundings taken from a boat or bridge as already described (*e.g.* Method 1, p. 356), and the section then considered as composed of a number of vertical strips, each say 10 ft. wide. At convenient vertical intervals of say 2 ft. on the centre lines of these strips, the velocity is measured by noting the number of revolutions of the current meter in a definite time, the meter being suspended from the up-stream side of the boat.

The product of the area of any particular vertical strip and the average or mean velocity on its centre line gives the discharge through that section, and the summation of these partial discharges over the whole width of the river gives the total discharge of the stream.

Much of this work may be done graphically, and at the same time

a considerable amount of information may be obtained regarding the distribution of velocity in the section.

To deduce the value of the mean velocity on any specified vertical section line, a curve may be plotted showing the velocities at different depths, as in Fig. 12.12, when

$$v_{\text{mean}} = \frac{\text{area enclosed by curve}}{\text{mean depth of section}}.$$

If the area is determined by means of a planimeter to be say a_1 sq. in., when the velocity scale is v ft. per sec. to 1 in. and the vertical linear scale is q ft. to 1 in., then

$$v_{\text{mean}} = \frac{a_1 \times v \times q}{d_1 \times q} \text{ or } \frac{a_1 v}{d_1},$$

where d_1 is the depth of the cross-section on the plan in inches or $d_1 q =$ the actual depth of the river at that point in feet. This means velocity, multiplied by the area of the vertical strip to which it refers, gives the discharge through that strip.

If a vertical line (cd) be drawn parallel to the datum line ab of the velocity curve, and, at a distance representing v_{mean} from it, the intersection of this line with the curve of velocities shows the depth or depths below the surface at which the actual velocity is equal to the mean.

The mean velocity on a vertical section may, as an alternative, be obtained, though not to the same degree of accuracy, by lowering the meter at a constant rate from the surface to the bottom of the river and noting the number of revolutions recorded with the corresponding time interval.

Or if it is known from a large number of similar experiments that a velocity equal to the mean velocity occurs at a certain proportion of the depth measured from the surface, a meter reading taken at this point may furnish a very close approximation to the true value. The mean position is usually at about ·6 of the depth, but the value varies considerably for different conditions, such as the nature of the bed, the ratio of depth to width, the proximity or otherwise of the banks, etc.

On the Severn, to which Figs. 12.12, 12.13 refer, v_{mean} was found to occur at about ·67 of the depth. The locus of the point for various parts of the cross-section is shown by the dotted line on Fig. 12.12, and the area enclosed by this divided by the area of the cross-section $= \frac{10{\cdot}44 \text{ sq. in. on plan}}{15{\cdot}55 \text{ sq. in. on plan}} = {\cdot}67.$

Example.—In Fig. 12.12, before reproduction, the vertical scale was 5 ft. to 1 in., and the velocity scale 5 ft. per sec. to 1 in. The area $a_1 = 1{\cdot}32$ sq. in., and the depth $ab = 1{\cdot}96$ in. (representing $5 \times 1{\cdot}96 = 9{\cdot}8$ ft.).

$$\therefore v_{\text{mean}} = 3{\cdot}36 \text{ ft. per sec.}$$

ac is equal to v_{mean} plotted to 5 ft. per sec. to 1 in., and cd represents 7 ft. or $cd = 71\frac{1}{2}\%$ of the depth ab.

Similarly d_1 is at 13·4% of the depth.

The discharge through the 10 ft. wide strip of the river at this section $= 3{\cdot}36 \times 9{\cdot}8 \times 10 = 330$ cub. ft. per sec.

When the mean velocity for each of a number of vertical sections has been determined by one of the above methods, the data may be

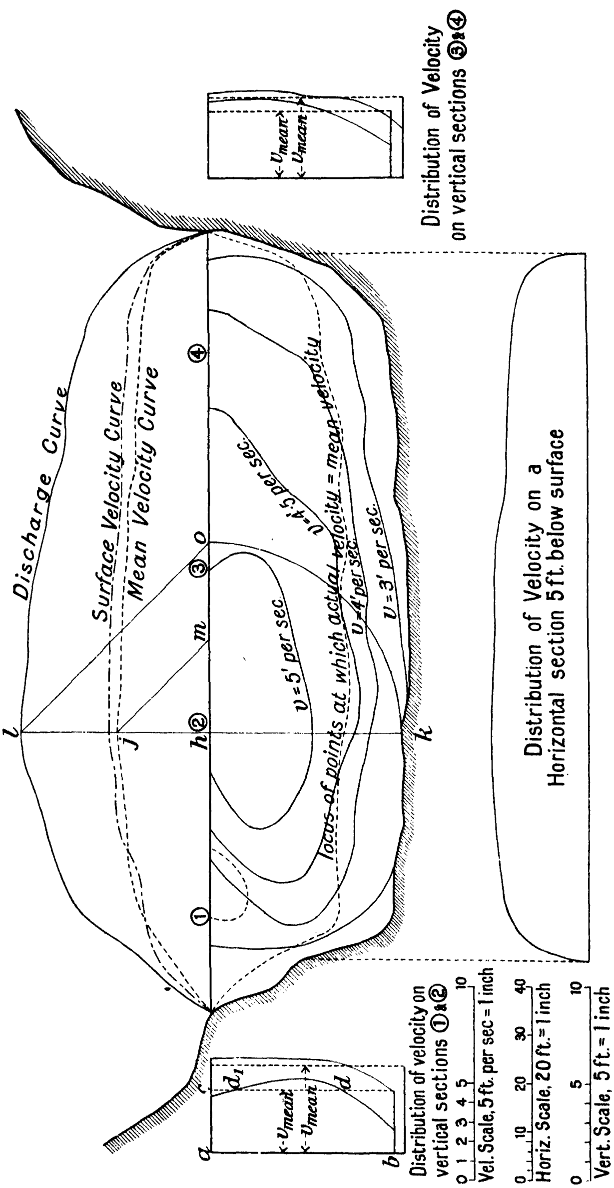


FIG. 12.12.—Current Meter Determinations

used to plot a mean velocity curve, as in Fig. 12.12, with a line representing the width of the river as a base. That is, the ordinate of the

curve at any point represents the mean velocity upon the vertical section of the river at that point.

From this curve the discharge may be obtained by the following graphical construction.

Let *ljhk* be any vertical line intersecting the surface at *h*, the mean

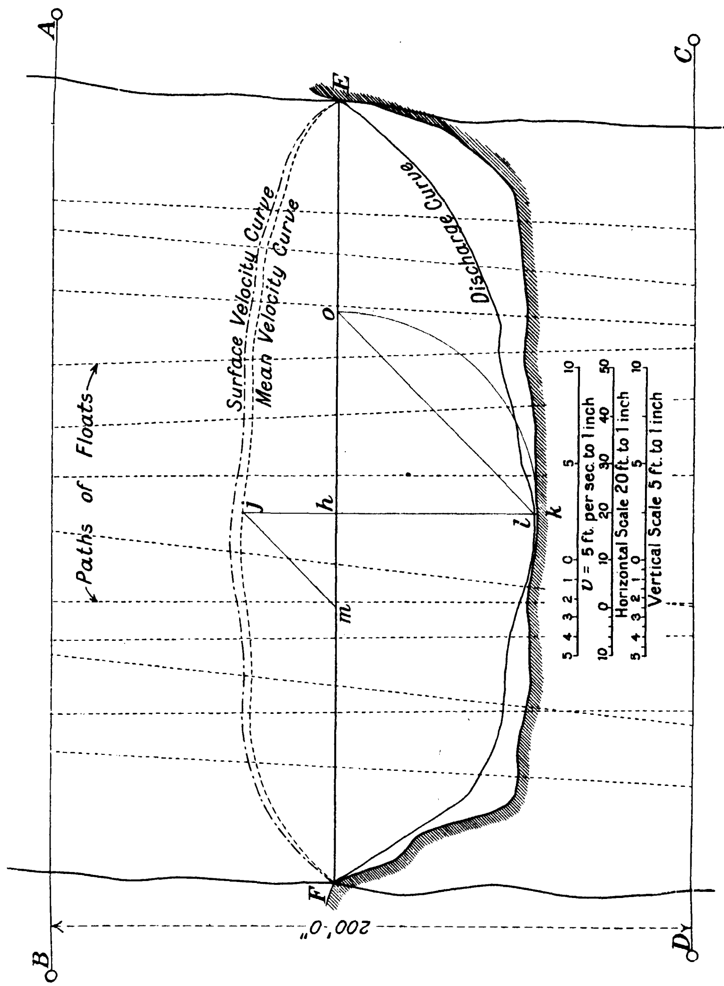


FIG. 12.13.—Surface Float Determinations

velocity curve at *j*, and the bed of the river at *k*; and let *hm*—any arbitrary and convenient length *x* (*e.g.* 2 in.)—be measured along the surface line from *h* in either direction.

With *h* as centre and *hk* as radius, *ho* is cut off equal to *hk*; *mj* is joined and *ol* drawn parallel to *mj* to cut *hk* in *l*. *l* is then a point upon the "discharge curve", which may be drawn completely by treating a number of vertical lines in this manner.

It may be proved that the area of this curve, to a certain scale, represents the total discharge Q of the river.

Because the triangles *hmj*, *hol* are similar,

$$\frac{hl}{hj}=\frac{ho}{hm},$$

i.e.

$$hl=\frac{hj \times ho}{hm}. \qquad . \quad . \quad . \quad (8)$$

But if the scales of the drawing are

horizontal scale $= p$ ft. to 1 in.,
vertical scale $= q$ ft. to 1 in.,
velocity scale $= v$ ft. per sec. to 1 in.,

and the distance $hm = x$ in.,

then

$$hj=\frac{v_{\text{mean}}}{v},$$

where v_{mean} is the mean velocity on the vertical *hl*,

and

$$ho=\frac{\text{depth}}{q}.$$

Therefore from (8)

$$\underset{\text{in inches}}{hl}=\frac{v_{\text{mean}} \times \text{depth}}{vqx}, \qquad . \quad . \quad (9)$$

i.e. *hl* to a scale of *vqx* cub. ft. per sec. to 1 in. represents the rate at which the discharge takes place per unit width of river at *hk*.

Thus if the width of a river is L, the discharge through a small strip, at any point, of width δL is $v_{\text{mean}} \times \text{depth} \times \delta L$.

But if δy is the corresponding strip upon the section, $\delta L = \delta y \times p$.

$$\therefore \delta Q = v_{\text{mean}} \times \text{depth} \times \delta y \times p,$$

or from (9)

$$\delta Q = (hl \times \delta y) \times pvqx.$$

By summing over the whole width of stream we find, therefore, that the total discharge Q is equal to the area of the discharge curve multiplied by *pvqx*, as $hl \times \delta y$ is the area of a strip of the discharge curve of width δy; or in other words, the area of the discharge curve represents the total discharge of the river to a scale of *pvqx* cub. ft. per sec. to 1 sq. in.

Example.—In Fig. 12.12 before reproduction the scales were as follows :

Horizontal scale, 20 ft. to 1 in.
Vertical scale, 5 ft. to 1 in.
Velocity scale, 5 ft. per sec. to 1 in.
$hm = 1$ in.

The area of the discharge curve as found with a planimeter was 12·71 sq. in.

Therefore the discharge of the river was $12{\cdot}71 \times 20 \times 5 \times 5 \times 1 = 6355$ cub. ft. per sec.

If desired, the distribution of velocity upon the cross-section may be shown by means of lines of equal velocity, corresponding to contour

lines for altitudes; a few such lines are shown on Fig. 12.12. The following are two of the methods which may be used for their determination:

(1) The values of all the "spot" velocities may be written upon the cross-section, and the lines interpolated as described in Method 2, p. 190; or

(2) The position on each vertical at which any particular velocity occurs may be found from the curves which show the distribution of velocity in a vertical plane. This method corresponds with Method 1, p. 189, for contours.

From the areas enclosed by the various velocity lines, by the application of the prismoidal formula (Chapter XI), the discharge of the river might be computed, but this method is seldom if ever employed.

Example.—The areas in square inches enclosed by the various velocity "contour" curves in Fig. 12.12 are:

5 ft. per sec. curve	2·61 sq. in.	representing	261 sq. ft.
$4\frac{1}{2}$,,	5·98	,,	598 ,,
4 ,,	9·44	,,	944 ,,
$3\frac{1}{2}$,,	11·83	,,	1183 ,,
3 ,,	13·40	,,	1340 ,,
$2\frac{1}{2}$,,	14·42	,,	1442 ,,
2 ,,	15·06	,,	1506 ,,
Whole section	15·55	,,	1555 ,,

Applying the prismoidal formula to the portion from 2 ft. to 5 ft. per sec., the partial discharge is

$$\frac{0{\cdot}5}{3}\{(261+1506)+4(598+1183+1442)+2(944+1340)\}$$

$$=3204 \text{ cub. ft. per sec.}$$

For the portion from 0 to 2 ft. per sec., as the velocity is at least 1 ft. per sec. over almost all the section, the partial discharge is

$$\tfrac{1}{3}\{1555+4(1555)+1506\}=3094 \text{ cub. ft. per sec.}$$

The total discharge is thus $3204+3094=6298$ cub. ft. per sec.

The quantity determined from the discharge curve was 6355 cub. ft. per sec., so that the discrepancy is only ·9%.

Curves may also be drawn to show the distribution of velocity upon horizontal planes; *e.g.* in Fig. 12.12 is shown the velocity curve upon a horizontal section at 5 ft. below the surface.

If the "spot" velocities are observed at the same intervals *below* the surface, over the width of the river, the same method as for the vertical curves may be employed, *i.e.* the values may be abstracted from the field notes and plotted directly. If, however, the observations are made at definite distances measured *upwards* from the bed of the river it may be more convenient to project the curves from the curves of equal velocity upon the cross-section, exactly in the same manner as in levelling, where a section is projected from contour lines upon a plan; or preferably the necessary data may be abstracted from the plotted vertical curves.

The data for Figs. 12.11, 12.12, and 12.13 were obtained by the fourth-year students of the University of Birmingham in connection with the river gauging experiments at Arley on the Severn in 1914.

2. **Pitot Tubes**[1] may be employed to ascertain the velocity at various points of a stream, and the same methods of calculating the discharge, or of determining curves showing the distributions of velocities in a cross-section, may be adopted as when using a current meter.

3. **Floats.**—A very important method often adopted for the determination of the discharge of a stream is that which involves the employment of floats.

Floats may be grouped under three headings:

(*a*) Surface floats;
(*b*) Sub-surface or double floats;
and (*c*) Rod floats.

(*a*) **Surface floats** are small pieces of wood, cork, or other material, which project slightly above the water. They should be sufficiently distinct to enable an observer on the banks clearly and easily to locate the course they follow, but otherwise they should be as small as possible in order to minimize the effect which the wind may have upon their progress.

An up-stream wind considerably retards the velocity of surface floats, while a down-stream wind increases it, so that, unless the weather is favourable, very misleading results may be obtained.

The mean velocity on any vertical section may be obtained from the formula,[2] due to Bazin,

$$u = V - \frac{K}{3}\sqrt{Hi}, \qquad (10)$$

where u = mean velocity in ft. per sec.;
V = surface velocity in ft. per sec.;
K = 36·2 approximately;
H and i = depth and inclination of the stream respectively;

or, more approximately, the mean velocity upon any vertical section is about

$$\cdot 9V \qquad (11)$$

In the example shown in Fig. 12.12 the area enclosed by the surface velocity curve was 7·26 in. on the plan, while that enclosed by the mean velocity curve was 6·42 in.

The mean velocity curve was constructed by abstracting the values from velocity curves similar to those at (1), (2), (3), (4), taken at 10 ft. intervals across the river.

The ratio in this case was therefore $\frac{6\cdot 42}{7\cdot 26} = \cdot 885V$.

(*b*) **Sub-surface floats** yield a result much more independent of the wind, as the main float is submerged a short distance below the surface by means of properly adjusted weights, and connected by a fine wire or cord to a small surface float, which enables the motion to be observed from the shore or other convenient position.

The velocity is then very little different from that of the stream

[1] *Hydraulics*, F. C. Lea. [2] *Ibid.*

at the depth of the lower float, since the resistance offered by the small float and the cord is made as low as possible.

In this manner the velocity at any depth may be ascertained, and the mean velocity calculated from an equation such as that of Cunningham :

$$u = \frac{1}{4}(V + 3V_{\frac{3}{8}}),$$

where $V_{\frac{3}{8}}$ is the velocity at $\frac{3}{8}$ths of the depth.

Or, if the velocity is determined at half the depth, the mean velocity may be taken as

$$u = \cdot 98V_{\frac{1}{2}},$$

while for ·6 of the depth the mean velocity is approximately equal to the observed velocity.

If the depth of immersion is small, the result is practically the same as the surface velocity, whilst there is less danger of error from wind action.

Sometimes the two floats are of equal size, when a mean between the surface velocity and that at the depth of the lower float is obtained.

(*c*) **Rod floats** give better results than either of the above types, provided that the strain is of fairly regular cross-section and the bed is not choked with weeds.

The rods are of such a length that, when weighted at the lower end to give the required depth of immersion, the upper end projects above the surface of the water and may be observed. If the lower end is carried down as nearly as possible to the bed of the river, the velocity recorded is approximately the mean velocity on the vertical.

To ascertain the velocity of a current by means of floats, it is necessary to determine the exact length of the path followed by the float during an observed interval of time.

For tidal currents, the method explained on p. 357 may be resorted to, and in this way a series of points may be located by simultaneous theodolite or sextant observations made from two instrument stations on the shore, or in other suitable positions ; for instance, if more convenient, sextant observations may be taken from two boats, themselves located by the " Three-point " method.

The path of the float may then be plotted upon a chart, the length between specific points of observation, *e.g.* X and Y, determined, and the mean velocity calculated ; *i.e.*

$$v = \frac{XY}{t},$$

where t is the time in seconds which was observed to elapse between the passings of X and Y by the float.

To determine the discharge of a stream by means of float observations, a straight uniform length of 200 ft. or more is selected, and range posts A and B, C and D, set out on the same or on opposite banks (Fig. 12.13). The floats are released from a boat 50 feet or more upstream from AB, and the time taken to travel from AB to CD noted with a stop-watch.

If the rate of flow is small, the instants at which AB and CD are

passed may be noted by one observer, who may walk from AB to CD in the interval.

If the rate of flow is large, this may be impracticable, in which case the observer with the stop-watch may be stationed at AB and note the time of transit of AB himself, while the instant the float reaches CD is noted by a second observer at CD, and signalled by him to the first observer.

The path of the float on a straight length of river is generally assumed to be a straight line between AB and CD, so that the direction and length of the path may be determined, if the positions at which the float crosses these lines are observed.

An instrument station is chosen at some suitable point G, and its position relative to ABCD ascertained by any ordinary surveying method.

If a theodolite is employed, the vernier of the horizontal scale is clamped at zero, and the telescope directed to some arbitrary referring object such as B, and the lower axis clamped.

The upper clamp is then loosened, and the motion of the float as it approaches AB is followed with the telescope until a signal given by an observer at AB indicates the moment of transit. The scale plate is then clamped and the angle θ read, and from these data the position of the float may be plotted on the plan either by means of a protractor or by co-ordinates, as explained on p. 358.

The position at which the float crosses CD may be located in a similar manner by an observation from a second instrument station G_1, or, if the velocity is sufficiently slow, the second station may be dispensed with and the reading on CD noted by the same instrument man at G, after the reading for AB has been booked.

It is not always necessary, however, to use a theodolite, as sextant observations are capable of yielding very good results. In a suitable climate the positions ABC, etc. (Fig. 12.13), may be plotted directly on the plan by means of plane-table observations, and this method is very expeditious. The plan is made in the usual manner (Chapter IX), and the lines AB and CD shown upon it. The instrument is set up at a convenient position G, the corresponding point plotted on the paper, and the table oriented. The float as it approaches AB is followed by the telescope as in the case of a theodolite, the alidade rule being rotated about a pin which marks the position of G upon the plan, and which acts as a pivot.

At the moment of transit, signalled by an observer at AB, a ray is drawn through G to intersect AB on the plan at the required point, and without loss of time the telescope is redirected to observe the crossing of CD.

Each point on the plan is numbered or lettered as it is located, in order that the results may be combined afterwards with the time observations taken simultaneously by other operators.

To proceed to calculate the discharge of the river a cross-section at an intermediate position EF is determined by means of soundings, and plotted as in Fig. 12.13.

By joining corresponding points on AB and CD the positions at which the float crossed EF are found, and the mean velocity at these points may be calculated, as already explained, from the observed surface velocity.

By plotting these values, a mean velocity curve may be drawn, and from the corresponding discharge curve the discharge may be computed as shown in Fig. 12.13.

Example.—The area of the discharge curve on the plan (Fig. 12.13) was 12·88 sq. in., so that the discharge $= 12 \cdot 88 \times 20 \times 5 \times 5 \times 1 = 6440$ cub. ft. per sec.

where the horizontal scale = 20 ft. to 1 in.
,, vertical ,, = 5 ft. to 1 in.
,, velocity ,, = 5 ft. per sec. to 1 in.
,, *hm* = 1 in.

The result may be compared with that obtained with the current meter (Fig. 12.12), *i.e.* 6355 cub. ft. per sec.—the difference being only 1·34%.

The fact that a down-stream wind was blowing at the time would tend to make the surface-float results slightly too large—but the agreement is as close as could reasonably be expected.

4. An ingenious method for determining the surface velocity has been suggested by Mr. Thrupp.[1] He found that if two projections are placed in a stream, ripples are formed, and for any constant distance apart (d) of the two obstructions, the surface velocity may be deduced from the length (l) to the junction of the two ripples, *i.e.*

$$v = c + al,$$

where c and a are constants depending upon the value of d, v is the velocity in feet per second, and l is in inches.

Thus if $d = 4$ in. the formula becomes

$$v = \cdot 40 + \cdot 280l, \quad . \quad . \quad . \quad (12)$$

while if $d = 6$ in. the formula becomes

$$v = \cdot 40 + 206l. \quad . \quad . \quad . \quad (13)$$

A very simple apparatus for this purpose may be made with two nails or spikes projecting downwards from a board, which carries a scale on its edge.

Example.—When $d = 6$ in. and $l = 2 \cdot 9$ in., $v = 1$ ft. per sec.,
and when $d = 6$ in. and $l = 22 \cdot 4$ in., $v = 5$ ft. per sec.

5. For smaller streams very accurate results may be obtained by constructing a rectangular weir, and passing all the water over this: a method which is obviously impracticable for large rivers.

The discharge over a weir of length L feet, provided with a sharp edge, and having a free overfall, so that the vein discharges freely into the air, and does not cling to the down-stream side of the weir, may be calculated by means of Francis's formula[2]

$$Q = 3 \cdot 33\ (L - 0 \cdot 1NH)H^{\frac{3}{2}}, \quad . \quad . \quad (14)$$

[1] *Proc. Inst. C.E.* vol. clxvii. p. 217.

[2] For the derivation of formulae 14-16 the student is referred to a standard work on Hydraulics—for instance, *Hydraulics*, by F. C. Lea.

where Q is the discharge in cubic feet per second,

L = the length of the weir in feet,

N = number of side contractions.

e.g. N = 0 if the weir is the full width of the channel, or

= 2 if the weir is less than the full width of the channel, and has two sharp vertical edges.

H = Head of water in feet, measured at a short distance up-stream, above the sill of the weir, which should be as high as possible above the bed of the stream.

If the stream itself, as it approaches the weir, has an appreciable velocity v, the formula must be modified, *i.e.*

$$Q = 3{\cdot}33\ (L - 0{\cdot}1NH)\{(H + h)^{\frac{3}{2}} - h^{\frac{3}{2}}\}, \quad . \quad (15)$$

where $h = \dfrac{v^2}{2g}$.

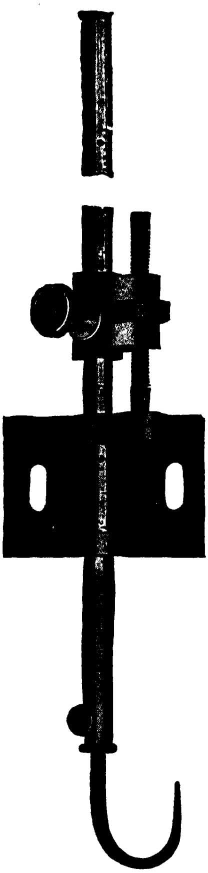


Fig. 12.14

The head H is measured at a short distance up-stream from the weir, on account of the curve adopted by the surface of the vein as it discharges over the edge.

The most simple method is to fix a graduated scale upon a post, driven into the bed of the stream near one of the banks, and notice the reading on this which coincides with the surface level of the water. If the reading which corresponds exactly with the level of the bottom edge of the weir is known, the " head " at any time can be found by subtraction.

A more accurate result is obtained by the use of a **hook gauge,** one form of which, by Messrs. Gurley, is shown in Fig. 12.14. This device consists of a hollow rod graduated in feet, tenths and hundredths, and carrying at its lower extremity a vertical sharp-pointed hook.

On the metal base frame, which is bolted through the elongated holes to a timber or other support, is fixed a vernier, which enables the scale to be read to thousandths of a foot.

To take a reading with the instrument, the clamping screw shown in the figure is loosened, and the rod raised until the point of the hook approaches the surface of the water from below; the clamp is then tightened, and the upward motion continued by means of the fine adjustment screw, until the point of the hook just raises a small projection on the surface film of the water. The reading of the scale is then noted, and by subtracting the reading which corresponds to the level of the horizontal edge of the weir, the " head " is obtained. On the instrument in Fig. 12.14, the hook, which is adjustable within the tube, may be moved through a range of 12 in. without altering the

reading on the scale. The point of the hook may thus be adjusted to the level of the sill of the weir while the scale reading is zero, after which the " head " may be read directly from the scale.

It is impossible to get a very accurate result unless the surface of the water is perfectly smooth and undisturbed by waves, so that generally it is advisable to enclose the gauge in a small chamber into which the water of the stream has free access, through a number of holes or pipes well below the surface. Very accurate and consistent readings may then be obtained.

An autographic diagram, showing the " head " at any instant, may be obtained from a float enclosed in such a chamber, exactly as in a tide gauge, though in this case it may not be necessary to reduce the vertical movements of the float in order to record them on a drum.

If it is not necessary to so reduce them the float may be counterbalanced by a weight to which it is attached by means of a light cord or chain passing over a pulley, and the pencil carriage is then attached directly to the cord.

Example 1.—If the head of water flowing over a sharp-edged rectangular notch 3 ft. long was found to be 10·26 in., what was the discharge in gallons per minute ?

Assuming that the stream is more than 3 ft. wide so that there are two end contractions, formula (14) may be applied, *i.e.*

$$Q = 3{\cdot}33(3 - {\cdot}2 \times H)H^{\frac{3}{2}} \text{ cub. ft. per sec.}$$

$$= 3{\cdot}33(3 - {\cdot}2 \times {\cdot}855){\cdot}855^{\frac{3}{2}} \text{ cub. ft. per sec. (as 10·26 in.} = {\cdot}855 \text{ ft.)}$$

$$= 3{\cdot}33 \times 2{\cdot}829 \times {\cdot}855^{\frac{3}{2}} \text{ cub. ft. per sec.}$$

$$= 3{\cdot}33 \times 2{\cdot}829 \times {\cdot}855^{\frac{3}{2}} \times 60 \times 6{\cdot}25 \text{ galls. per min.}$$

$$\begin{aligned}
\log 3{\cdot}33 &= {\cdot}52244 \\
,,\ 2{\cdot}829 &= {\cdot}45162 \\
,,\ {\cdot}855^{\frac{3}{2}} &= \bar{1}{\cdot}89795 \quad i.e.\ \tfrac{3}{2} \times \log {\cdot}855 = \tfrac{3}{2} \times \bar{1}{\cdot}93197 \\
,,\ 60 &= 1{\cdot}77815 \\
,,\ 6{\cdot}25 &= {\cdot}79588 \\
\hline
3{\cdot}44604 &= \log.\ \underline{2792{\cdot}8},\ i.e.\ Q = 2793 \text{ galls. per min. nearly,}
\end{aligned}$$

or by means of a slide rule Q may be found as 2790 galls. per min. say.

Example 2.—Solve the above question when the velocity of approach is very high, say 4 ft. per second.

From equation (15)

$$Q = 3{\cdot}33 \times (3 - {\cdot}2H)\left\{\left(H + \frac{v^2}{2g}\right)^{\frac{3}{2}} - \left(\frac{v^2}{2g}\right)^{\frac{3}{2}}\right\} \text{cub. ft. per sec.}$$

$$= 3{\cdot}33 \times 2{\cdot}829\{({\cdot}855 + {\cdot}25)^{\frac{3}{2}} - ({\cdot}25)^{\frac{3}{2}}\} \text{ cub. ft. per sec.}$$
$$= 3{\cdot}33 \times 2{\cdot}829 \times (1{\cdot}1614 - {\cdot}125) \text{ cub. ft. per sec.}$$
$$= 3{\cdot}33 \times 2{\cdot}829 \times 1{\cdot}0364 \times 60 \times 6{\cdot}25 \text{ galls. per min.}$$
$$= 3660 \text{ galls. per min. nearly.}$$

6. For still smaller streams a **V notch** may be employed, and the head H measured in the same manner as for a rectangular notch or weir.

If the notch has sharp edges and the apex be situated well above the bed of the stream, then if the vein has a free discharge the quantity for a 90° notch may be computed from the formula

$$Q = 2{\cdot}535H^{\frac{5}{2}}. \quad . \quad . \quad . \quad (16)$$

where Q = discharge in cubic feet per second,

H = the head of water in feet, flowing over the notch, measured upwards from the apex.

If the velocity of approach is considerable $(H+h)$ must be substituted for H in the above formula, where $h = a \, . \, \frac{v^2}{2g}$, a being a coefficient often taken as unity, though probably of a slightly higher value : and v being the velocity of the stream itself before it reaches the notch.

Example.—Calculate the discharge over a right-angled V notch, when the head is 10·26 in.

Here $Q = 2{\cdot}535\left(\frac{10{\cdot}26}{12}\right)^{\frac{5}{2}} = 2{\cdot}535 \times {\cdot}855^{\frac{5}{2}}$ cub. ft. per sec. or $2{\cdot}535 \times {\cdot}855^{\frac{5}{2}} \times 60 \times 6{\cdot}25$ galls. per min.

$= 642{\cdot}6$ galls. per min.

Had the velocity of approach been 2 ft. per sec.,

$$Q = 2{\cdot}535 \times ({\cdot}855 + {\cdot}0625)^{\frac{5}{2}}, \text{ taking } h = \frac{v^2}{2g} = \frac{4}{64} = {\cdot}0625$$

$$= 2{\cdot}535 \times ({\cdot}9175)^{\frac{5}{2}} \times 60 \times 6{\cdot}25 \text{ galls. per min.}$$

$$= 766{\cdot}5 \text{ galls. per min.}$$

7. There are several other methods which may be employed to determine the discharge of a stream, but they are hardly suitable for a Surveyor. One method suggested by Mr. Stromeyer consists in adding to the stream a definite quantity of some chemical such as pure sulphuric acid or a strong salt solution and then at a lower point on the stream ascertaining the amount of dilution from a number of samples.

Mr. Stromeyer claims that the discharge may be determined within 1 per cent of its true value by this method. For further details the reader is referred to his paper in the *Proc. Inst. C.E.* vol. clx.

EXAMPLES

1. (U. of L.) Explain carefully the method of gauging the flow of a stream by means of a current meter.

How are the constants for such meters determined ?

The greatest surface velocity of the water in a channel of rectangular section, 6 ft. wide and 3 ft. deep, is found to be 3·5 ft. per second ; using any approximate formula, deduce the mean velocity of the stream, and hence determine the quantity of water flowing down the channel in cubic feet per diem.

2. In rating a current meter, the following results were recorded :

Distance towed in metres	57·0	58·0	59·0	60·25
Time in seconds	1 m. 40 s.	2 m. 30 s.	3 m. 20 s.	4 m. 37·5 s.
Number of intervals between contacts	5	5	5	5

Deduce the equation for the meter, within the range of velocities tried.

3. A river is 80 ft. wide, and soundings are taken at the centre of each 10 ft. of the width. The depths are 3·2, 3·65, 4·4, 4·9, 5·6, 5·35, 4·05, 2·5 ft., and the mean velocities on these sections are 1·4, 2·05, 2·58, 2·84, 3·12, 2·90, 2·00, 1·2 ft. per sec. Calculate, without plotting, the approximate discharge of the river. Also, for the sake of comparison, plot the section and the mean velocity curve and deduce the result from a discharge curve.

4. Calculate the discharge in gallons per hour over a sharp-edged rectangular weir having a length of 10 ft., when the depth of water over the sill is 1·08 ft.
Assume (*a*) that the velocity of approach is negligible.
Assume (*b*) that the velocity of approach is 2 ft. per sec.

5. Calculate the discharge through a 90° V notch, when the depth of water above the vertex is 1·08 ft.

6. In order to locate the position O of a boat, observations were made with a sextant to three points A, B, and C on shore.
The angles AOB and BOC were found to be 50°-56′ and 27°-23′ respectively. From the map AB was scaled as 1330 feet and BC as 660 feet, while the angle ABC measured 163°-18′. What were the distances of O from A, B, and C respectively ?

7. (U. of L.) A Traverse Survey A, B, C is made along the shore line of an estuary just above high-water mark, and the following notes obtained :

AB, 389 ft. long, 122°-15′ azimuth.
BC, 511 ft. long, 100°-18′ azimuth.

A series of offshore soundings is then taken, each line being set out parallel to the meridian. The first line is projected seaward from the point A, and the remainder at 180-ft. intervals along ABC. The particulars of the soundings are given in the accompanying table,[1] as also the tide-gauge readings, which are taken simultaneously with the soundings. The maximum tidal range at Spring Tides is 15 ft. and the soundings are taken during that period. The zero of the tide gauge is 2 ft. below low water of ordinary Spring Tides.
Reduce the soundings given to datum of low-water ordinary Spring Tides and construct contour lines of the sea bottom at 2-ft. vertical intervals. Assuming that the beach between high and low water slopes approximately at 1 vertical to 12 horizontal, and with the aid of the soundings given, trace also the HW and LW marks on the plan. Show, by hatching, all sandbanks appearing at low water. All distances are measured outwards from lines ABC. Scale 40 ft. = 1 in.
A quay wall is to be built between the first and last lines of soundings, running parallel to ABC and 100 ft. seawards of it. It is required to have a minimum depth of water of 10 ft. below low water of ordinary Spring Tides from the front of the wall outwards. Estimate the quantity of dredging in cubic yards necessary to obtain this depth.

8. (U. of B.) An observer taking soundings from a boat O wished to locate his position, and measured with a sextant the angles subtended at O by three points A, B, and C on the shore. The lengths of AB and BC, scaled from a map, were 658′ and 720′ respectively, and an angle ABC was 105°-30′. The observed angles AOB and BOC were 40°-15′ and 34°-15′ respectively.
What can the observer deduce from these figures ?

[1] See pp. 382-383.

[TABLE

Gauge Reading (Feet)	Time of Gauge Reading	Time of Sounding	Depth of Sounding (Feet)	Distance (Feet)
1st Line				
	A.M.	A.M.		
12¾	8·0	—	0	51
		8·5	4½	100
			8½	150
			13	200
			20	250
13¼	8·15	8·15	30	300
			35½	350
			25¼	400
		8·25	27	450
			23	500
14	8·30		18¾	550
			10	600
		8·35	20	700
2nd Line				
14¾	8·45	8·45	0	40
			5	100
			9	150
			13½	200
15½	9·00	9·00	21¼	250
			35	300
			39¾	350
			32	400
		9·10	31	450
			25	500
			17	550
16	9·15		4	600
		9·20	17	700
3rd Line				
16¾	9·45			
		9·50	0	30
			6¼	100
			10	150
			14	200
			25¼	250
17	10·00	10·00	35	300
			45½	350
			32½	400
			22¼	450
		10·10	18	500
16¾	10·15		8	550
			17	600
		10·20	21	700

Gauge Reading (Feet)	Time of: Gauge Reading	Time of: Sounding	Depth of Sounding (Feet)	Distance (Feet)
4th Line				
	A.M.	A.M.		
16	10·30	10·40	0	60
			3½	100
			7½	150
			11¾	200
14	10·45			
		10·50	19½	250
			28¾	300
			35	350
			38	400
			33½	450
12¾	11·0	11·0	21¼	500
			9¼	550
			8¼	600
		11·10	18½	700
5th Line				
12	11·15			
		11·20	0	50
			4¼	100
			8¼	150
			13½	200
11½	11·30		22	250
		11·40	26	300
			33½	350
10	11·45		38	400
			42	450
		11·50	37	500
			26½	550
			13	600
9	12·00	12·00	23½	700
6th Line				
		12·10	0	40
8¾	12·15		5¼	100
			9¼	150
			13¼	200
		12·20	25	250
			24½	300
6	12·30		18	350
			13	400
		12·40	11	450
			3	500
			5	550
5¼	12·45		11	600
		12·50	16	700

CHAPTER XIII

TRIANGULATION

In Chapter I it was explained that the principle of a simple chain survey is triangulation, the country to be surveyed being covered with a skeleton framework of triangles, the three sides of each of which are determined by direct measurement.

Similarly, the principle of an ordinary plane-table survey is triangulation, each point being located by means of a triangle in which the base is known (by direct measurement or otherwise), and of which the two angles at the base are measured and plotted graphically in the field.

However, neither of these surveys is classified under the heading of Triangulation or Trigonometrical Surveys.

Triangulation Surveys are those in which the sides of the various triangles are ascertained by calculation from (*a*) a single base line measured directly, and from (*b*) the three angles of each triangle measured with a theodolite or similar instrument.

For less important work only two angles of each triangle are measured directly, and the third angle is deduced by subtracting their sum from 180°, but wherever possible it is advisable that all three angles should be observed.

The direct object of a triangulation or Trigonometrical Survey is not to provide a complete plan showing detail and topographical features, but simply to locate a number of isolated points over the surface of the country.

The degree of accuracy with which the relative positions of these points are fixed depends upon several considerations, amongst which may be mentioned :

(1) The extent of the survey and the scale to which the published plans are to be reproduced.

It is desirable that the maximum probable displacement of any point due to possible errors in the triangulation shall be so small that its displacement on the finished plan shall be inappreciable.

The greater the extent of the survey, the more opportunity is there for small errors to accumulate and become significant, so that the larger the survey, the more accurately should the work be executed.

Similarly the larger the scale, the more appreciable will be an error of given magnitude in the measurements.

Thus, a displacement on the paper of $\frac{1}{100}$th of an inch, which is about as small a quantity as can be easily plotted—represents an error

of 5 in. in the field if the scale is $\frac{1}{500}$, or an error of 25 in. when the scale is $\frac{1}{2500}$.

(2) The time and funds available.

(3) The purpose of the map.

Filling in.—When the trigonometrical points have been located, the topographical and other details are surveyed by means of one or other of the methods described in the previous chapters.

These subsidiary surveys—*e.g.* chain, traverse, plane-table, stadia, or other surveys—are conducted between the various points, which form checks as to their accuracy, prevent errors from accumulating, and provide data for their adjustment. They may be carried on simultaneously by different parties at various points upon the survey.

Procedure.—The first step is to choose a suitable position for the accurate measurement of a base line by one of the methods described in Chapter XIV.

The method to be adopted depends upon the degree of accuracy which it is desired to attain, and upon the extent of the survey which is to be developed from it.

The site should be approximately level, evenly sloping, or gently undulating, and as free as possible from obstructions, in order that a line of the required length may be accurately measured without undue expense.

The extremities should, if possible, be intervisible, and in positions sufficiently commanding to allow of their being used as instrument stations for the extension of the base, as on p. 427 ; or to allow the base line to be used as a side of one of the main triangles. The site must also be so chosen that it will be possible to obtain well-conditioned triangles from the ends of the adopted base to positions suitable for other station points, *i.e.* for the apices of the various adjacent triangles of the system.

The stations having been selected, the theodolite is set up at each in turn, and wherever practicable, the three angles of each triangle are observed, the methods of Repetition or Reiteration (pp. 118 and 119) being resorted to for the more precise surveys.

Selection of Stations.—In selecting positions suitable for trigonometrical stations, the following factors, among others, should be considered :

(1) The triangles formed with the adjacent stations should be well conditioned (see p. 396), *i.e.* they should be as nearly equilateral as possible ; in any case no angle of a triangle should be less than 30° or more than 120°.

In the preliminary location a box-sextant may be used with advantage, to ensure that such conditions are not violated.

(2) Each station should be clearly visible from all the adjacent stations. Consequently, commanding positions such as hill-tops are generally very suitable. Care must be taken, however, that the stations are visible from points that can be made use of by the topography surveyors, as well as from the other primary stations. This is particularly important for plane-table work, where positions have to be located by resection (see Chapter IX).

(3) The length of sight should be neither too large (when the signal may be too indistinct for accurate bisection) nor too small (when errors of centering and bisection become appreciable, the triangles too small to be economical, and the magnitude of the cumulative errors liable to be excessive).

The length of sight to be adopted in any particular case depends to some extent upon the magnitude of the survey, and also upon the instruments available. It is also largely dependent upon the nature of the country.

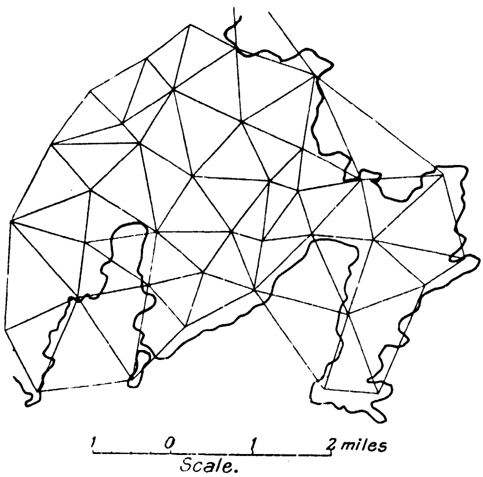


FIG. 13.1.—Portion of Triangulation System, Survey of Malta

In a flat country, for instance, a very long sight cannot be obtained owing to the effect of curvature, unless elevated signals or beacons are employed.

Sometimes on an extensive survey, in addition to the use of elevated signals, the instrument is raised on a scaffold to a considerable height above the ground. In this case an independent scaffold is usually erected for the observer.

In hilly country it is generally possible to obtain much longer sights—from one summit to another —than could be got on the plains without entailing prohibitive expense in the erection of towers and beacons.

On the Indian triangulation, for instance, the average length of side in the hilly country was 30 miles, while on the plains the average was about 11 miles only.

In the Secondary triangulation chains in the Himalayas sights as long as 200 miles were obtained.

(4) The expense in the erection of elevated scaffolds, beacons, etc.

(5) The amount of cutting and clearing necessary to remove obstructions from the line of sight.

(6) The difficulty or otherwise of access to the station.

(7) The permanence or otherwise of the station.

Form of Triangulation.—

(*a*) The triangulation for a relatively small topographical or photographic survey (see Chapter XVIII) consists of a network of triangles covering the whole area.

Such a network was employed, for instance, on the Survey of the Island of Malta [1] (Fig. 13.1), the base line being near the centre of the system.

[1] See *A Treatise on Surveying*, by Middleton and Chadwick.

On the Survey of Great Britain the method was somewhat similar

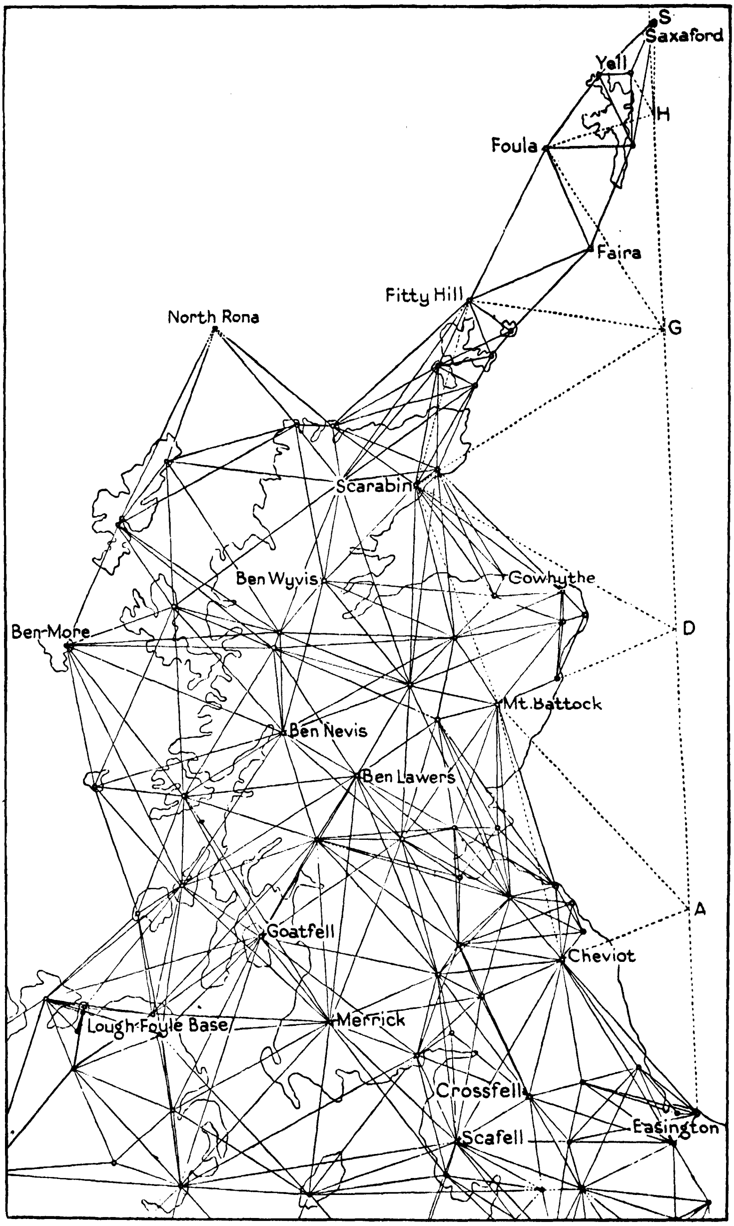
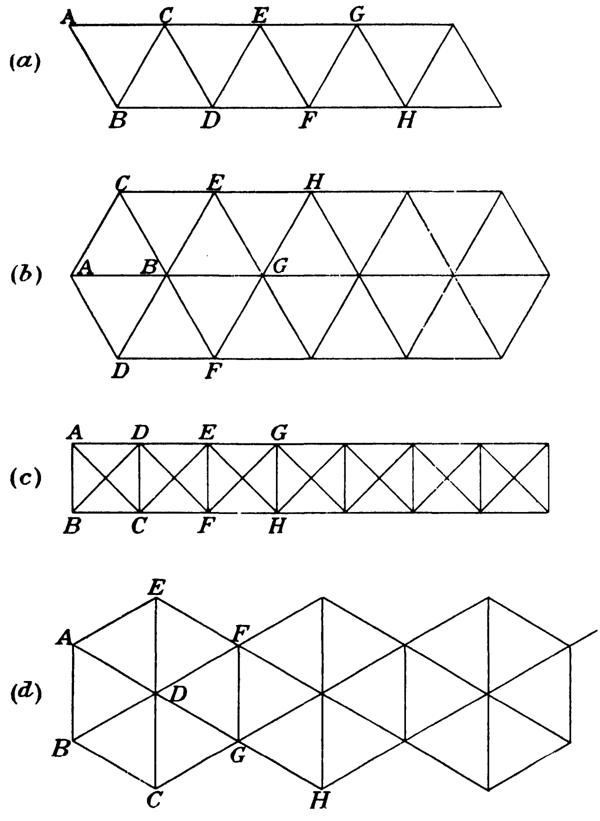


FIG. 13.2

except that a great number of interlacing rays were taken in addition, as shown in Fig. 13.2.

(*b*) On the Indian, French, and other surveys the whole country was not covered with a network in this way, but chains of triangles were run in directions approximately along and at right angles to the meridian.

On the Indian Survey, for instance, one main chain ran almost due N. and S. from Cape Comorin to the Dehra Dun Base in the extreme north, while another parallel chain ran N. from Mangalore.



(*a*) Simple chain of triangles, suitable for secondary or less important systems.

(*b*) Double triangles.

(*c*) Quadrilaterals or interlacing triangles.

(*d*) Polygons (theoretically hexagonal).

FIG. 13.3.—Theoretical Triangulation Chain Systems

Similarly E. and W. were several chains, *e.g.* Mangalore to Madras, Karachi to Calcutta, etc., while other chains ran down the East Coast from Calcutta to Cape Comorin, and down the West Coast of Further India.

The chains are composed of triangles arranged in one of the above ways (Fig. 13.3).

Fig. 13.1 is taken from the triangulation of Malta.

Fig. 13.2 is taken from the triangulation of Scotland.

Fig. 13.4 is taken from the triangulation of the Transvaal and Orange River Colony.

Fig. 13.5 is taken from the triangulation of India, to show the practical application of the theoretical forms in Fig. 13.3.

Nomenclature. — On the Ordnance Survey the first framework of triangles set out over the country constituted the "**Principal**" or "**Primary**" triangulation. These large triangles in turn formed the basis of a system of smaller triangles which constituted the "**Secondary**" triangulation. A further subdivision resulted in the "**Tertiary**" triangulation (see p. 392).

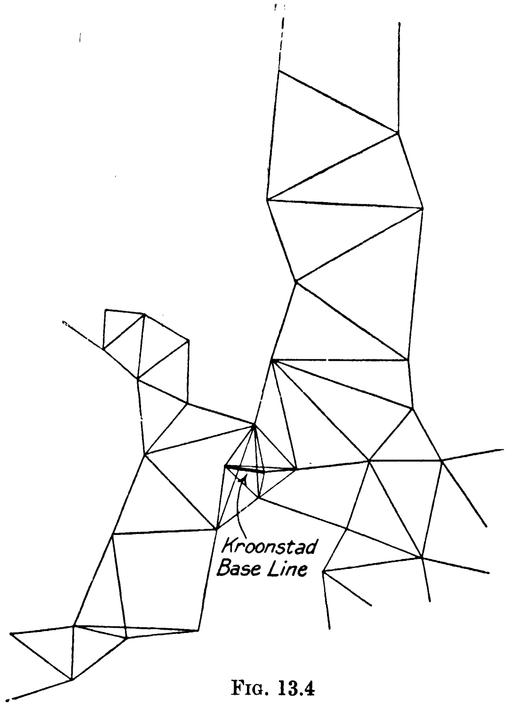


FIG. 13.4

Portion of Main Triangulation System of the Transvaal and Orange River Colony Survey

On the Survey of India also, the same terms are applied, the governing chains being known as the "Principal" or "Primary" chains, and the subordinate as "Secondary" and "Tertiary" chains.

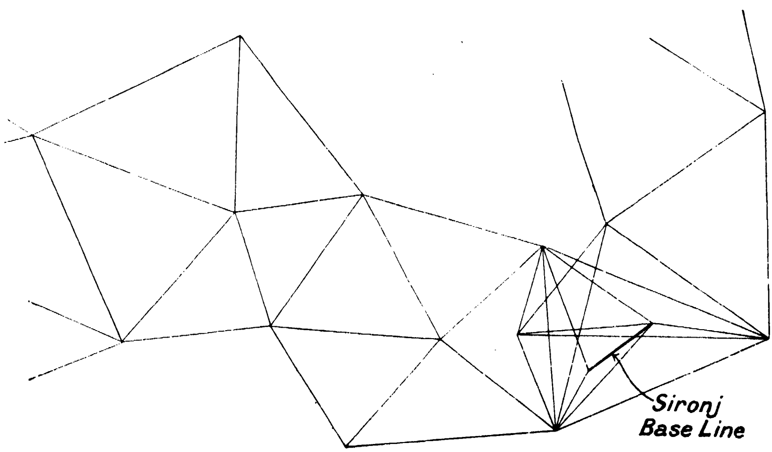


FIG. 13.5.—Portion of Main Triangulation System of the Survey of India

In many places, however, the terms " **Major** " and " **Minor** " have

been adopted, *e.g.* on the Orange River Colony Survey: or Major, Minor, and Tertiary on the East African Protectorate Survey.

Generally "Major" triangulation corresponds to Primary or Principal triangulation, but is usually not quite so elaborate or precise. Minor triangulation then corresponds to Secondary or Tertiary triangulation.

A **Topographical** triangulation is usually of limited extent, and is not necessarily based upon a Primary or Major system. It would generally be quite independent and be of the third or fourth order of precision (see p. 408).

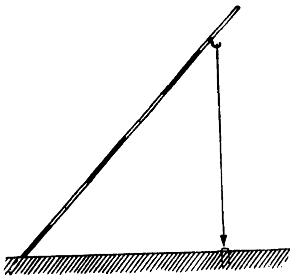


FIG. 13.6

A **Geodetic** triangulation is the term applied to a very precise system of triangles intended for the measurement of an arc of a meridian, or for the determination of other data concerning the shape of the earth as a whole.

Signals, etc.—For short sights, *e.g.* on a small topographical survey, the station is preferably marked by means of a ranging rod, the lowest visible point of which is bisected by the cross-wires of the theodolite. To avoid errors [1] which may be introduced when the rod is not exactly vertical, a plumb line is frequently used.

This may be suspended from a tripod, or from a single ranging rod placed in an inclined position over the peg as in Fig. 13.6. The height of the bob is regulated by means of a sliding knot near the top of the string, or preferably by twisting the string a few turns round the hook from which it is suspended. The point of the bob is brought exactly over the station point marked on the peg or pillar, and any point on the string may then be sighted.

For large surveys more elaborate signals must be used, as, to be clearly visible, an angle of 2 to 4 seconds should be subtended by the signal at the instrument.

A rope, weighted at its end, is sometimes suspended from a tripod formed by three stout spars lashed together, and the rope is painted in white or other colours, depending upon the nature of the background.

When the rope itself is not sufficiently distinct, a symmetrical signal is occasionally fixed on the rope. This might be a light framework of wire covered with white cloth.

On the Totley[2] tunnel alignment a 1-inch diameter blackened tube was suspended on a fine steel wire, heavily weighted.

On the India and other Surveys, for short sights, a staff, held in place by guys or stays if necessary, and carrying a symmetrical bundle of grass, brushwood, etc., was used.

Signals are also formed in the shape of Tripods or Quadripods (Fig. 13.7), built up of rough timbers, firmly lashed together, and

[1] See Examples, p. 106. *Proc. Inst. C.E.* vol. cxvi.

covered with thatch, battens, or calico. Sometimes there is a central mast held in the vertical position by means of stays.

Luminous signals are generally used now upon Geodetic or Primary triangulation surveys.

Arrangements—heliographs, heliostats, or heliotropes—for reflecting the sun's rays in the required direction have been much used, the object to be bisected, when viewed through the telescope, having the appearance of a bright star.

On the Ordnance Survey, Bengal lights were used, and these were succeeded by Argand lamps furnished with parabolic reflectors.

On the India Survey, Drummond's lights superseded the previous appliances, and proved very successful. A small ball of lime placed in the focus of a parabolic reflector is raised to an extremely high temperature by impinging upon it a stream of oxygen passing through a flame of alcohol.

Electric lights have also been employed, *e.g.* in connecting the triangulation of Spain with that of Algiers.

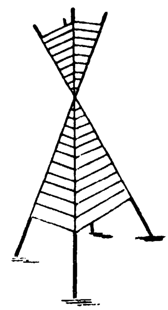


FIG. 13.7
Quadripod Beacon

The permanent mark which defines a station point is often fixed below the sighting signal, after this has been erected, as it is much easier to do this than to centre the beacon accurately over a mark previously fixed. The mark is usually sufficiently low to enable the instrument to be conveniently set up above it.

When a central mast is provided to a tripod or quadripod beacon, it is sometimes pivoted so that it may be pulled aside to afford a clear space for the instrument.

When it is necessary that the instrument should be raised to a considerable height above the ground, a scaffold or staging must be erected. The instrument is then centered over the station point by means of a long plumb line, protected from the action of the wind by being enclosed in a narrow vertical trough or tube, having one side open. This tube can be turned so that the open side is to leeward of the wind.

In such a case, a separate staging must be erected for the observer, and this must be entirely independent of the instrument scaffold.

On the Canadian Survey a portable erection built up of angle irons was employed.

The mark defining the station point should be of a permanent nature.

Sometimes a masonry pillar is erected and a mark made on the top surface by inserting a metal plug: if sufficiently high, the instrument may be set up on the pillar itself, and the tripod dispensed with.

Often two marks are adopted, one upon a slab of stone at some distance below the surface of the ground, and one at ground level, the latter often being protected by means of a cairn of stones. When stone is not available, concrete is frequently employed.

In the **Ordnance Survey** [1] of Great Britain and Ireland, commenced by General Roy, the first base to be measured was that on Hounslow Heath in 1784, the preliminary measurement being made with a 100-foot steel chain. Later, seasoned deal trussed rods 20 ft. 3 in. long were used, but, owing to changes in length due to absorption of moisture from the air, this method was considered unsatisfactory, and glass rods were next resorted to. In 1791 the base was again measured with Ramsden's steel chain (p. 411) and the apparent length corrected for temperature and reduced to mean sea level. Ramsden's steel chain was also used for other bases at Misterton Carr (1801), Rhuddlan Marsh in Flintshire (1806), Belhelvie Sands in Aberdeenshire (1817), and King's Sedgmore in Somerset. The main bases, however, were that on Salisbury Plain (1849) and that on the shore of Lough Foyle in Ireland (1827), and these were measured with Colby's compensation bars (p. 411), though the Salisbury Plain Base had previously been measured with the chain in 1794.

Upon these two fundamental bases, the one 6·93 miles and the other 7·89 miles in length, was set out a network of **primary triangles**. The difference between the measured lengths of the bases and their lengths as computed through the triangulation system was 0·4178 ft., and this error was distributed −0·202 ft. to the Salisbury Plain Base and +0·216 ft. to the Lough Foyle Base, to give the equivalent of a Mean Base. The remaining bases were regarded as bases of verification, including the Lossiemouth Base (p. 406), measured later (1909) with invar tapes (p. 430).

The angles of the primary triangles were observed with the great theodolite of Ramsden, 3 ft. in diameter, the telescope of which had a focal length of 3 ft. and a magnifying power of 54. There were two such instruments, in addition to one of 2 ft. diameter, and one of 1 ft. 6 in. diameter. Each was furnished with two verniers.

The average length of a side of a primary triangle was 35·4 miles, but the largest was 111 miles from Slieve Donard to Scaw Fell.

The main or primary triangles were subdivided into **secondary triangles** having an average length of side of 5 miles; many of the secondary station-points were fixed with the large instruments, while the primary triangles were being set out; others were located independently with a 12-in. theodolite. Similarly the secondary triangles were again split up into **tertiary triangles** having an average length of side of about 1¼ miles, 7-in. and 5-in. theodolites being employed. Smaller surveys were then conducted between the various tertiary triangulation points, the work being chiefly executed with the chain and prismatic compass. On other surveys the plane table is largely employed, and much detail surveyed by stadia observations, but these methods were not adopted on the Ordnance Survey of Great Britain.

On the United States Coast and Geodetic Survey the great theodolite was 2 ft. 6 in. in diameter and furnished with three equidistant micrometer microscopes, reading to 1 second of arc.

[1] *Ordnance Trigonometrical Survey of Great Britain and Ireland. Account of the Observations and Calculations of the Principal Triangulation*, by A. R. Clarke, 1858.

On the Great Trigonometrical Survey of India the theodolites, 3 ft. and 2 ft. in diameter, were furnished with five equidistant microscopes each reading to 1 second of arc.

Observation of Horizontal Angles.—For small surveys the method of " Repetition " (p. 118) is often adopted. For larger surveys the method of " Reiteration " is employed.

The errors to which theodolite observations are liable, and the methods of eliminating them, have been mentioned in Chapter IV.

To measure the angles at a station A (Fig. 13.8) with extreme accuracy the following method might be adopted :

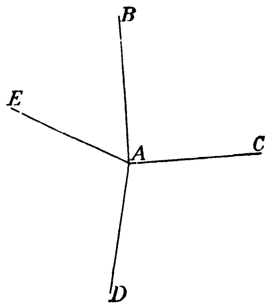


FIG. 13.8

(1) The instrument is set up, levelled and accurately centered at A, and the telescope directed to some referring object, say B. The instrument would be " face right " (F.R.), the leading micrometer adjusted to zero, and the readings of the remaining micrometers noted. The telescope would then be rotated towards the right (swing right) until C was bisected, the cross-hairs being brought into coincidence from the left (approach left).

All the micrometers having been read, the telescope would then be rotated farther to the right and D bisected from the left. Then E and finally B would be intersected.

In each case all the micrometers or verniers would be read, and coincidence would be obtained by always bringing up the cross-hairs in the same direction, *i.e.* from the left.

If the readings of the micrometers on again intersecting B do not agree with the original readings, the difference, if small, may be equally divided among the angles BAC, CAD, DAE, and EAB. If considerable, the observed values would be rejected and the observations repeated.

(2) The operations would then be repeated with the vertical circle still " face right," commencing from B as zero, but swinging the instrument to the left, *i.e.* observing the points in the order E, D, C, B. The intersections would be obtained in this case by bringing the cross-hairs into coincidence from the right instead of from the left.

(3) The telescope would be transitted without removal from its supports, and the face of the instrument converted into " face left ".

A round of observations would then be made swinging to the right and approaching from the left.

(4) A round of observations would then be made swinging to the left and approaching from the right, *i.e.*

(1)	Instrument	F.R.	swing	R.	approach	L.
(2)	,,	F.R.	,,	L.	,,	R.
(3)	,,	F.L.	,,	R.	,,	L.
(4)	,,	F.L.	,,	L.	,,	R.

Thus if the instrument is fitted with two verniers or microscopes, observations (1), (2), (3), (4) would furnish in all eight values for each angle.

The observations would now be repeated, commencing from a different " zero ", *i.e.* the leading vernier might be adjusted to 90° instead of to 360°, and a further eight values would be obtained.

With three verniers or microscopes each " zero " furnishes twelve values, and in the second of two sets of readings the leading vernier would be adjusted to 60°.

The total number of observations required will depend upon the accuracy to be obtained (see p. 384), and for geodetic triangulation four to six or more " zeros " may be used.

Thus if there are n verniers, the angle between them will be $\frac{360}{n}$, and if there are to be m different sets of observations, the approximate angle between the various " zeros " will be $\frac{360°}{nm}$; *e.g.* if there are three verniers, and four complete sets are required, the zeros will be $\frac{360}{12}=30°$ apart, *i.e.* 0°, 30°, 60°, 90°.

It is advisable—to avoid bias—not to set the verniers exactly to an even degree.

The effect of swinging the telescope right and left, and of bringing the cross-hairs into coincidence alternately from the left and right, tends to eliminate errors due to

(*a*) Twist of the instrument supports, due to the effect of the sun and wind ;

(*b*) Slip due to defective clamping and tangent screw arrangements.

" Face right and face left " observations tend to eliminate the errors due to

(*a*) The lack of adjustment in the line of collimation ;

(*b*) Inequality of the levels of the A supports.

Change of zero tends to eliminate errors due to defective graduation.

The reading of several verniers tends to eliminate errors due to the inaccurate centering of the instrument axes, and unequal graduations.

The repetitions also tend to eliminate personal errors in the bisection of the object, the reading of the verniers or microscopes, etc.

The procedure mentioned above is too tedious and unnecessarily laborious for smaller triangulations, so that for such surveys the work may be much simplified. For instance, for a topographical survey, or a tertiary triangulation, probably it would be quite sufficient to take two zeros only, with one " face right and swing right ", and one " face left and swing left " observation from each. This procedure would yield eight readings when the instrument is fitted with two verniers.

Or, for less precise results, one " face right and swing right " observation from one zero, and one " face left and swing left " observation from the second zero, would be sufficient. This would furnish four readings when two verniers are fitted.

For ordinary work the angles may be observed at almost any time

of the day or night, provided that the weather is favourable, *i.e.* not excessively windy or misty.

For geodetic work, or a primary triangulation, it is usual to confine the operations to those times when the effect of lateral refraction is a minimum. Day operations are best undertaken between 4 P.M. and dusk. Night operations are probably more reliable, and are now very generally adopted owing to the convenience and suitability of the modern luminous signals for long distances.

Vertical angles should be observed " face right " and " face left ", the cross-hairs being brought into coincidence with the object alternately from above and from below. Both microscopes or verniers should be read. The most suitable time for the observation of these angles is in the afternoon—say 2 to 4 P.M.—and reciprocal observations should be employed as described in Chapter VII.

Reduction to Centre.—As already considered in Chapter V. on traverse surveying, it is sometimes impossible to set up the instrument exactly over or under the signal which has been observed from other station-points. In such a case the instrument is set up near the signal at a " satellite station," and the observed angles reduced to the centre. In triangulation surveys there also occurs a modification of this problem, *e.g.* when a leaning beacon or eccentric signal is observed, the angles at each point are reduced to the values they would have, were the signal correctly placed.

The problem of locating a position O, by observations from O, to three known positions A, B, and C, has been referred to previously (see p. 359), and the brochure of the Royal Engineers on *The Resection Problem* may be consulted for further particulars and examples.

Example 1.—Taken from the Survey of the Transvaal and Orange River Colony.

In Fig. 13.9 W represents the Walschbank Peak station and beacon.
B represents Bendearg.
C represents Xuka.
O represents the instrument station which is eccentrically placed.

OW $= d = 37\cdot328$ ft.

Bearing OW $= 0°\text{-}0'$.

Bearing OB $= 121°\text{-}56'$. log WB $= 5\cdot2393$.

Bearing OC $= 144°\text{-}54'$. log WC $= 5\cdot2316$.

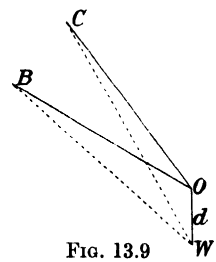


FIG. 13.9
Reduction to Centre

The angle WBO is obtained from the sine formula (1), p. 543, *i.e.*

$$\log \sin \text{WBO} = \log d + \log \sin 121°\text{-}56' - \log \text{WB}$$

and

$$\log \sin \text{WCO} = \log d + \log \sin 144°\text{-}54' - \log \text{WC}.$$

$\log d = 1\cdot5720$	$\log d = 1\cdot5720$
$\log \sin 121°\text{-}56' = \bar{1}\cdot9287$	$\log \sin 144°\text{-}54' = \bar{1}\cdot7597$
$1\cdot5007$	$1\cdot3317$
$\log \text{WB} = 5\cdot2393$	$\log \text{WC} = 5\cdot2316$
$\log \sin \text{WBO} = \bar{4}\cdot2614$	$\log \sin \text{WCO} = \bar{4}\cdot1001$

Therefore the correction $\angle WBO = 37{\cdot}66''$, and the correction $\angle WCO = 25{\cdot}97''$.

These corrections are both $+^{ve}$ and must be added to the bearings of B and C from O, to obtain the bearings of those points from the true station W.

Example 2.—From the same survey (Fig. 13.10).

V = position of instrument at Vlakfontein, where the beacon O is ·861 ft. out of centre. D = Tafel Kop signal.

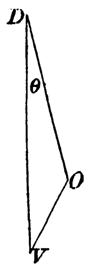


FIG. 13.10

Bearing of beacon top, *i.e.* VO = 0°-0′.
Bearing of beacon at D = 330°-30′.
Log of distance DV = 5·2901.

Then if θ is the angle of correction, *i.e.* the angle VDO,

$$\log \sin \theta = \log 0{\cdot}861 + \log \sin 29°\text{-}30' - 5{\cdot}2901,$$

$$\begin{aligned} \log 0{\cdot}861 &= \bar{1}{\cdot}9350 \\ \log \sin 29°\text{-}30' &= \bar{1}{\cdot}6923 \\ \hline & \bar{1}{\cdot}6273 \\ \log DV &= 5{\cdot}2901 \\ \hline \log \sin \theta &= \bar{6}{\cdot}3372, \\ \therefore\ \theta &= 0{\cdot}44''. \end{aligned}$$

i.e. the correction to the bearing of O to give that of V from D is $+0{\cdot}44''$.

Shape of Triangles.—The shape of the triangles should be such that any error in the measurement of the angles shall have a minimum effect upon the lengths of the calculated sides.

Thus if AB is the side originally known—by direct measurement or by computation from other triangles—and if ABC is the triangle built up upon it, then any fractional error in the assumed length of AB will cause a similar fractional error in the other two sides AC and BC, and also in all other linear measurements of the triangulation which depends upon it.

Similarly any additional fractional error in AC or BC, caused by an error in the assumed value of one of the angles A, B, or C, will be transmitted through the whole of the triangulation. It is thus desirable that the angles A, B, and C shall be such that any error in their measurement shall have a minimum effect upon the length of AC or BC.

But because the triangulation is dependent upon both AC and BC, it is not desirable that the accuracy of one shall be sacrificed in order to render the accuracy of the other more trustworthy, *i.e.* AC and BC should be equally accurate, and this condition is best attained by making the triangle isosceles.

To find the best values for the angles A and B and C we may proceed as follows :

By the sine formula $$a = c \,.\, \frac{\sin A}{\sin C},$$

therefore if δa_1 is the error produced in a by an error of δA in A,

$$\delta a_1 = \frac{c \,.\, \cos A \,.\, \delta A}{\sin C}$$

and $$\frac{\delta a_1}{a} = \frac{c}{a} \,.\, \frac{\cos A}{\sin C} \,.\, \delta A = \cot A \,.\, \delta A$$

by the sine rule.

Similarly if δa_2 is the error due to an error δC in C,

$$\delta a_2 = -\frac{c \sin A \cos C\ \delta C}{\sin^2 C}$$

and

$$\frac{\delta a_2}{a} = -\cot C . \delta C.$$

If δA and δC represent the probable errors in the angles, *i.e.* $\pm\theta$ radians, then the probable fractional error in the sides a or b

$$= \pm\theta\sqrt{\cot^2 A + \cot^2 C}. \qquad (1)$$

This expression is a minimum when $\cot^2 A + \cot^2 C$ is a minimum, or since $C = 180° - 2A$, when $\cot^2 A + \cot^2 2A$ is a minimum.

Differentiating this expression and equating to zero, we have after reduction

$$4 \cos^4 A + 2 \cos^2 A - 1 = 0,$$

from which $A = 56°\text{-}14'$ approximately.

From these conclusions, therefore, the best shape of triangle would appear to be isosceles, having the angles at the base $= 56°\text{-}14'$, or the apical angle equal to $67°\text{-}32'$.

It would, however, be impossible to make a network composed entirely of such triangles, besides which, in a chain of triangles of these proportions, the triangles would gradually diminish in size as the distance from the base was increased. This would be undesirable for many reasons, *e.g.* it would be uneconomical, and the centering and bisection errors would tend to become more and more appreciable as the sides diminished in length.

For these reasons an equilateral triangle is considered the most suitable shape, but of course in practice this is modified very considerably, and is governed almost entirely by the topographical features of the country. If possible, however, triangles having an angle smaller than 30° or greater than 120° are avoided, and triangles as nearly equilateral as is practicable are employed.

Fig. 13.11 shows graphically the probable fractional error in the sides when the angles are liable to a probable error of $\pm 1''$. It will be seen that when the angles of the triangle fall outside the limits above stated, the resulting linear error rapidly increases—particularly when the angle is smaller than 30°.

It is also evident from the curve that an equilateral triangle is almost as favourable a shape as the theoretical best shape as derived above.

The curve has a similar shape whatever the probable error in the angles.

When the probable fractional error in the side c, *i.e.* $\frac{\delta c}{c}$, is considered, the resultant probable fractional error in the sides a and b is

$$\pm\sqrt{\left(\frac{\delta c}{c}\right)^2 + \theta^2 (\cot^2 A + \cot^2 2A)}. \qquad (2)$$

The effect of the term $\left(\frac{\delta c}{c}\right)$ is to raise the vertex and flatten the curve.

Curve (2) shows $\left(\frac{\delta c}{c}\right) = \pm 2 \times 10^{-6}$ and $\theta = \pm 1''$.

Curve (3) shows $\left(\frac{\delta c}{c}\right) = \pm 10 \times 10^{-6}$ and $\theta = \pm 1''$.

Calculations.—The sine formula is applied in the calculation of the unknown sides (a and b) of the triangle ABC, when the side c is

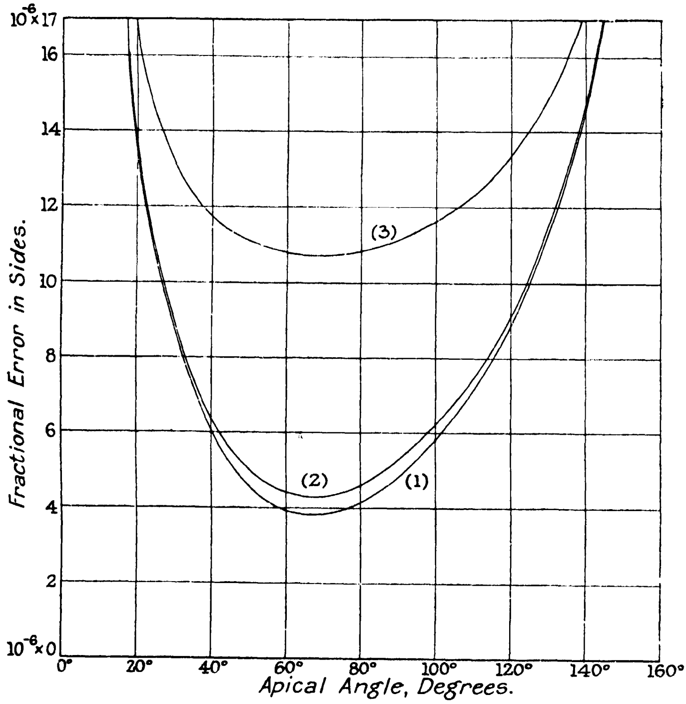


Fig. 13.11

known by direct measurement or by previous computations, and when the angles A, B, and C are determined, *i.e.*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Before applying this rule, however, the observations are adjusted so that any discrepancy in the results is distributed as evenly as possible.

In a simple chain of triangles, as in Fig. 13.3, a, the three angles of each triangle should sum exactly to 180°, unless the sides are of such a length that the figure cannot be considered as approximately plane,

in which case the sum of the three angles should exceed 180° by an amount known as the spherical excess.

This is due to the fact that each angle is measured in a plane tangential to the spheroid. If the triangle is small, these three planes may be considered as coincident: if not, then each angle will be slightly greater than the corresponding angle of the plane triangle, which is obtained by joining A, B, and C by straight lines or chords of the spheroid.

The spherical excess of a triangle may be calculated—on the assumption that the earth is a sphere—by the application of the formula derived in Chapter XV, *i.e.*

$$e^\circ = \frac{\text{area of triangle}}{\pi R^2} \times 180^\circ, \qquad (3)$$

where R is the radius of the sphere.

From this, the spherical excess in seconds,

$$e'' = \frac{A_s \times 360 \times 60 \times 60}{2\pi R^2}, \qquad (4)$$

where A_s is the area of the spherical triangle in the same units that R represents the radius of the earth.

For accurate work the mean radius of curvature through the centre of the triangle should be substituted for R.

Thus if R = 20,889,000 ft. say, solving (4) by logarithms, we get

$$\log e'' = \log A_s + \overline{10}\cdot6745902$$

or

$$\log e'' = \log A_s + \overline{2}\cdot1198580,$$

where A_s is the area of the spherical triangle in square feet or square miles respectively.

For a triangle having an area of 1 square mile, $\log A_s = 0$, and $\log e = \overline{2}\cdot1198580$, from which $e = 0\cdot01318$ second, a negligible amount for all ordinary work.

Usually it is sufficiently accurate to substitute for A_s the area of the triangle considered plane instead of spherical, *i.e.* the area A_s is calculated by the usual plane trigonometrical formulae.

Thus if one side c and the two angles A and B are known, as is usually the case,

$$A_s = \tfrac{1}{2}c^2 \frac{\sin A \sin B}{\sin (A+B)} \text{ or } \tfrac{1}{2}c^2 \frac{\sin A \sin B}{\sin C}; \qquad (5)$$

if two sides a and b and the included angle C are known,

$$A_s = \tfrac{1}{2}a \cdot b \cdot \sin C. \qquad (6)$$

For these computations the three angles are first adjusted to sum to 180° by dividing the excess (or deficit), including the unknown spherical excess, equally between them.

When the spherical excess has been computed, the **error of the observations** is easily deduced, and is divided equally between the three angles—provided each angle is assumed to have been observed with an equal degree of accuracy, *i.e.* the correction to be applied to each angle is

$$\frac{(180^\circ + \text{spherical excess}) \sim \text{sum of observed mean values of A, B, and C}}{3}$$

$$= \frac{D''}{3} \text{ say.}$$

If the angles are not equally reliable, they may be weighted, for instance, in proportion to the number of observations of each, or to the clearness or otherwise of sighting, etc.

Thus if the three angles are accorded weights ω_A, ω_B, and ω_C respectively, the discrepancy D may be distributed in the ratio of the reciprocals or of the squares of the reciprocals of the weights, *e.g.* the correction to A, by the first method, would be

$$\frac{\frac{1}{\omega_A}}{\frac{1}{\omega_A} + \frac{1}{\omega_B} + \frac{1}{\omega_C}} . D, \qquad . \qquad . \qquad (7)$$

or by the second method

$$\frac{\left(\frac{1}{\omega_A}\right)^2}{\left(\frac{1}{\omega_A}\right)^2 + \left(\frac{1}{\omega_B}\right)^2 + \left(\frac{1}{\omega_C}\right)^2} . D. \qquad . \qquad (8)$$

By correcting the angles A, B, and C in this way the values to be adopted of the spherical angles A, B, and C are decided upon.

These values are employed in the calculation of the sides a and b—unless Legendre's theorem is used as explained below—and also in the determination of the true azimuth of the various lines (see Chapter XV on the Convergence of the Meridians).

The next step is to calculate the lengths of the sides of the triangle.

For large triangles this may be done by pure spherical trigonometry, when one side (c) and the three spherical angles are known, but the process is laborious, and is not so frequently resorted to in practice. It was, however, used, with the two methods described below, upon the French Survey.

The two chief methods are those of Delambre and Legendre : these were employed on the Ordnance Survey.

Delambre's Method.—The length of the chord c_c of the spheroid, which corresponds with the side c_s of the spherical triangle, is calculated, and the three spherical angles A_s, B_s, and C_s are reduced to the angles A_c, B_c, C_c of the plane triangle formed by the chords joining A, B, and C.

The unknown sides a_c and b_c are then calculated by plane trigonometry, and afterwards converted into the corresponding spherical arcs a_s and b_s.

Legendre's theorem states that if a triangle is small in comparison with the surface area of the sphere, it is sufficiently correct to subtract

$\frac{1}{3}$rd of the spherical excess from each angle of the spherical triangle and apply the usual sine formula.

For tertiary and topographical triangulation the spherical excess is too small to be appreciable, and the triangles are solved by plane trigonometry.

Example.

		Corrections
Observed angles of equal weight—	A = 88°-34′-10·5″	·8″
	B = 42°-16′-18·0″	·8″
	C = 49°- 9′-33·9″	·8″
	180°- 0′- 2·4″	2·4″

		log sin
Corrected angles for preliminary calculation and for Legendre's theorem—	A = 88°-34′- 9·7″	$\bar{1}$·9998646
	B = 42°-16′-17·2″	$\bar{1}$·8277852
	C = 49°- 9′-33·1″	$\bar{1}$·8788258
Known length	c = 82211·02	log c = 4·9149300

Log area by formula (5)—

log ·5 =	$\bar{1}$·6989700
log c^2 =	9·8298600
log sin A =	$\bar{1}$·9998646
log sin B =	$\bar{1}$·8277852
	9·3564798
log sin C =	$\bar{1}$·8788258
log area =	9·4776540
constant =	$\overline{10}$·6745902
log e =	·1522442

Spherical excess e = 1·42 second nearly

The total error in observations = 2·4 − 1·42 = ·98″.

The spherical angles therefore, subtracting $\frac{1}{3}$rd from each,

A_s = 88°-34′-10·18″
B_s = 42°-16′-17·67″
C_s = 49°- 9′-33·57″
180°- 0′-01·42″

To calculate a and b by Legendre's theorem, when

$$b = \frac{c \cdot \sin B}{\sin C} \text{ and } a = \frac{c \cdot \sin A}{\sin C}.$$

log c =	4·9149300
log sin A =	$\bar{1}$·9998646
	4·9147946 ∴ a = 108634·75 ft.
log sin C =	$\bar{1}$·8788258
log a =	5·0359688

log c =	4·9149300
log sin B =	$\bar{1}$·8277852
	4·7427152 ∴ b = 73095·3 ft.
log sin C =	$\bar{1}$·8788258
log b =	4·8638894

For systems other than a simple chain of triangles, resort must be made to a more elaborate method of distributing the various errors of observation.

The full treatment of this branch of the subject is beyond the scope of the present volume, and the student is referred to one of the standard works on "Geodesy." A few notes on the subject are, however, given below, and reference may be made to Chapter XIX.

Considering the system of polygons shown in Fig. 13.3, *d*, the errors must be so distributed that the following conditions shall be satisfied:

(1) The sum of the three angles of each triangle must be equal to 180° + the ascertained spherical excess if the triangles are sufficiently large for this to be appreciable.

Thus if the polygon has *n* sides, this incurs *n* conditions.

(2) The sum of all the angles at the central station D must be equal to 360°, as all these are measured in the one plane which is tangential to the spheroid at D.

(3) From (1) and (2) it follows that the sum of the interior angles of the polygon must be equal to $(2n-4)$ right angles, plus the sum of the spherical excesses of all the individual triangles. This is not an independent condition.

(4) The apparent length of FG as calculated from AB, working through the sides BD, CD, GD, must be equal to the apparent length as calculated through the sides AD, ED, and FD. Similarly each of the other sides BC, CG, FE, EA, BD, CD, GD, etc., must yield coincident results by whichever route they are calculated.

Thus if A_R and A_L, B_R and B , etc., are the right- and left-hand angles at the various points of the polygon, and if the side AB is known by direct measurement, or from the proceeding polygon after adjustment, we have

$$AD=\frac{AB\ .\ \sin B_L}{\sin ADB}\quad\text{and}\quad BD=\frac{AB\sin A_R}{\sin ADB}.$$

Then if any side, *e.g.* FD, is calculated through the triangles to the right of BD, or through those to the left of AD, the same result should be obtained.

Thus

$$\frac{FD}{ED}=\frac{\sin E_L}{\sin F_R}\qquad\qquad \frac{FD}{DG}=\frac{\sin G_R}{\sin F_L}$$

$$\frac{ED}{AD}=\frac{\sin A_L}{\sin E_R}\qquad\qquad \frac{DG}{DC}=\frac{\sin C_R}{\sin G_L}$$

$$\frac{AD}{AB}=\frac{\sin B_L}{\sin ADB}\qquad\qquad \frac{DC}{DB}=\frac{\sin B_R}{\sin C_L}$$

$$\frac{DB}{AB}=\frac{\sin A_R}{\sin ADB}$$

By multiplying the various factors together,

$$\frac{FD\ .\ ED\ .\ AD}{ED\ .\ AD\ .\ AB}=\frac{FD}{AB}=\frac{\sin E_L\sin A_L\sin B_L}{\sin F_R\sin E\ \ \sin ADB}$$

and $$\frac{FD \cdot DG \cdot DC \cdot DB}{DG \cdot DC \cdot DB \cdot AB} = \frac{FD}{AB} = \frac{\sin G_R \sin C_R \sin B_R \sin A_R}{\sin F_L \sin G_L \sin C_L \sin ADB}.$$

Each of these expressions for $\frac{FD}{AB}$ should be equal.

Therefore by equating, cross-multiplying, and cancelling sin ADB we get $\sin A_L \sin B_L \sin C_L \ldots = \sin A_R \sin B_R \sin C_R \ldots$, or, using the logarithmic form as more convenient for calculations, and letting θ_L and θ_R represent the left- and right-hand angles respectively,

$$\Sigma \log \sin \theta_L = \Sigma \log \sin \theta_R$$

Thus if the polygons are hexagonal, the number of independent conditions to be satisfied for each polygon is 8.

If the instrument is set up N times, so that a number of complete polygons are formed, the number of conditions will be $\frac{8}{5}$ (N − 2); the distance covered, if D represents the length of one side of a triangle, will be $\frac{\sqrt{3}}{5}$ (N − 2) D, and the area enclosed by the various triangles will be $\frac{*3}{2} \cdot \frac{\sqrt{3}}{5}$ (N − 2) D^2, assuming that the triangles are equilateral.

In the system of squares or interlacing triangles in Fig. 13.3, *c*, each bay is adjusted separately, and the conditions to be satisfied are :

(1) The angles of each of the two triangles ABC, ADC, to be equal to 180° + the spherical excess.

(2) The angles of each of the two triangles ABD, CBD, to be equal to 180° + the spherical excess.

(3) As a result of (1) or (2), the four angles of each square to be equal to 360° + the spherical excess.

(4) The deduced lengths of the sides to have the same value, whatever portion of the data is employed in their computation.

This reduces to the same rule as stated above, *i.e.*

$$\sin A_R \sin B_R \sin C_R \sin D_R = \sin A_L \sin B_L \sin C_L \sin D_L.$$

The number of **independent** conditions for a single bay is thus 4, and the number of settings of the instrument is 4.

For N settings of the instrument, where N is an even number to ensure a complete series of squares, the number of conditions = 2 (N − 2).

The distance covered, if D is the length of a side of a square, is $\frac{D}{2}$(N − 2), and the area enclosed is $\frac{D^2}{2}$(N − 2).

If, however, the length of the diagonal, *i.e.* the maximum length of sight, is restricted to D, the distance covered for N settings is $\frac{D}{2\sqrt{2}}$(N − 2), and the area enclosed is $\frac{D^2}{4}$(N − 2). This is probably

* If the triangles such as HGC are included, this expression becomes $\frac{2\sqrt{3}}{5}$(N − 2)D^2.

the case which should be taken in any comparison between the systems.

For the purposes of comparison the expressions for the various systems are tabulated in Table I. In Table II are given the values of each when N = 12.

Curves may be plotted if desired to compare the three systems for any values of N.

From the results it will be seen that, to cover a given distance rapidly, the simple chain of triangles is the most expeditious and economical.

The system of interlacing triangles or squares gives the most equations of conditions to be satisfied, and consequently this is probably the most accurate and reliable method, though the individual triangles are then hardly as well conditioned as those in the polygonal series.

The advantage of the polygonal system is that the largest area is enclosed for a given number of stations. This method is therefore the most economical for a topographical survey of a large area, and as the number of equations of conditions is large, a high degree of accuracy might be expected.

TABLE I

System	Number of Stations	Distance covered	Area covered	Length of Lines	Number of Conditions
Simple triangles	N	$\frac{D}{2}(N-1)$	$\frac{\sqrt{3}}{4}(N-2)\,D^2$	$(2N-3)\,D$	$N-2$
Hexagons	N	$\frac{\sqrt{3}}{5}D\,(N-2)$	$\frac{3\sqrt{3}}{10}(N-2)\,D^2$	$\frac{(11N-17)}{5}D$	$\frac{8}{5}(N-2)$
Squares (base = D)	N	$\frac{D}{2}(N-2)$	$\frac{D^2}{2}(N-2)$	$(2{\cdot}914N-4{\cdot}828)\,D$	$2\,(N-2)$
Squares (diagonal = D)	N	$\frac{D}{2\sqrt{2}}(N-2)$	$\frac{D^2}{4}(N-2)$	$(2{\cdot}060N-3{\cdot}413)\,D$	$2\,(N-2)$

TABLE II

System	Number of Stations	Distance covered	Area covered	Length of Lines	Number of Conditions
Simple triangles	12	5·5 D	4·33 D^2	21 D	10
Hexagons	12	3·46 D	5·19 D^2	23 D	16
Squares (base = D)	12	5 D	5 D^2	30·14 D	20
Squares (diagonal = D)	12	3·54 D	2·5 D^2	21·3 D	20

The number of equations are not sufficient to determine the various errors by a very rigorous treatment, so that the adjustment is largely a question of trial and error, and the number of possible solutions is very large.

On the Ordnance Survey the number of equations of condition was 920, involving the same number of unknowns.

Example 1.—From G.S.[1] of Transvaal and Orange River Colony (Fig. 13.12).

Observed Angles:

1 = 62°-32′-14·13″	5 = 70°-51′-38·87″
2 = 51°-11′-31·34″	6 = 45°-25′-22·61″
3 = 25°-49′-58·10″	7 = 31°-36′-05·37″
4 = 32°-06′-55·67″	8 = 40°-26′-20·27″

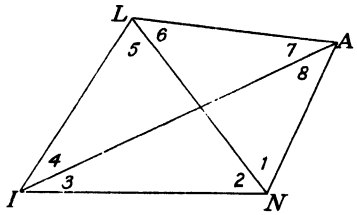


FIG. 13.12

Equations to be satisfied:

$$x_1 + x_2 + x_3 + x_8 - 0{\cdot}03'' = 0$$
$$x_4 + x_5 + x_6 + x_7 - 1{\cdot}37'' = 0$$
$$x_2 + x_3 + x_4 + x_5 - 0{\cdot}42'' = 0$$
$$10{\cdot}9x_1 - 17{\cdot}0x_2 + 43{\cdot}5x_3 - 33{\cdot}5x_4 + 7{\cdot}3x_5 - 20{\cdot}8x_6 + 34{\cdot}2x_7 - 24{\cdot}7x_8 - 9{\cdot}00 = 0$$

Resulting Corrections:

$x_1 = +0{\cdot}10''$	$x_3 = +0{\cdot}03''$	$x_5 = +0{\cdot}30''$	$x_7 = +0{\cdot}50''$
$x_2 = -0{\cdot}11''$	$x_4 = +0{\cdot}20''$	$x_6 = +0{\cdot}37''$	$x_8 = +0{\cdot}01''$

Example 2.[2]—From same course (Fig. 13.13).

Observed Angles:

1 = 27°-05′-52·31″	7 = 49°-39′-56·19″
2 = 61°-15′-38·52″	8 = 52°-08′-12·81″
3 = 55°-11′-01·68″	9 = 63°-33′-20·37″
4 = 13°-16′-37·37″	10 = 100°-45′-56·15″
5 = 31°-33′-48·62″	11 = 60°-31′-09·32″
6 = 69°-48′-53·38″	

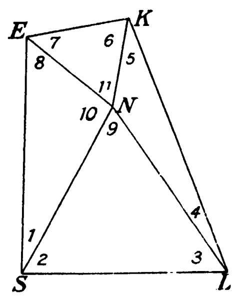


FIG. 13.13

Equations to be satisfied:

$$x_2 + x_3 + x_9 - 0{\cdot}13 = 0$$
$$x_1 + x_8 + x_{10} + 0{\cdot}85 = 0$$
$$x_6 + x_7 + x_{11} - 1{\cdot}28 = 0$$
$$x_4 + x_5 - x_9 - x_{10} - x_{11} - 0{\cdot}11 = 0$$
$$+41{\cdot}2x_1 - 11{\cdot}5x_2 + 14{\cdot}7x_3 - 89{\cdot}2x_4 + 34{\cdot}3x_5 - 7{\cdot}8x_6 + 17{\cdot}9x_7 - 16{\cdot}4x_8 + 134{\cdot}2 = 0$$

Resulting Corrections:

$x_1 = -0{\cdot}68''$	$x_4 = +0{\cdot}98''$	$x_7 = +0{\cdot}24''$	$x_{10} = -0{\cdot}16''$
$x_2 = +0{\cdot}17''$	$x_5 = -0{\cdot}44''$	$x_8 = -0{\cdot}01''$	$x_{11} = +0{\cdot}50''$
$x_3 = -0{\cdot}13''$	$x_6 = +0{\cdot}54''$	$x_9 = +0{\cdot}09''$	

Further adjustments of the triangle LNK, considering N or (12) as fixed, $= 360° - (9 + 10 + 11) = 135°\text{-}09'\text{-}33{\cdot}76''$, result in further corrections $e_4 = 0{\cdot}04''$ and $e_5 = 0{\cdot}01''$ to angles 4 and 5, the equations to be satisfied being

$$x_4 + x_5 + 0{\cdot}03 = 0$$
$$-89{\cdot}2\,x_4 + 34{\cdot}3\,x_5 - 3{\cdot}7 = 0$$

It is recommended that the student should carry out the complete

[1] *Report of Geodetic Survey of South Africa*, vol. v. p. 107.
[2] *Ibid.* p. 119. See also *Empire Survey Review*, No. 61, vol. viii. p. 264.

calculations for these two examples, using the methods described on pp. 538 and 540.

Map Projections.—As the earth is approximately spherical in form, it is impossible to represent without distortion any portion of the earth's surface upon a plane area. For small tracts of country the distortion is quite inappreciable and is neglected, but for large areas some system or method of projection must be adopted. There are a number of such systems, and each has its advantages. For a discussion of the properties and details of the most common of these, the reader is referred to *Map Projections* by Hinks, or *A Textbook of Topographical and Geographical Surveying* by Close and Winterbotham.

Accuracy.[1]—A few examples have been given in Chapter V upon the accuracy obtainable in angular measurements, and in Chapter XIV are given data of a few base-line measurements.

Ordnance Survey.—In a report made by General Ferrero (1892) to the International Geodetic Association, the Ordnance Survey of the United Kingdom was—judging by the angular errors—classed as inferior to the other European Surveys in precision.

The p.e. of a single angle on the Ordnance Survey has been given as about $\pm 1{\cdot}20''$, but the network is so complicated that the accuracy of the final results is quite comparable with the best modern work, in which, under favourable conditions, the p.e. in the angles may be as low as $\pm 0{\cdot}25''$.

On account of this opinion, however, and for other reasons, it was decided to measure a new base at Lossiemouth on the Moray Firth, connect it to the present triangulation, and to use it as a means of determining what errors had accumulated in the linear dimensions up to this point.

The results were very favourable, as will be seen by the following table, in which the fractions express the errors between the calculated and measured lengths in terms of the distance between the bases :

Bases	Salisbury Plain	Lough Foyle	Lossiemouth	Paris
Salisbury Plain .	..	1/93000	1/45000	1/505000
Lough Foyle .	1/93000	..	1/88000	1/79000
Lossiemouth . .	1/45000	1/88000	..	1/42000
Paris . . .	1/505000	1/79000	1/42000	..

The following table gives the values of three distances in Moray as calculated from the original triangulation and from the new Lossie mouth base :

[1] *British Assoc. Report*, 1913, Section E, Captain H. S. L. Winterbotham, R.E. ; *Engineering*, January 2, 1914 ; *Engineering*, January 23, 1914, Colonel Close.

	Principal Triangulation	New Deduction	Difference
	feet	feet	feet
Mormond Hill to Corriehabbie	247,660·01	247,655·91	4·10
Corriehabbie to Knock of Grange	120,524·16	120,521·03	3·13
Knock of Grange to Mormond Hill	145,861·40	145,859·85	1·55

If the error is considered as proportional to the square root of the distance covered, the discrepancy between the measured and calculated bases may be shown by the following table, which indicates that the Ordnance Survey compares very favourably with other National Surveys:

National Surveys of	No. of Comparisons	Discrepancies	Reference
Europe . .	36	1/83000	Report of International Geodetic Association
India . . .	8	1/197000	Account of G.T. Survey of India
South Africa . .	13	1/92000	Reports of Geodetic Survey of S. Africa
U.S. of America .	20	1/121000	Report of U.S. Coast and Geodetic Survey
Mean . .	77	1/99000	
United Kingdom .	6	1/152000	

With regard to the terms Principal, Primary, Secondary, Tertiary, Major, Minor, etc., Captain Lyons,[1] F.R.S., points out that they do not denote any definite degree of accuracy ; *i.e.* Secondary triangulation in one country is not necessarily of the same degree of precision as Secondary triangulation in another country.

He suggests the adoption of additional terms :

" First Order ", for the highest type of triangulation as used for geodetic purposes and for the main framework of large countries. In this the mean triangular error as determined by Ferrero's formula should not exceed 1″. The sides of such triangles would be in general 15 or more miles in length, and the angles measured with large

[1] *Brit. Assoc. Report,* 1913.

micrometer theodolites reading to say 2″ directly, or to 0·2″ by estimation.

"Second Order", in which the triangular error should not exceed 5″.
"Third Order", " " " " 15″.
"Fourth Order", " " " " 30″.

EXAMPLES

1. (U. of L.) In a certain small triangulation survey, wooden pegs were driven into the ground exactly 2 ft. towards the (magnetic) north of the stations. The signals were erected over the true stations. At one station (say A) it was desired not to disturb the signal, so the theodolite was set up over the centre of the wooden peg; it was directed towards magnetic north as zero, and the magnetic bearings of two stations, B and C, were observed to be 317°-21′ and 21°-17′ respectively. By a small-scale drawing plotted with a protractor, the distances AB and AC were found to be about 1360 ft. and 1870 ft. respectively. Find, to the nearest minute, the value of the angle BAC reduced to the centre of the true station.

(*N.B.*—An approximate system of calculation such as used in solar parallax, or in tacheometry will suffice, but you should indicate the more rigorous method of solution, supposing the peg and station were farther apart.)

2. On the Transvaal and Orange River Colony Geodetic Survey the instrument was eccentrically placed at a distance of 8·06 ft. from Manoutsa beacon (Fig. 13.14). What corrections must be applied to the bearings taken from O to the stations Thama Koosh, Mt. Anderson, and Iron Crown?

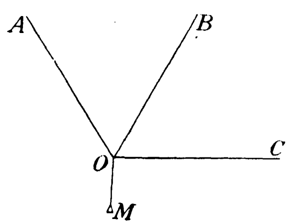


FIG. 13.14

In the figure M = Beacon.
O = Instrument.
A = Mt. Anderson.
B = Thama Koosh.
C = Iron Crown.

		Log Distance
Bearing of Beacon =	0°-0′	..
"	A = 145°-5′	5·3674
"	B = 206°-53′	5·3619
"	C = 267°-14′	5·4615

3. On the same survey an eccentric heliograph H (Fig. 13.15) at Johannesburg was observed from Blijvooruitzicht (B). At Johannesburg the instrument was set up in an eccentric position E, and the following data obtained.

What correction must be applied to the bearing of H from B to reduce it to the true station O at Johannesburg?

HE = 11·46 ft. EO = 4·33 ft.

		Log Distance
Bearing to Beacon =	0°-0′	..
"	B = 68°-14′	5·3825
"	H = 135°-55′	..

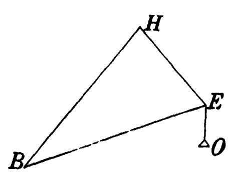


FIG. 13.15

CHAPTER XIV

BASE LINE MEASUREMENT

THE reasons for the accurate measurement of base lines in a triangulation survey have been dealt with in the preceding chapter.

The procedure to be adopted in the measurement of such a line depends upon the accuracy which it is desired to attain; and this, in turn, is influenced by the nature and extent of the triangulation which springs from it.

For example, (1) the larger the tract of country to be surveyed, the more accurately is the base measured; because if l is the maximum length of the survey, and b the length of the base, and e_b the error in the base, the resulting total error in the length l from this cause alone is $e_b \frac{l}{b}$, *i.e.* the error in the base is equivalent to an error in the scale to which the map is plotted, because it affects all measurements in the same proportion. Other errors, of course, influence the apparent length l, as explained in Chapters I, V, and XIII.

(2) The scale to which the resulting map is to be plotted is to be considered, as the larger the scale the more appreciable is the error of a given magnitude upon the paper.

(3) The expense, and the time at the disposal of the survey party are also important considerations; *e.g.* it would not be reasonable to spend the same amount of time or money upon the base line of a topographical survey of small extent as upon the base line of a triangulation survey for a large country.

The length of the base of a topographical survey may be as short as ¼ to ½ mile, while that of a large trigonometrical survey may be as much as 10 miles or more. It has been generally considered, however, that little is to be gained by the use of a longer line than 5 or 6 miles, as this can be extended with great accuracy, as explained on p. 427.

Nevertheless, owing to the facility and accuracy of measurements made by the more modern steel tapes and wires, this length is now very frequently exceeded.

The lengths of a few primary triangulation bases are given on pp. 392 and 430.

Standards.—The standard length of the Ordnance Survey is computed from that of the standard yard: it is 10 ft. in length, and is defined by means of two platinum dots, situated in the neutral axis of a rectangular iron bar, which measures 1½ in. wide and 2½ in.

O

deep. For a short space near each end, one-half of the depth of the bar is cut away, to enable the dots to be placed exactly upon the neutral axis. The bar is supported at $\frac{1}{4}$th and $\frac{3}{4}$ths of its length from either end.

A *yard* is defined by Act of Parliament, and is the distance at 62° F. between two marks upon a bar of bronze kept at the Exchequer.

Such a standard is known as " *à traits* " ; Continental standards are frequently defined as the distance from one extremity of the bar to the other, and the term " *à bouts* " is then applied.

It will be noticed from the following notes that there are several distinct methods of making the required measurements, *e.g.*:

(1) A base line may be measured by the employment of two or more standardised bars, the measuring length of each being defined by transverse lines scribed across the bar. In conducting the measurements, the forward mark upon the one bar is brought alongside and into exact coincidence with the hindermost mark upon the next bar.

Any fractional length of a bar at the end of the base is measured separately, and its value added to the length of the remainder of the base as deduced from the number of complete bar lengths included.

(2) A defining dot upon the forward end of one bar may be brought within a definite distance of the hindermost dot upon the second bar (*e.g.* Colby's compensation bars).

(3) The standard length of each bar may be defined as the distance between one extremity of the bar and the other, and in this case the forward end of one bar may be brought into contact with the rear end of the following bar, and so on.

(4) As an alternative, instead of bringing consecutive bars exactly into contact, the intervening space between the bars may be measured with a scale, tapering wedge, or other contrivance. The total distance is then comprised of a number of standard bar lengths and a number of measured intervals.

(5) When a steel tape is employed, special tripods carrying fixed marks may be used, and the intervals between consecutive marks are then measured individually, and the separate results combined to furnish the total length ; or

(6) The whole length of the tape or wire may be used for each application, and a mark scribed on each peg in turn to coincide with the forward end of the tape, the rear end being held upon the previous mark.

The chief appliances which have been adopted in the past for the accurate measurement of base lines are :

(1) Steel chains.
(2) Deal rods.
(3) Glass rods.
(4) Compound rods.
(5) Steel tapes or wires.

(1) **Steel Chains** afford an expeditious and economical method of measurement. For a small survey they may be laid upon the ground,

but for important work they are laid in accurately levelled coffers or troughs supported on trestles, and aligned with a theodolite or transit instrument. The temperature is taken at a number of points along each chain length and a constant pull applied.

The Hounslow Heath Base of the Ordnance Survey was re-measured in 1791 by this method, the chain in this case being 100 ft. long, subdivided into 40 links, ½ in. square in section. A constant pull of 28 lbs. was applied by means of a weight, and the temperature taken at five points in each 100 ft. length.

The measurements were taken between transverse lines inscribed one on each handle of the chain; and a second similar chain was kept for purposes of comparison.

Most of the other bases of the Ordnance Survey were measured in a similar way with the steel chain.

(2) **Deal Rods,** trussed and well seasoned, have been much used. Generally they were placed end to end, spherical metal tips being fitted to ensure accuracy of contact.

They are not used on modern surveys, as they have been superseded by steel tapes and wires, or other more refined apparatus.

They are liable to alter considerably in length owing to changes in the humidity of atmosphere; it is essential, therefore, that they should be well oiled or varnished.

An attempt was made in 1784 to measure the Hounslow Heath Base with such rods 20 ft. 3 in. in length, but the result was considered unreliable.

(3) **Glass Rods** placed end to end in accurately levelled and aligned coffers have also been used, *e.g.* upon the first measurement that was adopted of Hounslow Heath Base in 1784. In this case the length of each rod was 20 ft. The temperatures were noted, and proper corrections applied.

(4) **Compound and Special Rods.**—**Colonel Colby's** apparatus was designed to eliminate the effect of changes of temperature upon the measuring appliance.

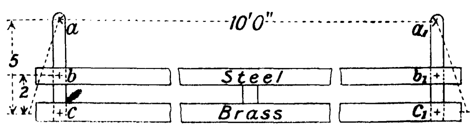


FIG. 14.1
Colonel Colby's Measuring Bars

Two bars, each 10 ft. long, are riveted together at the centre of their length: one bar is of steel and the other of brass, the ratio of the coefficients of linear expansion of these metals having been determined as 3 : 5.

Across the extremities of the two bars is fixed a metal tongue projecting on the side away from the brass rod, as shown diagrammatically in Fig. 14.1. In this tongue is placed a minute dot a of platinum, the distance ab to the junction with the steel being $\frac{3}{5}$ths of ac, the distance to the brass connection.

The brass is coated with a special preparation in order to render it equally susceptible to changes of temperature as the steel. Conse-

quently, if the distance between the ends c and c_1 of the brass rod alters an amount x due to a change of temperature of $t°$, the distance between the points b and b_1 will alter by an amount $\frac{3}{5}x$, and the points a and a_1 will be unaltered in position, *i.e.* the distance from a to a_1 has a constant value of 10 ft.

This apparatus was employed on the Ordnance Survey for the measurement of the Lough Foyle base in the north of Ireland.

Here coffers supported on trestles were aligned with a transit instrument, and accurately levelled.

A framework, on the same principle as the 10-ft. bars, was employed to carry two microscopes, each of which was furnished with cross-hairs, the distance from the centre of the one to the centre of the other being 6 in.

To commence the measurement, the cross-hairs of the one microscope were brought exactly over a platinum dot, let into the centre of a stone pillar to mark the one extremity of the base line. The platinum dot a of one of the bars was then brought exactly beneath the cross-hairs of the second microscope, the adjustment being made by moving the coffer with the aid of three slow-motion screws.

A second framework was then placed at the far end of the first length of compound bar, and, after levelling this, the one microscope was adjusted to its position over the platinum dot a_1 at the further end of the first measuring bar, while the second compound bar was placed under the second microscope.

In this way five lengths were laid at a time—*i.e.* a total length of 52 ft. 6 in. The coffers were then moved, and further lengths of 52 ft. 6 in. measured until the base was completed.

The Colonel Colby apparatus was used for the measurement of ten base lines of the Trigonometrical Survey of India, and also, in 1849, for the remeasurement of the Salisbury Plain base of the Ordnance Survey, in this case six successive lengths being used at one time.

In India, some difficulty was experienced owing to the fact that the bar on the side towards the sun became heated more quickly, and attained a higher temperature than the far bar. Thermometers were accordingly used to attempt to ascertain the temperature of each bar as the work was proceeding, and corrections were applied in the usual way to allow for any differences between the two bars.

It was calculated [1] from the experiments carried out on the Cape Comorin base, that without thermometers a probable error of $\pm 1{\cdot}5\mu$ might be expected with this apparatus, while with thermometers this error might be reduced to $\pm{\cdot}5\mu$, where μ represents 10^{-6} or one-millionth part of the whole length of base.

French System.[2]—**Borda** designed a compound bar of platinum and copper, which was carried upon a wooden support. The length of the lower platinum strip was two toises, while the upper copper

[1] *G.T. Survey of India.*

[2] For further particulars on the various systems, see Clarke's *Geodesy*.

strip was somewhat shorter. These two strips were firmly connected at one end, but they were free to expand or contract, one relatively to the other, at the opposite end.

Attached to the end of the copper bar was a scale which moved against a vernier fixed to the platinum. The reading of this vernier was then a measure of the relative expansions or contractions of the two strips, and consequently furnished a means of ascertaining the mean temperature of the platinum measuring strip.

The interval between successive strips was measured by means of a graduated slider moving against a second vernier upon the platinum component.

Both verniers were read by the aid of microscopes, and four sets of bars were employed at one time, though later one of these was reserved as a standard for reference and not employed in the actual field work.

Two iron tripods provided with levelling screws were used for each bar, and the inclinations were measured upon a graduated arc.

German System (Bessell's Apparatus).—The bars employed upon the German bases were arranged, like Borda's rods, to form a metallic thermometer, the materials used being, in this case, iron and zinc. The extremities of the zinc bars were furnished with horizontal knife edges, and the interval between these was measured by means of a glass wedge, instead of by the graduated slide working in a groove, of the French.

The compound bar was supported by seven rollers carried on an iron bar, and the whole apparatus, with the exception of the contact ends, was enclosed in a case for protection.

Russian System.—Struve's Bar, as used on seven of the Russian bases, was not a compound bar forming a metallic thermometer as were the French and German bars. Each rod consisted of a single wrought-iron bar, one end being fitted with a slightly convex and highly polished steel terminal, while the other end was fitted with a small lever. One end of this lever carried a highly polished hemisphere, which was brought into contact with the convex terminal of the next bar, while the upper end of the lever moved over a graduated scale, from the reading of which the length of the bar could be deduced. The pressure at contact was regulated by a spring which operated upon the small lever.

Four such bars were employed in series, and the temperature was judged from the readings of thermometers, whose bulbs were let into the metal—two in each bar.

The United States Coast and Geodetic Survey Apparatus consisted of a compound bar of iron and brass, 6 metres in length, firmly joined at one end, and free to expand at the other.

The masses of the two components were inversely proportional to their specific heats, while the surface areas were equal.

The relative movement of the upper iron bar, which rested upon small rollers on the surface of the brass, was measured by means of a vernier and scale, and afforded data from which the exact length of the apparatus could be deduced. At the free ends, a lever arrangement

was provided, pivoted on the lower brass bar, and bearing with a knife edge on the steel terminal of the iron bar.

The extremity of the lever was provided with a similar knife edge on the outer side, and against this was directed a constant pull. This pull regulated the pressure of the small lever against the extremity of the iron component of the apparatus : it was transmitted by means of a sliding rod (*a*) actuated by a spring, from a frame on the iron bar. This rod also carried an agate contact surface at its outer extremity.

At the opposite end of the bars was a somewhat similar sliding bar (*b*), carrying a knife edge at its outer extremity. Contact between successive bars was made by bringing this knife edge of the one sliding rod (*b*) (at the united end of the components) into contact with the agate surface upon the extremity of the other sliding rod (*a*) at the free end of the components.

The pressure was regulated until a bubble, actuated by the rod (*b*), was brought to the centre of its run.

The whole apparatus, with the exception of the sliding rod terminals, was enclosed in a double tubular case, fitted with glass windows for observation purposes.

Ice Bar Apparatus.—To eliminate as much as possible errors due to temperature variations, an ice bar apparatus, designed by Professor Woodward, has been used in U.S.A. Micrometer microscopes are employed for marking the distance between platinum iridium dots in the neutral axis of a steel bar, which is embedded in a V trough filled with melting ice.

(5) **Steel Tapes or Wires** are employed in the most modern work, as results can be obtained with as high a degree of accuracy as those furnished by the elaborate compensation bars previously used, while the work is executed much more expeditiously, and the expense is considerably less.

For very accurate work, an alloy of iron and nickel (36% nickel, and 0·2% carbon), and known as " Invar " steel, has been introduced. The chief advantage of this material lies in the fact that it has an extremely low coefficient of expansion—the lowest of any known metal. The value varies in different specimens, but may be about ·0000003 to ·0000004. Thus, as the chief source of error in base line measurement has previously been that due to the difficulty of correctly estimating the temperature of the measuring appliance, or of eliminating this source of error, it is evident that the possibility of error is here reduced to a minimum.

The delicacy of such tapes and wires precludes the use of this material for ordinary work.

The discovery of invar is due to Dr. Guillaume of Paris, whose investigations extended over several years. The ultimate strength [1] in tension, from tests at the Washington Bureau of Standards, is about 100,000 lb./sq. in., and the yield point is at about 70% (or more) of the breaking load.

[1] *Proc. Inst. C.E.* vol. clxxii.

A test[1] at the National Physical Laboratory on three invar wires gave the following results, in kilogrammes per sq. cm.:

Elastic Limit	Yield Point	Max. Load	E.
6600	7800	8740	1,520,000
5800	7000	8050	1,440,000
6200	7400	8270	1,445,000

For important bases the tape or wire would be 300 to 500 ft. long, though a 100-ft. or 150-ft. length is often employed. Lengths of 24 metres, 50 metres, etc., are also sometimes adopted.

The base is divided into bays, and the tape or wire is stretched between each two successive supports in turn, and allowed to hang freely in a catenary between them, the tension being regulated by means of a spring balance, a suspended weight, or other means. The difference in height of the supports is measured by means of an ordinary level, the supports are aligned with a theodolite or transit intrument, and the temperature of the tape is noted by means of attached thermometers. To ensure as true results as possible, a cloudy or dull day should be selected, otherwise the readings of the thermometers may give very erroneous figures.

For the best class of work, the supports are tripods, at the top of each of which is a small flat surface engraved with a fine mark, to which the measurements are taken.

At each end of the wire or tape is a small graduated scale, and the reading on this which corresponds with the mark on the tripod head is read with a microscope.

For less accurate measurements, the supports are timber stakes driven firmly into the ground, and accurately levelled. On the top of each peg a zinc plate is tacked, and upon this a longitudinal scratch is made with a steel scriber, the transit being used to fix the true alignment. The tape is then stretched from peg to peg, and a transverse scratch made to intersect the longitudinal scratch, at the extremity of each tape length.

Straining trestles or pegs are desirable to support the wire, to ensure that the tripods or marking pegs are not displaced during the stretching of the tape.

One end of the wire is held firmly in position while the other end is attached to the straining weight, or other straining appliance.

For a topographical base the rear end of the tape may be attached to a ranging rod and held as shown in Fig. 14.2, a second operator adjusting the zero of the tape to the mark upon the zinc plate. To the front of the tape may be attached a spring balance, held in the hand by one operator (and preferably passed over a special straining peg), while the mark is scribed on the zinc at the tape extremity by another operator. Care must be taken to eliminate friction over the straining and marking pegs as much as possible.

As an alternative, the tape may be attached to a cord, passing

[1] *The Geographical Journal*, "Base Measuring Apparatus", J. A. Agar Baugh, January 1912.

over a pulley upon a ranging or other rod, and carrying the straining weight as in Fig. 14.2.

Corrections.—The following are the corrections which it is necessary to make in the calculations :

(1) Correction for tension.
(2) Correction for sag.
(3) Correction for slope.
(4) Correction for temperature.
(5) Correction for alignment.
(6) Correction for standard.
(7) Reduction to sea-level.

(1) **Correction for Tension.**—When a tape is subjected to a tension of P lbs. weight, it stretches a small amount. This small amount of elongation may be easily calculated, because, as is stated by Hooke's Law, if the elastic limit of the material is not exceeded, strain (*i.e.* stretch per unit length) is proportional to the stress (*i.e.* pull per unit area) producing it.

The ratio, $\frac{\text{stress}}{\text{strain}}$, is thus equal to some constant, which is known

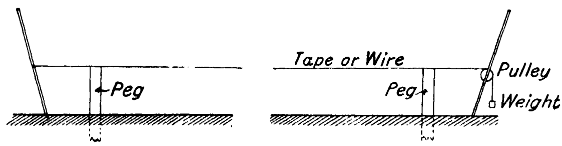


FIG. 14.2

as the stretch modulus of elasticity, or Young's modulus (E) of the material.

Thus if the sectional area of the tape is A sq. in., the stress is P/A lb./sq. in. ; and if the total elongation of the tape is l_1 ft. in a total length of L ft., the strain is expressed by the fraction l_1/L.

Consequently, $$\frac{P}{A} \div \frac{l_1}{L} = E,$$

and $$l_1 = \frac{PL}{AE} \text{ ft.} \qquad . \quad . \quad . \quad (1)$$

The value of E for normal steels is about 30,000,000 or 30×10^6 lb./sq. in. For Invar, a figure of 22×10^6 lb./sq. in. is often taken : in the tests quoted on the previous page for invar wires, E averaged about $20{\cdot}9 \times 10^6$ lb./sq. in.

The effect of the pull is, therefore, to lengthen the tape, and so to decrease the apparent length of the base line. Consequently, the correction for pull must be *added* to the apparent length in deducing the true length.

The mean sectional area of the tape may be deduced if the length and weight of the tape are known.

An error of $\pm\delta P$ in the determination of P will produce an error of

$$\frac{(P\pm\delta P)L}{AE}-\frac{PL}{AE}=\pm\frac{\delta P}{P}\cdot\frac{PL}{AE},$$

or

$$\pm\frac{\delta P}{P}\times(\text{correction for tension}).$$

Similarly, an error of $\pm\delta A$ in the determination of the cross sectional area A will produce an error of

$$\mp\frac{\delta A}{A}\times(\text{correction for tension}).$$

(2) **Correction for Sag.**—The distances that it is required to ascertain, eventually, are the successive horizontal distances between the peg or tripod supports.

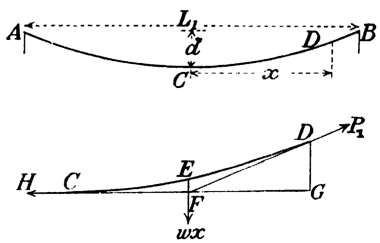


FIG. 14.3

Each of these distances, being a straight line, is obviously shorter than the length of the tape, and the difference between these quantities, *i.e.* between the curved length of the tape, and the distance from one support to the next is known as the Correction for Sag. The apparent length thus being too large, the correction must be subtracted, in determining the true length from the observations.

Thus assuming that the two supports A and B, Fig. 14.3, are of the same level, let L_1 be the horizontal distance between them, and let the total dip be d.

Then considering the equilibrium of a length CD of tape from the centre, the horizontal projection of CD being x ft., we have a horizontal pull at C of H lbs., say, a pull of P_1 lbs. along the tape at D (there can be no component at right angles to the tape as it is not sufficiently stiff to take any appreciable bending moment), and the weight of the portion CD, *i.e.* w . CD lbs., where w is the weight per foot run of the tape.

But, as the sag is very small in comparison with the span, $CD=x$ nearly, and the weight of CD may be considered as wx lbs. acting at its middle point E.

These three forces must be concurrent as CD is in equilibrium, *i.e.* H and P_1 meet at F vertically below E.

Draw DG parallel to EF (*i.e.* vertically) to meet CF produced in G.

Then by the triangle of forces

$$\frac{wx}{DG}=\frac{P_1}{\sqrt{DG^2+FG^2}}=\frac{H}{FG},$$

or
$$\frac{wx}{y}=\frac{P_1}{\sqrt{y^2+\left(\frac{x}{2}\right)^2}}=\frac{2H}{x}, \qquad (2)$$

where DG $=y$.

When $x=\frac{L_1}{2}$, *i.e.* at the supports, $y=d$, and P_1 becomes equal to P, the external pull applied with the spring balance, and

$$\frac{wL_1}{2d}=\frac{P}{\sqrt{d^2+\left(\frac{L_1}{4}\right)^2}}=\frac{4H}{L_1},$$

i.e.
$$H=\frac{wL_1^2}{8d}, \qquad (3)$$

therefore from (2)
$$\frac{wx}{y}=\frac{2wL_1^2}{8dx}$$

$$\therefore x^2=\frac{L_1^2}{4d}\cdot y \qquad (4)$$

which is the equation of a parabola, and may be written as

$$x^2=c\cdot y, \qquad (4a)$$

where
$$c=\frac{L_1^2}{4d}.$$

To determine the length of wire or tape in terms of the span, or *vice versa*, differentiate equation (4*a*)

$$2x=c\cdot\frac{dy}{dx},$$

or
$$\frac{dy}{dx}=\frac{2x}{c}.$$

The length of tape from the centre for a horizontal distance x_1 is then

$$\int_0^{x_1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx,$$

therefore L, *i.e.* the length of tape,

$$=2\int_0^{\frac{L_1}{2}}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx$$

$$=2\int_0^{\frac{L_1}{2}}\left\{1+\frac{4x^2}{c^2}\right\}^{\frac{1}{2}}dx$$

$$=2\int_0^{\frac{L_1}{2}}\left(1+\tfrac{1}{2}\cdot\frac{4x^2}{c^2}-\tfrac{1}{8}\left(\frac{4x^2}{c^2}\right)^2+\ .\ .\ .\ \right)dx$$

$$=2\left[x+\frac{2x^3}{3c^2}-\ .\ .\ .\ \right]_0^{\frac{L_1}{2}}$$

$$=L_1+\frac{L_1^3}{6c^2},\text{ approx.}$$

or substituting $c=\frac{L_1^2}{4d}$,

$$L=L_1+\frac{8}{3}\frac{d^2}{L_1}.$$

But from (3) $d=\frac{wL_1^2}{8H}=\frac{wL_1^2}{8P}$ nearly, because if the parabola is moderately flat, the pull P differs very slightly from its horizontal component H,

$$\therefore\ L=L_1+\frac{w^2L_1^3}{24P^2},$$

or, as the difference between L and L_1 is very small,

$$L_1=L-\frac{L}{24}\cdot\left(\frac{W}{P}\right)^2, \qquad (6)$$

where W is the total weight of the tape and $wL_1=wL$ nearly.

The difference between the curved length and the straight length may therefore be expressed by the equation

$$s=\frac{L}{24}\cdot\left(\frac{W}{P}\right)^2\text{ nearly.} \qquad (7)$$

Catenary Calculations.—It was assumed on p. 417 that the dip of the tape would be small in comparison with the span, and hence that w could be considered as constant per unit length of *span* rather than as is actually the case, constant per unit length of *tape.*

The curve taken up by the hanging tape is strictly a catenary, whereas equation (4a) is that of a parabola. The parabola is a simpler curve to deal with and is a close approximation to the catenary under these conditions.

The equation of the catenary can be derived from Fig. 14.3, F no longer being the centre point of CG, but the point below the c. of g. of the curved wire.

$$\frac{w\,.\,CD}{H}=\frac{DG}{GF}=\frac{dy}{dx}.$$

Let $\quad CD=z$ and let $H=wk$.

Then $\quad \frac{wz}{wk}=\frac{z}{k}=\frac{dy}{dx}$

and $\quad \frac{d^2y}{dx^2}=\frac{1}{k}\frac{dz}{dx}=\frac{1}{k}\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$

Writing $\frac{dy}{dx}=p$, say, this equation becomes

$$\frac{dp}{dx}=\frac{1}{k}\sqrt{1+p^2},$$

i.e.
$$\frac{dp}{\sqrt{1+p^2}}=\frac{1}{k}\,dx,$$

from which, on integrating,

$$\sinh^{-1} p=\frac{x}{k}+\text{constant}.$$

As $p=o$ when $x=o$, the constant term vanishes and

$$p,\ i.e.\ \frac{dy}{dx}=\sinh\frac{x}{k},$$

from which, on integrating again

$$y=k\cosh\frac{x}{k}-k, \quad . \quad . \quad . \quad . \quad (8)$$

and this is the equation of a catenary. Also as

$$\frac{dy}{dx}=\frac{z}{k}=\sinh\frac{x}{k},$$

$$z=k\sinh\frac{x}{k}.$$

The total length of cable is twice that for the half span, *i.e.*,

$$L=2k\sinh\frac{L_1}{2k}.$$

The correction for sag is the difference between L and L_1, and is calculated by using the expansion for sinh $L_1/2k$ and neglecting the third and following terms, *i.e.*

$$L=2k\left\{\frac{L_1}{2k}+\frac{1}{3!}\left(\frac{L_1}{2k}\right)^3+\frac{1}{5!}\left(\frac{L_1}{2k}\right)^5+\ .\ .\ .\right\}$$

$$=L_1+\frac{L_1^{\,3}}{6\,.\,(2k)^2}+\ .\ .\ .$$

$$L-L_1=\frac{L_1^{\,3}}{24\,k^2}=\frac{L_1^{\,3}w^2}{24\,H^2}=\frac{W^2}{24}\,\frac{L_1}{H^2}=\frac{W^2L}{24P^2}$$

as before, when the sag is small.

An error of $\pm\,\delta P$ in the determination of P will produce an error of

$$\frac{LW^2}{24(P\pm\delta P)^2}-\frac{LW^2}{24P^2}=\frac{LW^2}{24P^2}\left\{\frac{P^2}{(P\pm\delta P)^2}-1\right\}$$

$$=\frac{LW^2}{24P^2}\left\{\left(1\pm\frac{\delta P}{P}\right)^{-2}-1\right\}$$

$$=\frac{LW^2}{24P^2}\left\{\frac{3\delta P^2}{P^2} \mp \frac{2\delta P}{P}\right\} \text{ nearly}$$

$$=\left(\frac{3\delta P^2}{P^2} \mp \frac{2\delta P}{P}\right) \times (\text{correction for sag.})$$

(3) **Correction for Slope.**—When the supports cannot conveniently be arranged at exactly the same heights, the relationship between the length of tape (L) and the horizontal distance (L_1) between the supports may be derived from first principles. But as the dip and the differences in level are generally very small, the application of the above formula (7) will be found to determine the *slope* distance (L_2) between the supports with sufficient accuracy for the purpose.

The horizontal distance is then deduced as follows :

Let L_2 be the slope distance deduced by formula (6), *i.e.*

$$L_2 = L - \frac{W^2L}{24P^2},$$

and let L_1 be the required horizontal distance ; h the difference in level between the supports, and θ the angle of slope.

If θ is measured, we have

$$L_1 = L_2 \cos \theta,$$

so that the correction to be applied is

$$c = L_2 (1 - \cos \theta). \quad . \quad . \quad . \quad (9)$$

If h is determined by direct levelling, the sine of the angle of inclination θ is known, *i.e.*

$$\sin \theta = \frac{h}{L_2}$$

and

$$\text{cosine } \theta = \frac{\sqrt{L_2{}^2 - h^2}}{L_2} = \left\{1 - \left(\frac{h}{L_2}\right)^2\right\}^{\frac{1}{2}}$$

$$= \left(1 - \tfrac{1}{2}\frac{h^2}{L_2{}^2} + \ . \ . \ .\right),$$

but the correction to be applied is $L_2(1 - \cos \theta)$, *i.e.* neglecting higher powers of the expansion than the second, and substituting for $\cos \theta$ we get

$$c = \tfrac{1}{2}\frac{h^2}{L_2}. \quad . \quad . \quad . \quad (10)$$

This, being always negative, must be subtracted from the apparent length L_2 to yield the true length.

An error of $\pm \delta\theta$ in the determination of the angle of slope θ will produce in L_1 an error of

$$\pm L_2 \sin \theta \cdot \delta\theta$$

by differentiation of equation (9) ; or an error of $\pm \delta h$ in the determination of h will produce in L_1 an error of

$$\pm \frac{h \cdot \delta h}{L_2},$$

a result which may be obtained by the differentiation of equation (10), or which may be derived from the expression $\pm L_2 \sin\theta \,.\, \delta\theta$, *i.e.*

$$\pm L_2 \frac{h}{L_2} \cdot \frac{\delta h}{L_2} = \pm \frac{h \,.\, \delta h}{L_2}.$$

(4) **Correction for Temperature.**—The length of a tape is increased as its temperature is raised, with the result that apparent measurements are then too small. Thus, when the temperature of the tape is above the normal, a correction must be *added* to the apparent to obtain the true length of the base, and conversely when the temperature is below the normal, the correction has to be *subtracted.*

The estimation of the temperature of a tape, wire or bar is the most fruitful source of error in the accurate measurement of base lines.

If the tape is exposed to the sun, thermometer readings are most unreliable, so that to ensure accuracy such measurements should be confined to cloudy and dull days.

Even under such circumstances, if the temperature is rising or falling, the change recorded on the thermometer is not necessarily that experienced by the measuring appliance—particularly in the case of bars or rods. Here, if the thermometer bulb is outside the bar, the change of temperature—either rise or fall—is later in the metal than in the bulb, *i.e.* there is a distinct lag : when the thermometer bulb is enclosed in a pocket in the bar, the reverse takes place.

The effect of wind, too, upon the temperature of a tape is very difficult to estimate.

The compound bar types of apparatus were designed to eliminate as far as possible the effects of temperature, but the results were not altogether satisfactory (see p. 412).

The introduction of invar steel has considerably reduced the effect of any possible errors in the estimation of the temperature. The coefficients of expansion (a) for 1° F. are about

0·00000625 for ordinary steel,
0·00000661 for Chesterman steel tapes,
0·0000003 to 0·0000004 for invar steel.

Thus if the length of a tape is L at T°, then at a temperature of $(T \pm t)°$ F. the length will be approximately $L\,(1 \pm at°)$, and the correction consequently

$$\pm at°L. \qquad (11)$$

An error $\pm\delta a$ will thus produce an error of $\pm\frac{\delta a}{a} \times$ (correction for temperature), and an error of $\pm\delta t$ will produce an error of $\pm\frac{\delta t}{t} \times$ (correction for temperature).

(5) **Correction for Alignment.**—Generally a base is set out in one continuous straight line, but sometimes it is necessary to resort to a bent base line, composed of two or more straight portions subtending an angle other than 180° as in Fig. 14.4.

The length AB in this case $= \text{BC} \cos \theta_1 + \text{AC} \cos \theta_2$; θ_1 and θ_2 being measured directly with a theodolite.

The correction to be subtracted from the apparent length of line is thus

$$\{\text{BC}(1 - \cos \theta_1) + \text{AC}(1 - \cos \theta_2)\}. \quad . \quad . \quad (12)$$

If A and B are not mutually visible, the angle ACB (γ) or the angle DCB (γ_1) must be measured and relied upon.

Then $$\text{AB} = \sqrt{\text{AC}^2 + \text{BC}^2 - 2\text{AC} \cdot \text{CB} \cos \gamma}. \quad . \quad (13)$$

Also, θ_2 and θ_1 may be computed from the well-known sine formula, *i.e.*

$$\frac{\sin \theta_2}{\text{BC}} = \frac{\sin \theta_1}{\text{AC}} = \frac{\sin \gamma}{\text{AB}}. \quad . \quad . \quad . \quad (14)$$

Or, as an alternative, $\theta_2 - \theta_1$ may be determined from the equation

$$\tan \frac{\theta_2 - \theta_1}{2} = \frac{\text{BC} - \text{AC}}{\text{BC} + \text{AC}} \cot \frac{\gamma}{2}. \quad . \quad . \quad (15)$$

Then, as $$\theta_2 + \theta_1 = 180° - \gamma = \gamma_1,$$

θ_2 and θ_1 may be calculated, and, by the application of equation (12), the correction deduced.

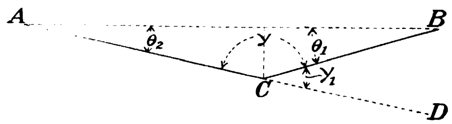


FIG. 14.4.—Base Line

Even when A, B, and C are mutually visible it would be advisable to measure θ_1, θ_2, and γ, and to calculate AB from both equations (12) and (13), and adopt the mean result.

It may be interesting here to consider in what way an error in the adopted values of the angles θ_1, θ_2, and γ may effect the results.

Considering the equation

$$\text{AB} = \text{BC} \cos \theta_1 + \text{AC} \cos \theta_2, \quad . \quad . \quad (16)$$

let the effect of a small error $\delta\theta_1$ in θ_1 produce an error of $\delta(\text{AB})'$ in AB; and a small error $\delta\theta_2$ in θ_2 produce a similar error $\delta(\text{AB})''$ in AB.

By differentiation then

$$\frac{\delta \text{AB}'}{\delta\theta_1} = -\text{BC} \sin \theta_1$$

and $$\frac{\delta \text{AB}''}{\delta\theta_2} = -\text{AC} \sin \theta_2.$$

The negative signs indicate that if the errors of $\delta\theta_1$ and $\delta\theta_2$ are $+^{\text{ve}}$ (*i.e.* tend to increase the apparent values of θ_1 and θ_2), their effect is to decrease the apparent length of AB. Similarly, if the errors $\delta\theta_1$ and $\delta\theta_2$ are $-^{\text{ve}}$, the apparent length of AB will be increased.

If now $\pm\delta\theta_1$ and $\pm\delta\theta_2$ are the probable errors in θ_1 and θ_2, the probable error δAB in AB, due to the two sources, is

$$\delta AB = \pm\sqrt{(BC\sin\theta_1\delta\theta_1)^2 + (AC\sin\theta_2\delta\theta_2)^2}. \qquad . \quad (17)$$

But the smaller θ_1 and θ_2, the smaller will be the values of $\sin\theta_1$ and $\sin\theta_2$, and therefore the smaller will be δAB.

That is the effect of an error of given magnitude in θ_1 or θ_2 produces the minimum effect upon the length of AB, when θ_1 and θ_2 are as small as possible, *i.e.* when ACB differs as little as possible from a straight line.

Similarly, by the differentiation of equation (13), *i.e.*

$$AB^2 = AC^2 + CB^2 - 2AC \,.\, CB \,.\, \cos\gamma,$$

$$2AB \,.\, \delta(AB) = 2AC \,.\, CB \,.\, \sin\gamma \,.\, \delta\gamma,$$

or

$$\frac{\delta(AB)}{\delta\gamma} = \frac{AC \,.\, CB \sin\gamma}{AB}. \qquad . \quad . \quad . \quad (18)$$

This expression is a minimum when γ is 180° ($\gamma = 0$ is not practicable). That is, the effect of an error of given magnitude in γ has the least effect upon the length of the line AB, when ACB approaches a straight line.

Similarly, it may be shown that the effect upon AB of errors in AC and CB is a minimum when AC = CB, *i.e.* when the triangle is isosceles.

Example.—A broken base is deflected through an angle of 5°, at $\frac{3}{5}$ths of its length from one extremity. What is the maximum error which can be made in the measure of this angle, so that the resulting error in the length of the base is not more than 1 in 1,000,000 ?

Applying equation (18) and considering the length AB = 5 units,

$$\delta\gamma = \frac{\delta AB \,.\, AB}{AC \,.\, CB \,.\, \sin\gamma} = \frac{5 \times 10^{-6} \times 5}{3 \times 2 \times \cdot 08716} = \frac{25}{6 \times 87160} \text{ radians}$$

$$= \frac{25 \times 180 \times 60 \times 60}{6 \times 87160 \times \pi} \text{ seconds}$$

$$= 9{\cdot}86 \text{ seconds.}$$

(6) **Correction for Standard.**—A standard or reference tape whose length has been accurately determined must be kept for purposes of comparison, as the field tapes are liable to suffer some permanent elongation, due to the continual tension to which they are subjected. For accurate work a comparison should be made immediately before and immediately after each day's work in the field.

In order to compare the tapes, they may be laid side by side on a flat surface such as a railway line, or suspended between two pegs in the usual manner.

The exact distance between the marks on the pegs or tripods must then be deduced by correcting the standard tape for pull and temperature, if these vary from the values at which the tape is correct.

The same length is measured with the field tape, and similarly corrected.

The length of the field tape in terms of the standard tape may then be deduced.

The standard tape may be tested either at the National Physical Laboratory, Teddington, or at the Ordnance Survey Office, Southampton.

(7) **Reduction to Sea-Level.**—In order that the survey of a large area may plot correctly upon the plans, it is usual to reduce the length of the base line to its equivalent length at mean sea-level. If this is done, with every base on a survey, it is possible to compare the results (see Base of Verification, p. 392): otherwise if the bases at various altitudes were not reduced to mean sea-level, comparison would be impossible.

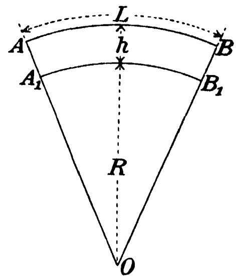


FIG. 14.5

Thus if L feet be the length of the measured base AB (Fig. 14.5), at an altitude of h feet above sea-level, the equivalent length A_1B_1 at sea-level will be $\frac{R}{R+h}$. L feet, where R is the radius of the earth in feet.

The correction to be applied is therefore

$$L\left(1-\frac{R}{R+h}\right)=L\frac{h}{R+h} \text{ or } L\frac{h}{R} \text{ nearly}, \quad . \quad . \quad (19)$$

as R is very large in comparison with h. R may be taken as 20,890,000 ft. nearly.

An error of $\pm\delta h$ in the assumed value of h above sea-level will thus produce an error of $\pm\frac{L}{R}\,.\,\delta h$ ft. in the calculated length of the base at sea-level.

Example 1.—The following base was measured for a topographical survey:

The standard tape, correct at 60° F., was stretched over two supports at a temperature of 76° F., with a pull of 30 lbs.

The field tape, with 20 lbs. pull at 72° F., was apparently 0·0455 ft. longer than the standard.

The measured apparent length of base going out was 1900 ft.	
The measured apparent length of base returning was 1900·0360 ft.	
The mean temperature going out	=65·4° F.
The mean temperature returning	=63·5° F.
The weight of the standard tape	=3·4916 lbs.
The weight of the field tape	=1·4759 lbs.
The sectional area of the standard	=·01036 sq. in.
The sectional area of the field	=·00439 sq. in.

Comparison with Standard.—The true length between the scratches made when comparing the standard with the field tape is calculated from the standard tape:

Standard Tape.

Correction for pull $=\frac{PL}{AE}$ ft. $=\frac{30\times100}{\cdot01036\times30\times10^6}=\cdot00965$ ft.

Correction for sag $=\frac{L\,.\,W^2}{24P^2}=\frac{100\times(3\cdot4916)^2}{24\times30^2}=\cdot05644$ ft.

Correction for temperature $=(L\,.\,a\,.\,16°)=(100\times\cdot00000625\times16)=\cdot01000$ ft.

The distance between the scratches was therefore 100 + ·00965 − ·05644 + ·01000 = 99·9632 ft.

Field Tape during Standardisation.—For the purposes of the corrections, the ength may be assumed to be 100 ft. at 60° F., so that

$$\text{Correction for pull} = \frac{PL}{AE} \text{ ft.} = \frac{20 \times 100}{\cdot 00439 \times 30 \times 10^6} = \cdot 0152 \text{ ft.},$$

$$\text{Correction for sag} = \frac{100 \times (1\cdot 4759)^2}{24 \times 20^2} = \cdot 0227 \text{ ft.},$$

Correction for temperature = L . a . 12° = ·0075 ft.

If L is the true length of the field tape at 60° F., then L + ·0152 − ·0227 + ·0075 = 99·9632 + ·0455 ft.

$$\therefore \text{L} = 100\cdot 0087 \text{ ft.}$$

But 1° F. corresponds to a difference of 100 × ·00000625 = ·000625 ft., so that ·0087 is equivalent to a temperature effect of 14°.

That is, the field tape is of the correct length under no pull at a temperature of (60 − 14)° = 46° F.

Measurement of Base.

Correction for pull = 19 × ·0152 = ·2888.

Correction for sag = 19 × ·0227 = ·4313.

Correction for temperature (going out) = 1900 × ·00000625 × (65·4 − 46) = ·2304.

Correction for temperature (returning) = 1900 × ·00000625 × (63·5 − 46) = ·2078.

Correction for inclination. In the first bay the difference in level between the pegs = 0·35 ft. Therefore correction in 100 ft. $= \frac{1}{2} \times \frac{(\cdot 35)^2}{100} = \cdot 00061$.

Similarly the other corrections for slope yield a total of ·1186 ft.

Reduction to Sea-Level. The height of the base above mean sea-level being about 100 ft., the correction to be applied

$$= \frac{100 \times 1900}{20{,}890{,}000} = \cdot 0091 \text{ ft.}$$

The total corrections may now be tabulated :

	Going out	Returning
Tension	+ ·2888	+ ·2888
Sag	− ·4313	− ·4313
Temperature and Standard .	+ ·2304	+ ·2078
Inclination	− ·1186	− ·1186
Altitude	− ·0091	− ·0091
Apparent length . . .	1900·0000	1900·0360
Corrected length . . .	1899·9602	1899·9736

Mean = 1899·9669 ft.

The difference between the two measurements is thus ·0134, and the "accuracy" is usually expressed by the fraction

$$\frac{\cdot 0134}{3800} = \frac{1}{353{,}000} \text{ nearly.}$$

Applying formula 10 on p. 532 we get a probable error of

$$\pm \frac{\cdot 674 \sqrt{\frac{2 \times (\cdot 0067)^2}{2 \times 1}}}{1899\cdot 9669} = \pm \frac{1}{523{,}000} \text{ approximately.}$$

It must be noticed, however, that

(1) The theory of errors is not reliable when applied to a small number of values—in this case only two.

(2) The figure $\frac{1}{523{,}000}$ only takes into account compensating errors. It

gives some idea of the accuracy with which the actual work was carried out, but is no criterion as to the true length of the base. For instance, the same figure would be obtained were there an appreciable error in the standard tape, or if the assumed altitude were incorrect, etc.

Example 2.—If the spring balance used in the example above was incorrect, and recorded 1 lb. in excess of the true pull in each case, what would be the resulting error in the computed length of the base ?

Standard Tape.—The applied correction for pull will be too large by an amount

$$\frac{\delta P}{P} \times \cdot 00965 = \frac{1}{30} \times \cdot 00965 = \cdot 00032 \text{ ft.}$$

The applied correction for sag will be too small by an amount

$$\left(\frac{2\delta P}{P} + \frac{3 \,.\, \delta P^2}{P^2}\right) \times \cdot 05644 = \cdot 00395 \text{ ft.}$$

The apparent distance between the marks on the pegs will therefore be too large by (·00032 + ·00395) = ·00427 ft.

This would make the apparent length of the field tape ·00427 ft. too great.

Field Tape.—The applied correction for pull will be too large by an amount

$$\frac{1}{20} \times \cdot 0152 = \cdot 00076 \text{ ft.}$$

The applied correction for sag will be too small by an amount

$$\left(\frac{2}{20} + \frac{3}{400}\right) \times \cdot 0227 = \cdot 00244 \text{ ft.}$$

In the comparison with the standard, both these errors would tend to make the apparent length of the field tape too small, and the computed length of the tape would be ·00320 too small.

The combined effect of the errors in the two tapes is to make the apparent length of the field tape ·00427 − ·00320 = ·00107 ft. too great.

In measuring the base we now have two sources of $+^{ve}$ error :

(*a*) From mistaken length of field tape	= 19 × ·00107 =	·0203
(*b*) From corrections for tension and sag	= 19 × ·00320 =	·0608
The apparent length of the base is thus too long by		·0811 ft.

A probable error of ± 1 lb. in the pull would produce an error of about ± ·022 ft. See also Example 5, p. 431.

Extension of a Base.—In the case of topographical triangulation, the lengths of the sides of the triangles are not excessively large, being often of the same order as the base itself. Consequently in such cases it is usually possible to employ the measured base directly as a side of one of the triangles.

In a primary triangulation, however, the sides of the main triangles are 30 to 60 or more miles in length, while it is not usual to measure directly a longer base than 6 to 10 miles. For one reason, it is often difficult to secure a suitable site upon which a longer base can be measured ; but even when it is quite possible, it is not advisable, as an accurately measured base of shorter length can be extended by means of a subsidiary triangulation with as great a degree of accuracy as a long line can be measured directly.

There are several methods of arranging this subsidiary triangulation.

(1) To prolong the base AB. On each side of AB (Fig. 14.6) choose

stations C and D, which are clearly visible from A and B, and which form well-proportioned triangles. Then in the line AB prolonged, choose a favourable position E from which C and D are both visible. and which forms well-shaped triangles ACE and ADE. Thus if the angles at A, C, D, and E are approximately 45°, and those at B are about 90°, both sets of triangles (on AB and on AE) will be well proportioned.

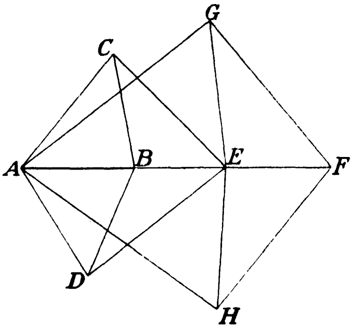


FIG. 14.6.—Verification Triangulation of a Base Line

The theodolite is then set up at each of the stations A, B, C, D, E, in turn, and the angles subtended between all the other stations are observed.

From the triangle ABC, the three angles being known, and the side AB measured, CB can be calculated ; after which BE can be calculated from the triangle CBE.

Similarly BD can be determined and BE calculated from the triangle BDE.

Also if AC is calculated from the triangle ABC, AE, and hence BE, can be found from the triangle ACE. Similarly AD can be calculated, and AE and BE found by solving the triangle ADE.

Thus a number of values for BE can be obtained, and the mean of these will enable the length to be determined with considerable accuracy.

The base may then be further extended to F, and the process repeated as many times as is required.

The method shown in Fig. 14.6 may also be employed as a check upon the accuracy of a base line ; *e.g.* let AE represent a section of the measured base, and B be a point on this, whose position was determined during the measurement. Then, assuming the length AB to be correct, the length BE can be calculated through the triangles ACB, ACE, ADB, ADE, as explained above.

A comparison between the measured and calculated values of BE then affords a check upon the accuracy of both AB and BE.

(2) A more common method of extension is shown in Fig. 14.7, where AB represents the measured base as before.

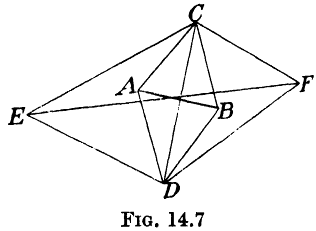


FIG. 14.7
Extension of a Base Line

Two stations C and D, visible from A and B, and from each other, are chosen on opposite sides of AB, to form well-proportioned triangles—in this case, as nearly equilateral as possible because there is no point upon the extension of the base to be considered as in Fig. 14.6. The angles at each of the four points, subtended by the remaining three, are then observed ;

the length CD calculated from the different triangles, and a mean result adopted.

Then, by the use of such stations as E and F, the line CD may be extended in a similar manner ; and the distance EF computed.

Proceeding in this way, a suitable extended base may be determined, its length being comparable with those of the sides of the primary triangles, and the positions of its extremities such that they are adapted for the observation of the primary triangulation angles.

The exact method of triangulating depends very largely upon the circumstances of each particular case, and it is impossible to lay down any exact method of procedure to be adopted in every case.

It should be noted, however, that the whole extension must be treated as one problem, and not as a series of separate problems. Thus C and D should not be finally fixed until the positions of E, F, and further points have been definitely located, because it is important that the final points should be as favourably situated as possible with reference to the main triangulation.

Fig. 13.4 shows the manner in which the Kroonstad Base of the Transvaal and Orange River Colony Survey was extended.

Fig. 13.5 shows the manner in which the Sironj base of the Survey of India was extended.

Accuracy.—The chief sources of error in base-line measurement may be summarised :

(1) Error in the standard measure or reference tape or bar.

(2) Error in deducing the length of the field tape or measuring bars from the reference, and in deducing the length of the reference from the standard measure.

(3) Error due to the pull in a tape not being exactly that allowed for in the calculations ; or the pressure at contact in the case of rods being variable.

(4) Error in the determination of the cross-section of a tape, affecting the correction for pull.

(5) Error due to the incorrect determination of temperature :

(*a*) In standard measure,
(*b*) In reference measure,
(*c*) In field measure.

This error may occur in the comparison of the reference with the standard, or that of the field measure with the reference, or in the actual field operations.

(6) Error due to the coefficients of expansion of the various metals not being accurately known.

(7) Errors in alignment.

(8) Errors in the determination of the inclination of the line.

(9) Errors in reduction to mean sea-level.

(10) Personal errors in reading scales, bisecting and marking terminal dots and scratches, etc.

(11) Displacement of the marking pegs in tape measurements, unless straining pegs are provided.

The following are a few examples of the accuracy obtained in various cases :

The p.e. of seven bases in Russia,[1] measured with Struve's bars, varied from $\pm 0{\cdot}73\mu$ to $\pm 0{\cdot}91\ \mu$.

The Belgian [1] bases at Beverloo and Ostend, measured with Bessel's apparatus, gave p.e.'s of $\pm 0{\cdot}59\ \mu$ and $\pm 0{\cdot}45\ \mu$ respectively in lengths of 2300 and 2488 metres.

The Atalanta [1] base in Georgia (U.S.A.) was measured three times with the U.S. apparatus, p. 414. The line was divided into six segments, each about 1 mile in length. The p.e. was calculated to be $\pm 1{\cdot}76\ \mu$. For seven previously measured bases the p.e. varied from $\pm 1{\cdot}8\ \mu$ to $\pm 2{\cdot}4\ \mu$.

From a number of measurements of the Cape Comorin [2] Base, 1·7 miles in length, in South India, it was deduced that the p.e. (*exclusive* of all constant errors) for a single measurement with the Colby apparatus was $\pm 1{\cdot}5\ \mu$ under ordinary conditions, but that if thermometers were let into the bars, and the differences in temperature between the two components determined in that way, a p.e. of $\pm 0{\cdot}5\ \mu$ could be determined.

An analysis of the p.e.'s deduced for the five bases of the Geodetic Survey of the Transvaal and Orange River Colony [3] is given below. They were measured with invar tapes, strained with a pull of 10 kilogrammes.

Base	Length in Miles	No. of Sections	Probable Effect of Systematic and Accidental Errors in Measurement	Probable Effect of Error in Height	Probable Effect of Systematic Errors of Bars and Thermometers	Probable Error of Base
Belfast .	11·8	8	$\pm 0{\cdot}725\mu$	$\pm 0{\cdot}278\mu$	$\pm 1{\cdot}065\mu$	$\pm 1{\cdot}318\mu$
Ottoshoop .	10·8	8	$\pm 0{\cdot}526\mu$	$\pm 0{\cdot}383\mu$	$\pm 1{\cdot}065\mu$	$\pm 1{\cdot}248\mu$
Wepener .	13·5	6	$\pm 0{\cdot}295\mu$	$\pm 0{\cdot}484\mu$	$\pm 1{\cdot}065\mu$	$\pm 1{\cdot}206\mu$
Kroonstad .	12·3	6	$\pm 0{\cdot}311\mu$	$\pm 0{\cdot}436\mu$	$\pm 1{\cdot}065\mu$	$\pm 1{\cdot}192\mu$
Houts River	21·1	8	$\pm 0{\cdot}526\mu$	$\pm 0{\cdot}354\mu$	$\pm 1{\cdot}065\mu$	$\pm 1{\cdot}239\mu$

In the measurement of the geodetic [4] arc of the 30th meridian west of Lake Victoria, a base $16\frac{1}{2}$ kilometres or $10\frac{1}{4}$ miles long was measured with invar wires. The line was subdivided into sections of about 1 kilometre, and each was measured twice, employing different wires. Some sections were measured three times.

The computed p.e. of the base, taking into account all sources of error, was $\pm 14{\cdot}92$ mm., or 1 in 1,108,000, *i.e.* $\pm 0{\cdot}90\ \mu$.

In the Lossiemouth [5] base, measured in 1909, with 100-ft. invar

[1] Clarke, *Geodesy*. [2] *Report of G.T. Survey of India.*
[3] *Report on G. Survey of S. Africa*, vol. iii.
[4] *Brit. Assoc. Report*, 1910.
[5] *Ordnance Survey Professional Papers*, New Series, No. 1.

tapes, aligned with 5-in. transit theodolites, the p.e. was ± 1 in 900,000, or $\pm 1{\cdot}11\ \mu$. The total length was nearly 23,526 ft.

EXAMPLES

1. (U. of L.) Describe, in detail, the field operations necessary for measuring a long base line with extreme accuracy by means of a steel tape or wire. Enumerate the corrections that must be made.

A line, 2 miles long, is measured with a tape of length 300 ft., which is standard under no pull at 60° F. The tape in section is $\frac{1}{8}$ in. wide and $\frac{1}{20}$ in. thick. If one half of the line is measured at a temperature of 70° F. and the other half at 80° F., and the tape is stretched with a pull of 50 lbs., find the correction on the total length. Coefficient of expansion $= 0{\cdot}0000065$; weight of 1 cub. in. of steel $= 0{\cdot}28$ lb. ; $E = 29{,}000{,}000$ lbs. per square inch.

2. (U. of L.) A steel measuring band 200 ft. long is used for measuring a line about 1000 ft. in length. The band is held clear of the ground at every setting, and the ends are supported on pegs at the same level and are pulled by hand, through a damaged spring balance, the readings on which are uncertain, but it is definitely known that the pull is not less than 30 lbs. nor more than 50 lbs. The operation is repeated five times. Within what limits is the length known after allowing for the elastic stretch of the tape and for the sag ?

Width of band, 0·20 in.
Thickness of band, 0·02 in.
Young's Modulus of Elasticity, 25,000,000 lbs. per square inch.

3. (I.C.E.) A steel tape is 100 ft. long between the end graduations at a temperature 60° F. when lying horizontally on the ground. Its sectional area is 0·0103 sq. in., its weight is 3·49 lbs., and the coefficient of expansion is 0·0000065 for 1° F. The tape is stretched over two supports, approximately 100 ft. apart, and is also supported in the middle, the three supports being at the same level. Calculate the actual length between the end graduations under the following conditions :

$$\text{Temperature} = 76^\circ\ \text{F., pull on tape} = 20\ \text{lbs.}$$

$$\text{Sag correction} = \frac{W^2L}{24P^2}\ \text{ft.,}$$

where W = weight of "unsupported" length in feet, L of the tape, P = pull in lbs.

$$\text{Pull correction} = \frac{P \times 100}{aE}\ \text{ft.,}$$

where a = sectional area in square inches, E = 30,000,000 lbs. per square inch.

4. (U. of B.) A base line was measured and found to be 4518·36 ft. long before any corrections were applied. The measuring tape, supposed 100 ft., was 0·03 ft. long on the reference tape at 69° F., while the reference tape was 0·04 ft. short of standard at 63° F.

The temperature of base during measurement was 73° F., and the height of the base above sea-level was 4520 ft.

For the first 2000 ft. the base had a down slope of 1°, and for the remainder an upward slope of 1°-30′.

Find the true length of the base line.

The same pull, namely 20 lbs., was applied during measurement and standardisation.

The weight of the tape was $1\frac{1}{2}$ lbs., and the radius of the earth may be taken as 20,900,000 ft.

5. What errors in the computed length of the base line would result from the following $+^{ve}$ errors in the data of the example on p. 425 ?

(i) 1° F. in the temperature of the standard tape (*i.e.* if the true temperature was 75° F. instead of 76° F. as recorded).

(ii) 1° F. in the temperature of the field tape during standardisation.
(iii) 1° F. in the temperature of the field tape during the measurement of the base.
(iv) 10^6 lbs. in the value of E (*a*) in the standard tape, (*b*) in the field tape.
(v) 30 ft. in the assumed height of the base above sea-level.
(vi) 0·05 ft. in determining the difference of level of the first bay pegs.
(vii) A negative error of ·00000025 in the value of α (*a*) for the standard tape. (*b*) for the field tape.

CHAPTER XV

SPHERICAL TRIGONOMETRY

It is not proposed in the present chapter to attempt much more than the brief proofs of those formulae for the solution of spherical triangles which are involved in the determination of azimuth, latitude, etc. For further information the reader is referred to any of the well-known treatises upon Spherical Trigonometry.

A spherical triangle is that triangle which is formed upon the surface of a sphere by the intersection of three "great" circles. There are eight such figures formed on the sphere by the three circles, and that which is considered is the one which has two if not three sides less than a quadrant: or, otherwise, if three planes meet at a point O, the intercepted space is termed a solid angle as in Euclid XI; so that if a sphere is then described with O as centre, the traces of the three sides of the solid angle upon the surface of the sphere will form a spherical triangle.

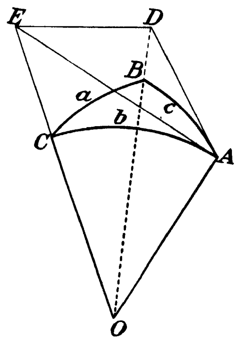


FIG. 15.1
Spherical Triangle

Before proceeding to the derivation of the required formulae, it may be useful to state the following theorems:

(1) The arc of a great circle is the shortest distance between any two points upon the surface of a sphere.

The trace of any plane which cuts the sphere is a circle; and the *greatest* circle is that which has a radius equal to that of the sphere, *i.e.* a great circle. Consequently, of all planes which can pass through two points A and B on the surface of a sphere, that which passes through the centre of the sphere traces the arc of least curvature between A and B, and this is obviously the shortest line.

Thus, as no parallel of latitude, except the equator, is a great circle, the shortest distance between two points A and B of the same latitude is *not* along the parallel, but along a great circle through A and B (see Example, p. 441).

(2) Any two sides of a spherical triangle are together greater than the third, because any two of the plane angles which form the solid angle at O are together greater than the third (Euclid XI. 20). Consequently, as the arcs AB, BC, CA, are proportional to the magnitudes of these angles, any two are greater than the third.

(3) The greater angle is opposite the greater side, and ***vice versa.***

This may be proved from the equations deduced below, *e.g.* the sine rule (4).

Let ABC (Fig. 15.1) be a spherical triangle, and O the centre of the sphere. The " parts " of the triangle are then

(*a*) The three spherical angles A, B, and C, measured in planes tangential to the sphere at those points.
(*b*) The three sides, *i.e.* the arcs BC, CA, and AB, denoted by the small letters *a*, *b*, and *c*, respectively, as in plane trigonometry, and expressed in terms of the angles BOC, COA, AOB, subtended at the centre O of the sphere.
(*c*) The area S of the triangle ABC.

Draw AE and AD at right angles to OA, *i.e.* tangent to the sphere at A, and in the planes OAC and OAB respectively.

Join OB and produce this line to meet AD in D, and similarly join OC and produce this line to meet AE in E.

Then the angle EAD is the angle A of the spherical triangle.

Now in the triangle DOE we have by plane trigonometry

$$DE^2 = OD^2 + OE^2 - 2 \,.\, OD \,.\, OE \,.\, \cos EOD. \quad (1)$$

Similarly, in the triangle DAE we have

$$DE^2 = AD^2 + AE^2 - 2 \,.\, AD \,.\, AE \,.\, \cos EAD. \quad (2)$$

Subtracting (2) from (1)

$$(OD^2 - AD^2) + (OE^2 - AE^2) + 2 \,.\, AD \,.\, AE \cos EAD - 2 \,.\, OD \,.\, OE \,.\, \cos EOD = 0.$$

But $OD^2 - AD^2 = OA^2 = R^2$, where R is the radius of the sphere. Similarly $OE^2 - AE^2 = R^2$.

Also $AD = R \,.\, \tan AOD$, *i.e.* $R \tan c$, $OD = R \sec c$, $AE = R \tan b$, and $OE = R \sec b$, because OAD and OAE are both right angles.

Substituting in the above equation,

$$R^2 + R^2 + 2R^2(\tan c \tan b \cos A - \sec b \sec c \cos a) = 0,$$

or cancelling $2R^2$,

$$1 + \tan c \tan b \cos A - \sec b \sec c \cos a = 0,$$

or multiplying throughout by $\cos b \cos c$,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \quad (3)$$

By transposing equation (3) we get

$$\cos A = \frac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c}.$$

But

$$\sin^2 A = 1 - \cos^2 A$$
$$= 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2$$
$$= \frac{\sin^2 b \sin^2 c - \cos^2 a + 2 \cos a \cos b \cos c - \cos^2 c \cos^2 b}{\sin^2 b \sin^2 c}.$$

Substituting in the numerator $(1-\cos^2 b)$ for $\sin^2 b$ and $(1-\cos^2 c)$ for $\sin^2 c$, and simplifying,

$$\sin^2 A = \frac{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}{\sin^2 b\sin^2 c},$$

$$\therefore \frac{\sin A}{\sin a} = \frac{\sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}}{\sin a\sin b\sin c}.$$

But the expression on the right-hand side of this equation is symmetrical in a, b, c, so that if these values are interchanged in equation (3), we shall get exactly the same expression for $\frac{\sin B}{\sin b}$ and for $\frac{\sin C}{\sin c}$,

$$\therefore \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \qquad (4)$$

This equation is known as the Sine Formula, or the Law of Sines. By transposing from equation (3) again,

$$\cos A = \frac{\cos a-\cos b\cos c}{\sin b\sin c},$$

$$\therefore 1-\cos A = \frac{\sin b\sin c+\cos b\cos c-\cos a}{\sin b\sin c}$$

$$= \frac{\cos(b-c)-\cos a}{\sin b\sin c}.$$

Applying the plane trigonometrical formula
$\cos S-\cos T = -2\sin\frac{S+T}{2}\sin\frac{S-T}{2}$, where $S=(b-c)$ and $T=a$,

$$1-\cos A = \frac{-2 \,.\, \sin\frac{b-c+a}{2}\sin\frac{b-c-a}{2}}{\sin b\sin c},$$

or writing $2\sin^2\frac{A}{2}$ for $(1-\cos A)$, dividing by 2, and writing s for $\frac{a+b+c}{2}$,

$$\sin^2\frac{A}{2} = \frac{\sin(s-c)\sin(s-b)}{\sin b\sin c}. \qquad (5)$$

Similarly $$1+\cos A = \frac{\sin b\sin c-\cos b\cos c+\cos a}{\sin b\sin c}$$

$$= \frac{\cos a-\cos(b+c)}{\sin b\sin c}.$$

Applying the formula $\cos S - \cos T = -2 \sin \frac{S+T}{2} \sin \frac{S-T}{2}$, and writing $\cos^2 \frac{A}{2}$ for $\frac{1}{2}(1 + \cos A)$, and s for $\frac{a+b+c}{2}$,

$$\cos^2 \frac{A}{2} = \frac{\sin s \sin (s-a)}{\sin b \,.\, \sin c}. \qquad (6)$$

Dividing (5) by (6) we get

$$\tan^2 \frac{A}{2} = \frac{\sin (s-c) \sin (s-b)}{\sin s \sin (s-a)}. \qquad (7)$$

From equation (3)

$$\cos a - \cos b \cos c = \sin b \sin c \cos A,$$

and similarly $\cos b - \cos a \cos c = \sin a \sin c \cos B$,

or writing $\sin b = n \sin B$, and $\sin a = n \sin A$,

$$\cos a - \cos b \cos c = n \sin B \sin c \cos A,$$

and $\cos b - \cos a \cos c = n \sin A \sin c \cos B.$

By adding we get

$(\cos a + \cos b)(1 - \cos c) = n \,.\, \sin c \,.\, \{\sin B \cos A + \sin A \cos B\}$,

or substituting $\frac{\sin a + \sin b}{\sin A + \sin B}$ for n $\left[\text{as } \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin a \pm \sin b}{\sin A \pm \sin B} = n\right]$,

$$(\cos a + \cos b)(1 - \cos c) = \frac{\sin a + \sin b}{\sin A + \sin B} \,.\, \sin c \,.\, \sin (A+B),$$

or
$$\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{\sin A + \sin B}{\sin (A+B)} \,.\, \frac{(1 - \cos c)}{\sin c}.$$

Applying the plane trigonometry formulae, $\sin S + \sin T = 2 \sin \frac{S+T}{2} \cos \frac{S-T}{2}$ and $\cos S + \cos T = 2 \cos \frac{S+T}{2} \cos \frac{S-T}{2}$,

$$\frac{2 \sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)}{2 \cos \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)} = \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A+B)} \cdot \frac{2 \sin^2 \frac{1}{2} c}{2 \sin \frac{1}{2} c \,.\, \cos \frac{1}{2} c},$$

therefore cancelling

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \frac{c}{2}. \qquad (8)$$

Similarly if $\frac{\sin a - \sin b}{\sin A - \sin B}$ is substituted for n above,

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \frac{c}{2}. \qquad (9)$$

Another formula which is sometimes required is

$$\cot A \sin B = \cot a \sin c - \cos B \cos c.$$

This may be derived as follows:

$$\cot A \sin B = \frac{\cos A \sin B}{\sin A} = \frac{\cos A \sin b}{\sin a}$$

$$= \frac{(\cos a - \cos b \cos c) \sin b}{\sin a \sin b \sin c}.$$

Multiplying $\cos a$ in the numerator by $(\sin^2 c + \cos^2 c)$, *i.e.* 1, and cancelling $\sin b$, we get

$$\cot A \sin B = \frac{\cos a\,(\sin^2 c + \cos^2 c) - \cos b \cos c}{\sin a \sin c}$$

$$= \cot a \sin c - \cos B \cos c. \qquad . \qquad . \quad (10)$$

When the angle A is 90° this reduces to

$$\cos B = \cot a \tan c. \quad . \qquad . \qquad . \quad (10a)$$

The above formulae are collected and tabulated with others in the Appendix.

Spherical Excess.—The three angles of a spherical triangle do not sum exactly to 180°, as do those of a plane triangle, but their sum exceeds two right angles by an amount known as the Spherical Excess.

The magnitude of this spherical excess is expressed in degrees by the formula

$$e = \frac{\text{area of triangle}}{\pi R^2} \times 180°, \quad . \quad (11)$$

where R is the radius of the sphere.

This may be proved as follows:

Let APTV and AQTU (Fig. 15.2) be two intersecting great circles, and let PQVU be a third great circle, at right angles to the planes of the first two.

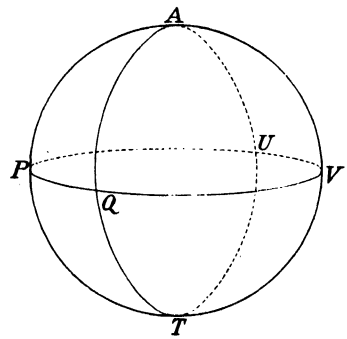


FIG. 15.2

Then the arc PQ is a measure of the spherical angle at A.

The surface area S′ of the whole sphere is $4\pi R^2$, and consequently the area of the portion APTQA is

$$\frac{PQ}{2\pi R} \cdot S' \cdot = \frac{A}{2\pi} S' \text{ where A is in radians}$$

$$= \frac{A}{360} \cdot S' \text{ where A is in degrees.}$$

Now, in a general case when three great circles intersect to form a spherical triangle ABC (Fig. 15.3), the whole surface of the sphere is divided into eight divisions, and, owing to the symmetry of the figure, the four on the one hemisphere are similar to the four on the opposite hemisphere.

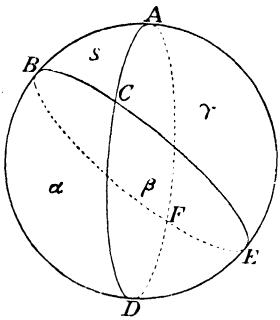


FIG. 15.3

From the relationship shown above, it is easily seen that in the figure where the area ABC = S, BCD = α, CDE = β, and ECA = γ,

$$S+\alpha=\frac{A}{360}\cdot S',$$

$$S+\gamma=\frac{B}{360}\cdot S',$$

$$S+\beta=\frac{C}{360}\cdot S', \text{ because FDE} = \text{ABC} = \text{S}.$$

Therefore, by addition,

$$2S+(S+\alpha+\beta+\gamma)=\frac{S'}{360}(A+B+C).$$

But $S+\alpha+\beta+\gamma$ comprises the area of a hemisphere, *i.e.* $\frac{S'}{2}$,

$$\therefore\ 2S+\frac{S'}{2}=\frac{S'}{360}(A+B+C),$$

or

$$2S=\frac{S'}{360}(A+B+C-180°).$$

Again, $A+B+C-180°$ is the spherical excess e of the triangle,

$$\therefore\ e=\frac{2\,.\,S\,.\,360}{S'},$$

$$e=\frac{\text{area of triangle}}{\text{area of quadrant}}\,.\,180°$$

$$=\frac{S}{\pi R^2}\,.\,180°.$$

For the application of this formula see Chapter XIV, p. 399.

A few miscellaneous applications of Spherical Trigonometry—apart from the astronomical observations in the following chapters—will now be considered.

Convergence of the Meridians.—If A and B (Fig. 15.4) represent two positions upon the earth's surface, the straight line AB, being the shortest distance between the two points, is an arc of a great circle.

The **Azimuth** of B from A is the angle included between the plane of this great circle and the plane of the meridian at A, while the azimuth of A from B is the angle included between the plane of the great circle through AB and the plane of the meridian at B.

But the meridians through A and B are not parallel: they converge and intersect at the pole P, and the two angles A and B of the spherical triangle PAB are not together equal to 180°. In other words, the azimuth of AB, as determined at A, is not identical with the azimuth of AB (not BA), as determined at B.

The **Bearing** of B from A is the angle included between the plane of the great circle through AB and that of *any* other standard great circle. This " standard " great circle may be the meridian through A, in which case the bearing of B from A is identical with the azimuth of B from A.

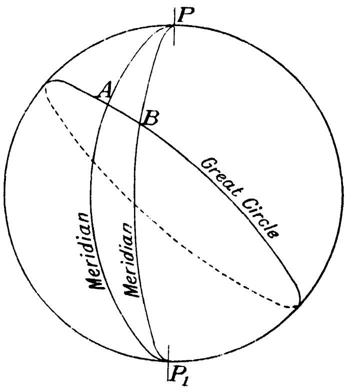


FIG. 15.4
Convergence of the Meridian

But the bearing of A from B is the angle made by BA with a plane, through B, *parallel* to that of the " standard " great circle through A. Consequently it is not equal to the *azimuth* of A from B, unless A and B both lie upon the same meridian, or both upon the equator.

The bearing of AB is identical whether determined from A or from B.

Suppose a traverse or triangulation survey commences with a line AN, and the azimuth (or bearing) of this from the meridian at A is determined. Then, after working through a considerable distance, it is desired to check the accuracy of the work by means of an astronomical observation of the azimuth of a line, say BM.

The *bearing* of BM with reference to an axis through B parallel to the original meridian AP is calculated by the method of co-ordinates as explained in Chapter V; but the *azimuth*, as determined by an astronomical observation, refers not to this axis, but to a meridian through B, which converges and meets the original meridian at the poles.

To make use of the check, therefore, it is necessary to compute the angle between the true meridian through B and that line through B which is parallel to the original meridian AP.

Thus let the latitude of $A = l$, and that of $B = l_1$, and let the difference in longitude be l_2.

Then the " convergence " of the meridians is the difference between 180° and A + B *i.e.* $c_1 = 180° - (A + B)$, where A and B are the angles of the spherical triangle formed by AB and the two meridians AP and BP.

The known data in the triangle are, therefore,

$$\angle P = l_2$$
$$AP = 90° - l$$
$$BP = 90° - l_1$$

By the application of formula 7, p. 544,

$$\tan\frac{A+B}{2}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}P,$$

i.e.
$$\tan\frac{A+B}{2}=\frac{\cos\frac{1}{2}(90-l_1-90+l)}{\cos\frac{1}{2}(90-l_1+90-l)}\cot\frac{l_2}{2},$$

i.e.
$$\cot\left(90-\frac{A+B}{2}\right)=\frac{\cos\frac{1}{2}(l-l_1)}{\sin\frac{1}{2}(l+l_1)}\cot\frac{l_2}{2},$$

i.e.
$$\cot\frac{c_1}{2}=\frac{\cos\frac{1}{2}(l-l_1)}{\sin\frac{1}{2}(l+l_1)}\cot\frac{l_2}{2},$$

or
$$\tan\frac{c_1}{2}=\tan\frac{l_2}{2}\cdot\frac{\sin\frac{1}{2}(l+l_1)}{\cos\frac{1}{2}(l-l_1)}.$$

When AB is small compared with the radius of the earth, we may substitute for the tangent of the angle the value of the angle in radian measure, *i.e.*

$$c_1=l_2\cdot\frac{\sin\frac{1}{2}(l+l_1)}{\cos\frac{1}{2}(l-l_1)}. \qquad (12)$$

This equation is equally true when c_1 and l_2 are both expressed in minutes, or both in seconds, as this result is obtained by multiplying each side of the equation by the same factor.

When A and B lie upon the same parallel, *i.e.* when $l=l_1$, the equation reduces to

$$c_1=l_2\,.\,\sin l. \qquad (12a)$$

When A and B lie upon the same meridian $l_2=0$, and $c_1=0$. Also when A and B lie upon the equator $l=l_1=0$, and hence from equation (12) $c_1=0$.

c_1 increases in value as l and l_1 approach 90°, and tends to the limit $c_1=l_2$.

If the latitude and longitude of the first point A are known, those of the last point B can be calculated from the "latitudes" and "departures" (or "co-ordinates") by means of specially prepared tables.

These may give (for various latitudes, from the equator to the poles) the lengths in feet or other units, of 1 minute or 1 second of arc in both latitude and longitude. The values tabulated will depend upon the particular spheroid which has been adopted (see p. 443). For small areas it may be sufficient to consider the earth as a sphere, and to calculate the required values upon this assumption.

According to Clarke's spheroid, as adopted by the U.S. Coast and Geodetic Survey, 1 minute of latitude varies from 6045·95 ft. at the equator to 6107·85 ft. at the pole, while 1 minute of longitude varies from 6087·15 ft. at the equator to nil at the pole.

A rough value is that a minute of latitude or a minute of longitude at the equator corresponds to an arc of 1·15 miles.

Example.—In a survey conducted from A in lat. N. 48°-35′-5″ and long. 118°-54′-6″ W. to a point E, lat. N. 48°-57′-25″, long. 119°-15′-12½″ W., what would be the theoretical difference between the azimuth of the final line at E, determined astronomically, and the bearing as calculated by means of co-ordinates from the meridian at A ?

In formula (12) $l_2 = 21'\text{-}6\tfrac{1}{2}'' = 1266''\cdot 5,$

$$\tfrac{1}{2}(l + l_1) = 48°\text{-}46'\text{-}15'',$$
$$\tfrac{1}{2}(l - l_1) = 11'\text{-}10'',$$

$$\log 1266\cdot 5 = 3\cdot 10261$$
$$\log \sin 48°\text{-}46'\text{-}15'' = \bar{1}\cdot 87626$$
$$\log \sec 11'\text{-}10'' = \cdot 000002$$

$$\log c_1 = 2\cdot 97887$$
$$\therefore c_1 = 952\cdot 42 \text{ seconds}$$
$$= 15'\text{-}52\tfrac{1}{2}'' \text{ about}$$

Example (Lond. B.Sc.).—At a point A in latitude 50° N, a straight line is ranged out, which runs due east at A. This straight line is prolonged for 60 nautical miles to B. Find the latitude of B, and, if it be desired to travel due north from B so as to meet the 50° parallel again at C, find the angle ABC at which we must set out, and the distance BC.

(*N.B.—A nautical mile subtends one minute at the earth's centre. Regard the earth as a sphere.*)

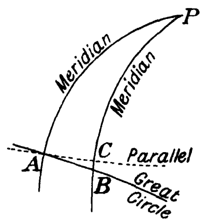


FIG. 15.5

The line AB (Fig. 15.5) being a straight line is a portion of a great circle, and subtends 60′ = 1° at the centre of the earth.

At the point A it makes an angle of 90° with the meridian AP, where P represents the Pole.

The distance BC, being due north and along a straight line, is a portion of the great circle, *i.e.* of the meridian, through B. PAB is therefore a spherical triangle.

The arc PB may be calculated by means of equation 3, p. 434, because AP = 90° − 50° = 40°, AB = 1°, and ∠A = 90°,

$$\therefore \cos PB = \cos 40° \cos 1° + \sin 40° \sin 1° \cos 90°$$
$$= \cos 40° \cos 1° \text{ as } \cos 90° = 0$$
$$\log \cos 40° = \bar{1}\cdot 8842540$$
$$\log \cos 1° = \bar{1}\cdot 9999338$$

$$\log \cos PB = \bar{1}\cdot 8841878$$

and $PB = 40°\text{-}0'\cdot 624$ nearly.

But $PC = 40°,$

$$\therefore BC = 0'\cdot 624,$$

which represents ·624 nautical mile.

The angle at B may be found by the application of the sine formula, *i.e.* equation 4, p. 435.

$$\frac{\sin B}{\sin 40°} = \frac{\sin 90°}{\sin 40°\text{-}0'\cdot 624}.$$
$$\log \sin 90° = 0\cdot 0000000$$
$$\log \sin 40° = \bar{1}\cdot 8080675$$
$$\log \operatorname{cosec} 40°\text{-}0'\cdot 624 = 0\cdot 1918386$$

$$\log \sin B = \bar{1}\cdot 9999061$$
$$B = 88°\text{-}48'\text{-}40''$$

EXAMPLES

1. In the example on p. 441 find the difference in longitude between A and B, and from this deduce the convergence of the meridians, and the value of the angle at B.

2. (U. of L.) If, starting from a point in latitude 50° north, it is desired to lay out a "straight line" on the earth's surface, which will be again in latitude 50° at the end of 60 nautical miles, in which direction will it be necessary to proceed? Determine the angle with the meridian of the starting-point. At the end of thirty nautical miles along this line, how far (in nautical miles) would the observer be from (*a*) the 50° parallel of latitude, (*b*) a "straight line" running due east (or west as the case may be) from the starting-point, assuming the earth spherical?

(*N.B.* One nautical mile subtends 1 minute at the earth's centre, and in spherical trigonometry if $C = 90°$, $\cos A = \tan b \cot c$; $\cos c = \cos a \cos b$; and $\sin a = \sin c \sin A$.)

3. (I.C.E.) What is the difference between "bearing" and "azimuth"?

Explain what course on the earth's surface would be followed by the following lines:

(*a*) A line laid out with a fixed azimuth of 85°.

(*b*) A line laid out with a fixed bearing of 85°. The starting-point of the line is in lat. 50° N. in each case.

4. (I.C.E.) A certain boundary line has been defined as a line running due east from a given point. Describe the operations needed so as to trace the line for 15 miles, and fix monuments on it at intervals of 1 mile.

5. One method of determining azimuth is from a knowledge of the hour angle (t), the latitude of the place (l) and the declination (δ) of an observed star (p. 476).

In the spherical triangle PZS, the angle $P = t$ is known, and the sides PZ and PS are $(90° - l)$ and $(90° - \delta)$ respectively. It is required to calculate the angle Z.

Z can conveniently be calculated by first determining an auxiliary angle θ from the equation

$$\tan \theta = \cot \delta \cos t$$

and then substituting in the equation

$$\cot Z = \cot t \cos (l + \theta) \operatorname{cosec} \theta.$$

Derive these equations.[1]

[1] The construction of two elegant circular nomograms to represent these equations is given in "The Nomogram" by H. S. Allcock and J. R. Jones. See also "Graphical Determination of Azimuth", by W. N. Thomas and D. N. Shorthose, *Proc. M. & Cy. E.*, vol. 48, p. 619.

CHAPTER XVI

ASTRONOMICAL TERMS

The Shape and Size of the Earth.—It has already been explained in Chapter VI that the figure of the earth as defined by the imaginary mean level of the sea is very irregular. This figure, termed the "geoid," has the distinctive property that its surface at any point lies in a plane tangential to the direction of gravity at that point.

To simplify calculations, however, some more or less regular figure must be assumed as an approximation, and consequently the globe is usually considered either as an oblate spheroid, or, for many purposes, as a sphere.

A *sphere* may be defined as that figure which is formed by the revolution of a circle or a semicircle about its diameter: every point on its surface is thus equidistant from the centre.

Similarly a *spheroid* is the figure traced by the revolution of an ellipse about one of its axes; when the major axis becomes the axis of revolution a *Prolate Spheroid* is formed, and when the minor axis is the axis of revolution an *Oblate Spheroid* is the result.

The earth, then, is often considered as an oblate spheroid. The dimensions used in the *Nautical Almanac* are those adopted by the Congrès International des Éphémérides Astronomiques (Paris, 1911), the semi-major axis of the ellipse, *i.e.* the equatorial radius a of the earth), according to Hayford's spheroid being 6378·388 km. and the semi-minor axis (*i.e.* the polar radius b of the earth) 6356·912 km.

The ellipticity is expressed by the ratio $(a-b)/a$ which reduces to 1/297·0; the eccentricity is expressed by the ratio $\sqrt{a^2-b^2}/a$, which reduces to 0·0820.

For the Survey of India, Everest's First Constants were used: $a=20{,}922{,}932$ feet, $b=20{,}853{,}375$ feet, and ellipticity 1/300·80.

Everest's Second Constants, though not used on the Indian Survey, give $a=20{,}920{,}902$ ft., and $b=20{,}853{,}642$ ft., the ellipticity being 1/311·04.

The standard of the U.S. Coast and Geodetic Survey is Clarke's spheroid, in which $a=20{,}926{,}062$ ft., and $b=20{,}855{,}121$, the ellipticity being 1/294·98.

In Clarke's later spheroid (1880) $a=20{,}926{,}202$ ft., and $b=20{,}854{,}895$, while in his first figure of 1858 $a=20{,}926{,}348$, and $b=20{,}855{,}233$, and the ellipticity 1/294·26.

During the years 1903–6 the U.S. Coast and Geodetic Survey calculated that the best values for the United States would be, $a=$

6,378,283 metres, with a probable error of ± 74 metres, and ellipticity of $1/(297{\cdot}8 \pm 0{\cdot}9)$.

When considered as a sphere the mean radius may be assumed as 20,890,000 feet.

Motion of the Earth.—The earth revolves about its minor or shorter axis, on an average, once in twenty-four hours. This axis is known as the Polar Axis, and the points at which it intersects the surface of the earth are termed the north and south *Geographical or Terrestrial Poles* ; while the trace upon the surface of the globe, which is made by any plane containing this axis, is termed a *meridian.* Thus, if the earth is considered as spherical, each meridian constitutes a complete circle, while if the earth is considered as spheroidal, each meridian traces an ellipse. The intersection with the surface of the globe, of any plane perpendicular to the polar axis, is a circle, whether the earth be considered as a sphere or as an oblate spheroid.

The particular plane which passes through the centre of the polar axis, and at right angles to it, is known as the *Equatorial Plane,* while its trace with the surface of the globe is known as the *Equator.*

If the earth is treated as a sphere, *any* plane passing through its centre traces out upon the surface a circle known as " a *great* circle ", and other planes trace out " small circles ".

In addition to the motion of rotation about its own polar axis, the earth has a motion of rotation relative to the sun, in a plane inclined at an angle of about $23°\text{-}27'$ to the plane of the equator, the time of a complete revolution being a year (see below).

According to **Kepler's First Law**, the earth's orbit or path is elliptical in shape, and the sun is so situated that it occupies a position in one focus of the ellipse.

When on December 31 the earth is at the nearest point of its orbit to the sun, it is said to be in *Perihelion* ; and when on July 1 it is at the most distant point of its orbit from the sun, it is said to be in *Aphelion.*

Kepler's Second Law states that the velocity of the earth in its orbit is not constant, but the motion is such that the radius vector, which joins the sun to the earth, sweeps over equal areas about the sun in equal intervals of time.

The phenomena of night and day are the result of the earth's rotation about its polar axis, while the " seasons " are due to the annual motion of the earth in its orbit round the sun.

Solar System.—Like the earth, other heavenly bodies move in distinct orbits, and in the same direction round the sun. These bodies are known as *Planets,* the chief or largest being Mercury and Venus—which, being nearer the sun than the earth, are termed " inferior planets",—and Mars, Asteroids, Jupiter, Saturn, Uranus, Neptune, and Pluto, which, being more remote from the sun, are termed " superior planets ".

The " periodic " time of a revolution varies, increasing as the planet is more distant : thus Mercury has a periodic time of about 88 days, and Neptune of nearly 165 years.

Again, travelling round some of the planets are smaller celestial

bodies known as *satellites*, *e.g.* the earth has one such body, the *moon*; Mars has two; Uranus, four; Jupiter, nine; while Saturn has nine, in addition to three rings, which are probably composed of innumerable very small satellites.

The sun, then, together with its planets and satellites, comprise what is known as the *solar system.*

The sun, however, is merely one star in the universe, which includes all the "fixed stars", and many of these are vastly greater than the sun, and probably have planetary systems of their own.

The Stars.—The "fixed stars" are not actually at rest in space,

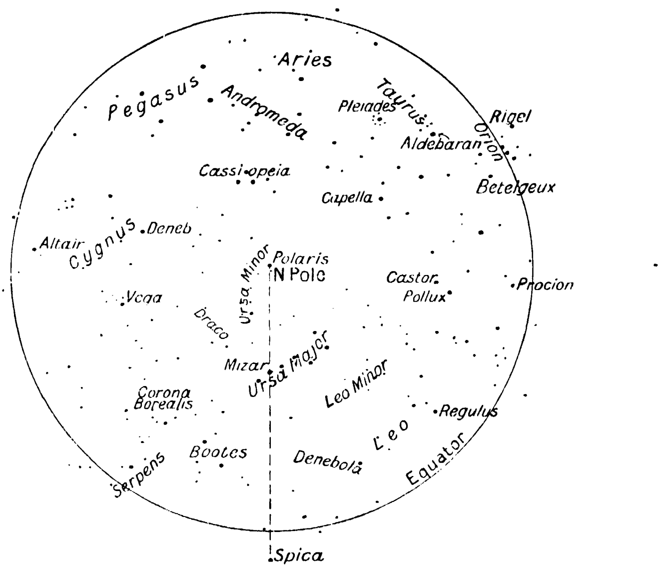


FIG. 16.1.—The Stars

but have definite motions of their own, though, on account of their great distance from the earth, their relative positions appear to alter exceedingly little during long intervals of time.

For the purposes of classification the fixed stars have been arranged into groups known as *Constellations*, the stars in each group being either lettered or numbered.

Fig. 16.1 shows some of the chief stars in the Northern Hemisphere, as seen by an observer looking upward to the sky. For identification purposes, however, various Star Charts or Planispheres are available and will be found useful.

Ursa Major is known by such popular names as "The Plough", "The Waggon", "Charles's Wain", "The Great Bear" or "The Great Dipper", while Ursa Minor is known as "The Little Bear", "The Little Dipper", etc.

The star α in the constellation of Ursa Minor is known as *Polaris* or *The Pole Star*: it may be located from *The Pointers,* β and α Ursa Major (see Fig. 17.2).

The circle of twelve constellations through which the " ecliptic " passes constitutes the *Signs of the Zodiac.*

These constellations taken in order are :

(1) Aries (Ram).	(7) Libra (The Balance).
(2) Taurus (Bull).	(8) Scorpio (Scorpion).
(3) Gemini (Twins).	(9) Sagittarius (The Archer).
(4) Cancer (Crab).	(10) Capricornus (Goat).
(5) Leo (Lion).	(11) Aquarius (Water-bearer).
(6) Virgo (Virgin).	(12) Pisces (Fishes).

The nearest of the fixed stars is at a distance (about 25×10^{12} miles) of nearly 270,000 times that (about 93×10^{6} miles) of the sun from the earth, and in consequence of this enormous distance the relative positions of the stars appear exactly the same from every point of view upon the earth's surface. Further, the relative positions, with the exception of a comparatively small number of the nearer stars, appear unaltered, even from different positions upon the earth's orbit. These nearer stars, however, appear to have a small motion relatively to the other " fixed stars ", and to trace out small elliptical orbits in the heavens, as the earth travels along its orbit.

This apparent movement, which is quite distinct from any real movement that the star or other body may possess, is known as *Parallax,* which may be defined generally as the apparent displacement of an object due to the real displacement of the observer. For instance, in this case, the apparent movement of the stars is due to the real displacement of the observer from one position to another upon the earth's orbit.

Diurnal Parallax or Geocentric Parallax (see p. 453) is the correction to be applied to reduce observations from various points upon the earth's surface, to the centre of the earth. For the fixed stars this is negligible, but it is appreciable when observations are taken to the sun or the moon.

Annual Parallax is the correction necessary to reduce observations on the nearest stars to the centre of the sun. It is the angle subtended at the observed star, by the mean radius of the earth's orbit, and is usually very small, being less than one second for the nearest of the fixed stars.

To an observer upon the earth the fixed stars seem to be dotted over the surface of a vast sphere, known as the **Celestial Sphere** (Fig. 16.2), at the centre of which the earth is apparently situated.

Owing to the real rotation of the earth about its polar axis every twenty-four hours, the celestial sphere *appears* to rotate about the same axis during that time, and hence in astronomical problems it is convenient to imagine the earth as stationary and consider the apparent movements of the various celestial bodies as real movements, relative to the earth. Actually, of course, the fixed stars are at varying distances from the earth ; they only *appear* to lie upon the surface of a sphere.

The **Celestial Poles** are the points at which the polar axis, when produced, intersects the celestial sphere.

The **Celestial Equator** is the great circle traced upon the celestial sphere by that plane which passes through the centre of the earth and is perpendicular to the polar axis.

The Ecliptic.—As mentioned above, the earth travels in its orbit round the sun, making a complete circuit in the year. Consequently, the sun appears to move relatively to the fixed stars in a direction from west to east, and the periodic time of a complete revolution is one year.

But the polar axis of the earth is not perpendicular to the plane of its orbit round the sun ; it is inclined to the normal at an angle of about 23°-27′. Consequently, the apparent path of the sun among

Z = Zenith
N = Nadir
PP_1 = Celestial Poles and Polar Axis
E_2EE_1 = Equator
CC_1 = Ecliptic
♈ = First Point of Aries
Ω = First Point of Libra
HH_1 = Horizon
$PZHNH_1$ = Meridian (in plane of paper)
ZENW = Prime Vertical, perpendicular to Plane of Meridian (shown distorted in figure)
E and W = East and West Points, at junction of Horizon and Prime Vertical

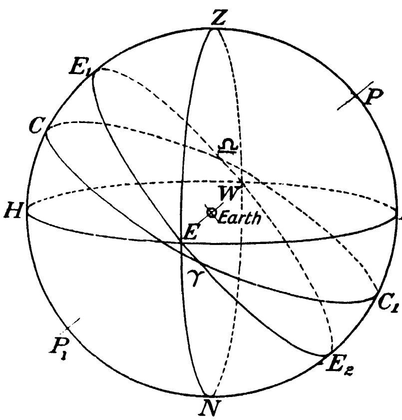


FIG. 16.2.—Celestial Sphere

the fixed stars is not in the equatorial plane, but is in a plane inclined at about 23°-27′ to this.

The trace of the plane upon the celestial sphere is a great circle which is known as the *Ecliptic*.[1]

The two points at which the Ecliptic intersects the equatorial plane are known as the *Equinoctial Points*. That point through which the sun passes from south to north of the Equator, *i.e.* at the *Spring or Vernal Equinox*, is termed the *First Point of Aries* (♈), while the other, *i.e.* at the *Autumnal Equinox*, is known as the *First Point of Libra* (Ω).

The poles of the Ecliptic lie at the intersection of the celestial sphere, with an axis through the centre, and perpendicular to the plane of the Ecliptic.

Nutation is the nodding of the Celestial Poles, to and from the poles of the Ecliptic ; this causes a variation of several seconds in

[1] The Mean Obliquity of the Ecliptic in 1946 was 23°·26′·46″·71, and the mean annual diminution = 0″·468 (*Nautical Almanac*).

the obliquity of the Ecliptic from the equatorial plane. The effect is due chiefly, though not entirely, to the action of the moon.

The **Moon** rotates about the earth in an elliptical orbit, inclined at an average angle of about 5°-8′ to the plane of the Ecliptic, which is intersected at points called the *Nodes*. The nodes are not stationary, but have a retrograde motion of about 19° per year.

Relatively to the earth, the moon appears to move from east to west, but it has a motion from west to east, relatively to the fixed stars.

Like the earth, the moon rotates about its polar axis ; but since in this case the time of one such revolution corresponds with the time of a complete circuit of its orbit, the same face of the moon is always directed towards the earth.

The phenomena of *New* and *Full Moon* occur when the moon, the sun, and the earth lie in the same vertical plane, though not necessarily in the same straight line. At new moon the moon lies between the sun and the earth ; it has the same longitude as the sun, and is said to be in *conjunction*. At full moon the earth lies between the sun and the moon ; the latter is then said to be in *opposition*.

During the interval between new moon and full moon the illuminated limb increases in size, and the moon is said to be waxing. During the interval between full moon and new moon the illuminated limb decreases in size, and the moon is said to be waning.

The interval between two successive new moons, *i.e.* the time of a complete revolution relative to the sun, is known as a *Lunar Month* (about 27⅓ days). A *Sidereal Month* (about 29½ days) is the time of a complete revolution, relative to the fixed stars.

When the moon's disc passes between a star and the earth, the star is said to be *occulted*, and the phenomenon is known as an *occultation*.

When the moon passes in front of the sun's disc, a *Solar Eclipse* is the result, while a *Lunar Eclipse* is caused by the passing of the shadow of the earth over the moon, *i.e.* when the earth passes directly between the sun and the moon.

Zenith, Nadir.—If a person occupies any position upon the earth's surface, and a straight line is drawn upwards in the direction of the force of gravity at that point, the intersection of such a line with the celestial sphere is termed the *Zenith* of the observer, *i.e.* the Zenith is the point on the celestial sphere immediately overhead.

Similarly, if the straight line be produced downwards through the centre of the earth, the other intersection with the celestial sphere is termed the *Nadir*.

The **Celestial or Rational Horizon** is the great circle traced upon the celestial sphere by that plane which is perpendicular to the Zenith Nadir line, and which passes through the centre of the earth.

The **Sensible Horizon** is the circle in which the plane through the position of the observer and tangential to the earth's surface (or perpendicular to the Zenith-Nadir line) cuts the celestial sphere.

The **Visible Horizon** is the line in which the more distant visible features of the earth's surface appear to cut the celestial sphere : on land this will consequently be of very irregular form owing to the existence of hills, etc.

In Fig. 16.4 the plane OC marks the Celestial Horizon ; the plane PR the Sensible Horizon, and the cone PV the Visible Horizon.

A **Celestial Meridian** is the great circle formed on the celestial sphere by the intersection of a plane passing through the two Celestial Poles. The *meridian* of any particular point is that great circle which passes through the Zenith and Nadir of the point as well as through the poles.

Great circles which pass through the Zenith and Nadir are known as *vertical circles* : thus the meridian of a place is also a vertical circle.

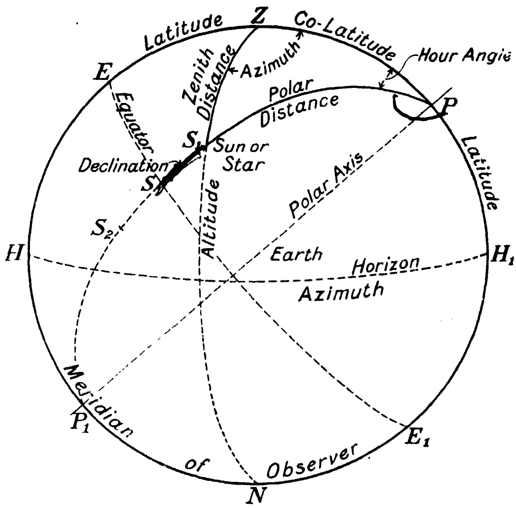


FIG. 16.3
Celestial Sphere, illustrating Astronomical Terms

The vertical circle which passes through the east and west points is known as the *Prime Vertical.* The plane which traces the Prime Vertical on the celestial sphere thus passes through the Zenith and Nadir, and is at right angles to the meridian of the place.

The Prime Vertical does not pass through the Celestial Poles unless the observer happens to be situated at one of the Terrestrial Poles, when the elevated pole is also his Zenith.

Latitude, Longitude.—The position of a point upon the earth's surface is defined in terms of " Latitude " and " Longitude ". *Latitude* is the angular distance measured from the equatorial plane towards the nearer pole, along the meridian of the place. The latitude of a point upon the Equator is thus 0°, while at the North and South Poles it is 90° N. and 90° S. latitude respectively ; and all other points lie between these limits.

It will be seen that, on the celestial sphere, latitude is the arc of the meridian intercepted between the Zenith and the Equator. The remainder of the quadrant between the Zenith and the pole is known as the *Co-latitude, i.e.* co-latitude = 90° − latitude.

The **Longitude** of a place is the angular distance which the meridian of the place makes with some standard meridian, and is measured from 0° to 180° either towards the east or towards the west.

The standard meridian adopted internationally as the zero is that which passes through Greenwich.

Longitude may be otherwise defined as the spherical angle at the poles, included between the standard meridian and the meridian of the place.

Declination (δ) **and Right Ascension** define the positions of heavenly bodies on the celestial sphere, exactly as Terrestrial Latitude and Longitude define positions upon the earth.

Thus the *Declination* of a star is the angular distance from the Equator to the star, measured along the celestial meridian, while *Co-declination* is the remainder of the quadrant; *i.e.* it is the arc of the meridian intercepted between the star and the nearer or elevated pole.

Thus co-declination $= 90° -$ declination.

Co-declination is also known as *Polar Distance.*

Right Ascension is the angular distance between the meridian of the star and the "standard" meridian. It is measured from the west towards the east, and is expressed in hours, minutes, and seconds, twenty-four hours representing 360°.

The "standard" meridian is that which passes through the "First Point of Aries" (see p. 447). This point is not fixed in space, but is gradually moving in a retrograde direction at a rate of about 50″·25 per annum, with the result that, although it still retains the original name, the point is no longer in the constellation Aries, but is now in the adjoining constellation—Pisces.

Similarly, the other equinoctial point, the "First Point of Libra", has gradually moved into the constellation Virgo.

This gradual movement is known as the *Precession of the Equinoxes.*

At the Vernal Equinox, *i.e.* on March 21, the declination of the sun is zero, and the right ascension (R.A.) also zero.

At the Autumnal Equinox, on September 23, the declination of the sun is again zero, and the right ascension is 180° or 12 hours.

On June 21 the sun has travelled 90° from the First Point of Aries, along the ecliptic, and has reached its most northerly declination of N. 23°-27′ nearly, the R.A. being 90° or 6 hours. This is called the *Summer Solstice.*

The *Winter Solstice* occurs on December 22, when the sun has described an arc of 90° from the First Point of Libra. The declination then has its maximum value of about 23°-27′ south, and the R.A. is 270° or 18 hours.

The positions of the sun at the Summer and Winter Solstices are known as the *Solstitial Points*, while the meridian passing through these points, along which the declination is measured, is known as the *Solstitial Colure.* Similarly the *Equinoctial Colure* is the meridian which passes through the equinoctial points.

The **Altitude** of a body is the angular distance from the plane of the horizon, measured along the great circle, or "vertical", which

passes through the body and the zenith and nadir of the observer. When this great circle also passes through the poles, the " vertical " becomes a meridian, and the altitude is a " Meridian Altitude ".

The **Co-altitude** is the remainder of the quadrant, *i.e.* (90° – altitude), and it is sometimes called the *Zenith Distance*, being the angular distance from the object to the observer's zenith.

In Fig. 16.4, if S represents a star or other celestial body, the true altitude is the angle SOC at the centre of the earth: but before this can be deduced from the observed altitude it may be necessary to apply some or all of the following corrections :

(1) Refraction.
(2) Dip.
(3) Index Error.
(4) Parallax.
(5) Semi-diameter.

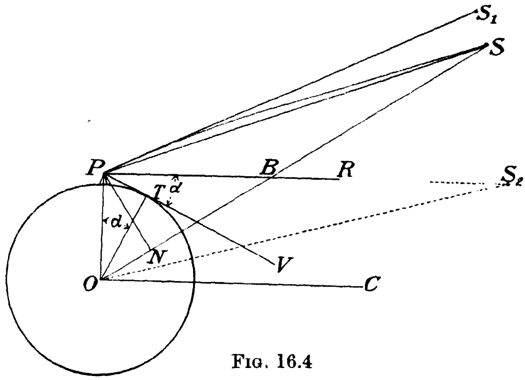


FIG. 16.4

For instance, in Fig. 16.4 let P represent the position of the observer and S the position of the celestial body, the true altitude, the angle SOC, of which is required.

Refraction.—Then because the density of the air decreases as the distance from the earth increases, it follows that a ray of light which is emitted from S, and traverses the different strata, in an oblique direction, becomes refracted, as explained in Chapter II. Hence the particular ray which reaches the observer at P has travelled in some curved path, such as that indicated by the curved line PS in Fig. 16.4 ; and as the eye cannot correct for this, the ray appears to come in a straight line from S_1, where the straight line PS_1 is tangential to the curve PS at P.

The angle SPS_1 is known as the correction for refraction, and the value may be obtained from such tables as Chambers's Mathematical Tables.

An approximate formula for Astronomical Refraction may be deduced as follows :

It is found by experiment that if a ray of light passes from an original stratum through a series of parallel strata of various densities, the final deviation of the ray is exactly the same as that which would result from the direct passage of the ray from the initial through the final stratum. Consequently, it is only necessary to consider one deviation of the ray.

When the altitude of the body is considerable, the spherical shape of the different strata of the atmosphere may be neglected, and they may be assumed plane. Hence in Fig. 16.5, where Z is the zenith,

P the position of the observer, S that of the star, and S_1 the apparent position, by the law of refraction (p. 48)

$$\frac{\sin \text{ZQS}}{\sin \text{PQN}} = \mu,$$

where μ is the index of refraction,

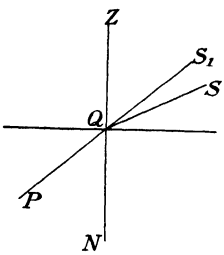


FIG. 16.5.—Refraction

but $\text{ZQS} = \text{ZQS}_1 + \text{S}_1\text{QS}$ and $\text{PQN} = \text{ZQS}_1$,

$$\begin{aligned} \therefore \mu \sin \text{ZQS}_1 &= \sin (\text{ZQS}_1 + \text{S}_1\text{QS}) \\ &= \sin \text{ZQS}_1 \cos \text{S}_1\text{QS} \\ &\quad + \cos \text{ZQS}_1 \sin \text{S}_1\text{QS}, \end{aligned}$$

but as S_1QS is very small, $\cos \text{S}_1\text{QS} = 1$ and $\sin \text{S}_1\text{QS} = \text{S}_1\text{QS}$ in radian measure, approximately,

$$\because \mu \sin \text{ZQS}_1 = \sin \text{ZQS}_1 + \text{S}_1\text{QS} \,.\, \cos \text{ZQS}_1,$$

and $$\text{S}_1\text{QS} = (\mu - 1) \tan \text{ZQS}_1,$$

i.e. the refraction is proportional to the tangent of the apparent zenith distance.

Reducing $\angle \text{S}_1\text{QS}$ to seconds and substituting for $(\mu - 1)$ the formula may be written :

Refraction in seconds at a pressure of 30 inches of mercury, and a temperature of 50° F. $= 58'' \times$ tangent of apparent zenith distance, or $58'' \times$ co-tangent of apparent altitude.

The above formula should only be used for rough calculations, however, as the index of refraction varies with the density of the air, *i.e.* it is dependent upon temperature and pressure or altitude.

For more accurate results, tables should be consulted, and allowance made for these factors : the spherical shape of the strata is also taken into account in the preparation of the tables. Below about 15° to 20° altitude the correction for refraction is large and not very reliable : as the object approaches the zenith, or the altitude approaches 90°, the correction diminishes until it is zero for a body exactly overhead.

Dip.—Now if a theodolite is employed, the horizontal direction PR is known, and the angle S_1PR is observed ; so that the altitude of S from the sensible horizon is equal to the observed angle S_1PR minus the correction for refraction S_1PS. But at sea it is impracticable to use a theodolite, and the altitude is observed from the visible or sea horizon, *i.e.* the angle S_1PV is observed (Fig. 16.4).

From this the altitude of S from the sensible horizon is deduced, *i.e.* $\text{SPR} = \text{S}_1\text{PV} - \text{S}_1\text{PS} - \text{RPV}$, where the latter angle RPV is known as the correction for dip.[1] The values of this correction for various heights of the observer above sea-level are tabulated for the convenience of seamen.

[1] See also Chapter VI, p. 200.

An expression for the angle of dip, d, may be easily deduced from Fig. 16.4. The angle $d = \angle RPV = \angle POT$, where PT is tangential to the sphere at T and OT is radial,

$$\therefore \tan d = \frac{PT}{OT} = \sqrt{\frac{(r+h)^2 - r^2}{r^2}}$$

$$= \sqrt{\frac{h\,(2r+h)}{r^2}}, \text{ or approximately } \sqrt{\frac{2h}{r}},$$

where r is the radius of the earth OT, and h is the height of the point P above sea-level.

From this d may be deduced, and corrected for refraction, because the line PT is actually slightly curved.

Example.—An observer is 30 ft. above the level of the sea—what is the correction to be applied for dip ?

$$\tan d = \sqrt{\frac{2\,.\,h}{20{,}890{,}000}} = \sqrt{\frac{h}{10{,}445{,}000}} = \sqrt{\frac{1}{348{,}170}} = \cdot 001694,$$

$$\therefore d = 5'\text{-}48\tfrac{1}{2}'' \text{ about.}$$

In Molesworth's tables d is given as $5'\text{-}14''$, and is calculated by the empirical formula d in seconds $= 57{\cdot}4\sqrt{h}$. This includes an allowance for refraction of about $\frac{1}{10}$th, as the equation above reduces to $d = 63{\cdot}8\sqrt{h}$, when d is expressed in seconds.

Index Error.—If the reading on the instrument verniers is not zero when the altitude is zero (or when the same body is observed by reflection and by direct vision with a sextant), but has a value say e, the values of any vertical angles observed with the instrument will be too large or too small by this amount, according as e is above or below the zero, for the zero angle.

The index error is said to be $+e$ when this amount must be added to the observed altitude, *i.e.* when the actual reading is $-e$ for an angle of zero altitude.

Parallax is the correction to be applied in order to deduce the true altitude SOC from the altitude SPR above the sensible horizon.

In Fig. 16.4 the angle SOC is equal to the angle PBO, because PR and OC are parallel, each being perpendicular to OP (I. 29). But the angle PBO, being the exterior angle of the triangle PSB, is equal to the sum of the interior and opposite angles SPR and PSB (I. 32), *i.e.* the true altitude SOC is equal to the apparent altitude SPR + the angle PSB.

The angle PSB is termed the correction for parallax, and it is added to the corrected apparent altitude in order to obtain the true altitude.

When the apparent altitude SPR is nil, *i.e.* when S lies on the horizontal line PR, *e.g.* at S_2, the angle of parallax is the angle whose tangent is $\frac{PO}{PS_2}$, and this is known as the horizontal parallax.

Or, since the angle is very small, its value is

$$\frac{R}{L} \text{ radians nearly,}$$

where R is the radius of the earth, and L is the distance from the earth to the observed celestial body S_2.

When the observed body has a greater altitude, and is at S say, the angle of parallax is approximately $\frac{PN}{L}$ radians, where PN is the perpendicular distance from P to OS, *i.e.* it is

$$\frac{R \cos a}{L} \text{ radians,}$$

because the angle OPN is equal to the altitude SOC (*i.e.* a). Thus the parallax in altitude is

Horizontal Parallax × cos . altitude.

For celestial bodies, as the parallax is very small, the apparent altitude SPR may be used in this formula.

Obviously, then, the correction for parallax decreases as the altitude increases, and becomes zero when the observed body is at the zenith.

For observations upon the fixed stars the correction for parallax is negligible, but it must be taken into account for observations upon the sun and the moon.

In the *Nautical Almanac* the values of the sun's horizontal parallax are given for intervals of ten days. The parallax for angles of altitude can be deduced from these by multiplying by the cosine of the apparent altitude.

Semi-diameter.—A further correction which it is necessary to apply in the case of observations upon the sun is that for semi-diameter. It is obvious that if it were attempted to adjust the cross-hairs of a telescope exactly to the centre of the sun's disc, there would be considerable liability to error. Consequently, observations are always taken either to the upper or the lower limb for measurements of altitude, and to the east and west limbs for measurements in azimuth, and an allowance made afterwards.

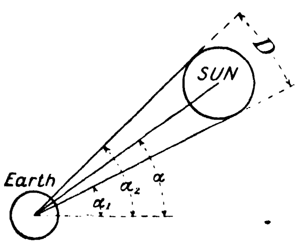


FIG. 16.6
Correction for Semi-diameter

For instance, in Fig. 16.6, if the altitude to the lower limb of the sun is a_1, that of the centre of the sun will be $a_1 + \frac{D}{2}$.

Similarly, if the altitude of the upper limb of the disc is observed to be a_2, that of the centre will be $a_2 - \frac{D}{2}$, where $\frac{D}{2}$ is the correction for

semi-diameter, and D is the angle subtended, at the centre of the earth, by the diameter of the sun.

The value of this correction depends upon the position of the earth upon its orbit, *i.e.* upon the apparent position of the sun on the ecliptic, and may be obtained from the *Nautical Almanac.*

According to Professor Auwers,[1] the semi-diameter of the sun at the earth's mean distance $= 16'\text{-}01''{\cdot}18$.

The **Azimuth** of a point is the angle between the plane of the meridian, and the " vertical " plane through the point; *i.e.* it is the spherical angle at the Zenith, between the plane which passes through the Zenith, Nadir, and the poles, and the plane which passes through the Zenith, Nadir, and the point in question. The arc of the horizon, intercepted between these two planes, is thus a measure of the azimuth of the point.

The Celestial Pole, which is above the observer's horizon, is known as the " Elevated Pole ", and the value of the azimuth computed from astronomical observations is the nearest angle from the elevated pole to the object, measured either to the east or to the west: it is accordingly not greater than 180°.

The computed value is then transformed into a " whole circle ", azimuth measured from 0° to 360° in a clockwise direction. Sometimes the S. is taken as zero, and the angle measured through the west: sometimes the N. is taken as zero, and the angle measured through the east.

As the azimuth of a point is the angle between the two vertical planes which pass through the position of the observer, it may be measured in a plane tangential to the earth, *i.e.* in a horizontal plane through the observer's position. Consequently, other angles measured in this plane—though they may not be true " azimuths "—are often called measurements " in azimuth ".

The **Hour Angle** of a star is the spherical angle at the pole, between the declination circle of the star and meridian of the observer. The arc of the Equator intercepted between these two planes is thus a measure of the hour angle.

When expressed in time, the hour angle of a star is the time interval in sidereal time that has elapsed since the star crossed the meridian at upper transit and is measured 0 *h* to 24 *h*.

Culmination, Elongation.—As already explained, each of the fixed stars appears to describe a circle round the earth, and to cross the meridian of the observer twice every twenty-four hours.

When a star or other celestial body appears to cross the meridian, it is said to " transit ". Thus in one revolution of the earth upon its axis, each star transits twice.

At each " transit " the star is said to *culminate*, the upper culmination being on that side of the pole which contains the Zenith. In other words, a star culminates at the *Lower Transit* when its altitude is a minimum (*Lower Culmination*), and at the *Upper Transit* when its altitude is a maximum (*Upper Culmination*).

[1] *Nautical Almanac.*

When the circle described by a star in the heavens does not fall below the horizon of the observer, the star is said to be a circumpolar star (see Fig. 16.7). In such a case the arc PS must be less than the arc PH_1, *i.e.* less than EZ, because $EZ = PH_1$; so that a circumpolar star is one the co-declination (*i.e.* PS) of which is less than the latitude (*i.e.* EZ) of the observer.

In the *Nautical Almanac*, the term *circumpolar* is restricted to stars within 10° of either pole. The Pole Star, or Polaris, which is classified as *α* in the constellation Ursa Minor, is an example of a circumpolar star, as it describes a circle of less than 1° radius from the pole.

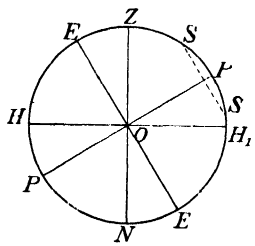


FIG. 16.7

When at its greatest distance E. or W. of the observer, the star is said to *elongate* or to be at *elongation*.

Time.—As the earth rotates from W. to E., the apparent motion of the fixed stars across the heavens is, like that of the sun, from E. to W. As, however, the earth is also travelling round the sun, the sun appears to move relatively to the stars, in a direction from W. to E., and to make a complete circuit of the heavens in one year.

The time interval between two successive upper transits of the " First Point of Aries " over the same meridian is known as a *Sidereal Day*, and the instant of crossing is termed *Sidereal Noon*. The day is divided into 24 hours, reckoned consecutively from 0 at one noon to 24 hours at the following noon. Each hour is divided into 60 minutes, and each minute is again subdivided into 60 seconds.

Thus (*a*) the sidereal time at any instant is obviously the hour angle (when expressed in time) of the First Point of Aries.

(*b*) The sidereal time at which a particular star transits is the measure of its right ascension, or the R.A. of a star is the sidereal time of its upper transit.

(*c*) The hour angle of a particular star at any instant is equal to the sidereal time − the star's R.A.

An **Apparent Solar Day** is the time interval between two successive lower transits of the centre of the sun across the same meridian, and the time of the upper transit is known as *Apparent Noon*.

The time taken for the earth to make a complete circuit of its orbit is known as a year. There are several definitions, but they are not very important from a surveyor's point of view.

A **Sidereal Year,** for instance, is the time interval between two successive transits of the sun through the meridian of any one of the fixed stars ; a *Solar* or *Tropical Year* is the time interval between two successive vernal equinoxes ; an *Anomalistic Year* is the time interval between two successive passages of the sun through perigee ; a *Civil Year* is composed of an exact number of mean solar days as defined in the *Gregorian Calendar*, etc.

According to Bessel, there are 366·24222 sidereal days in a tropical

year, *i.e.* during that time the First Point of Aries appears to have made 366·24222 revolutions from east to west, relatively to the earth. But in the meantime the sun appears to have moved through one complete revolution from west to east, relatively to the First Point of Aries. That is to say, during a tropical year the sun appears to have made (366·24222 − 1) revolutions from east to west, relatively to the earth, and therefore the sum of 366·24222 sidereal days = the sum of 365·24222 apparent solar days.

Although of constant value, it would be very inconvenient to adopt the sidereal day as the unit of civil time, because of the lack of agreement with the phenomena of night and day.

Apparent solar time would likewise be impracticable, as, for the following reasons, the length of an apparent solar day is variable, and cannot be subdivided by means of a clock whose rate is uniform.

(1) The sun does not move at a uniform rate along the ecliptic (see Kepler's Second Law, p. 444), as equal areas are swept over by the radius vector, and not equal angles in equal intervals of time; and

(2) Even if the rate along the ecliptic were uniform, the rate parallel to the Equator (*i.e.* the rate of change of R.A. of the sun) would not be uniform, owing to the obliquity of the two planes.

Mean Solar Time is therefore used for civil purposes, and a **Mean Solar Day** is an average of all the apparent solar days which occur during a revolution of the earth on its orbit. A mean solar day is thus equal to 1·0027379 sidereal days.

An alternative method of defining the mean solar day is to state that it is the time interval between two successive lower transits of the meridian by the *Mean Sun*, the mean sun being an imaginary point which travels round the ecliptic in exactly the same time as the real sun, but its motion is such that the rate of change of R.A. is uniform. This is not equivalent to saying that the mean sun moves at a uniform rate along the ecliptic, but that the projection of its movement along the Equator is uniform. Mean midnight and mean noon, then, occur at the instants the mean sun crosses the meridian at lower and upper transit respectively.

Greenwich Mean Time (G.M.T.) is measured from the lower transit of the Greenwich meridian by the mean sun, *i.e.* from Greenwich Mean Midnight, 0 *h.* to 24 *h.* It is identical with **Universal Time** (U.T.), a term which was recommended by the International Astronomical Union, and which has now been adopted by the *Nautical Almanac.*

Local Mean Time (L.M.T.) is measured from the lower transit of the meridian of the place of observation.

Greenwich (G.A.T.) and **Local Apparent Time** (L.A.T.) are measured from the lower passage of the real sun over the Greenwich or local meridian respectively, *i.e.* from Greenwich Apparent Midnight or Local Apparent Midnight.

It will be noted, therefore, that Sidereal time is measured 0 *h.* − 24 *h.* from the *upper* transit of the First Point of Aries and Solar Time 0 *h.* − 24 *h.* from the *lower* transit of the sun. Hence the Local Sidereal

time (L.S.T.) of Local Mean midnight = the Right Ascension of the Mean Sun (R.A.M.S.) ± 12 *h.* (or ± 180° if expressed in angle).

$$\text{L.S.T.} = \text{L.M.T.} + \text{R.A.M.S.} \pm 12\ h.$$

i.e.

$$\text{L.S.T.} = \text{L.M.T.} + \text{R.}$$

This quantity R, *i.e.* R.A.M.S. ± 12 *h.* or the difference between L.S.T. and L.M.T., is tabulated in the *Nautical Almanac Abridged for the Use of Seamen.*

In the example illustrated in Fig. 17.2, p. 467, the data refer to September 3, 19—, at 1 *h.* 30 *m.* of sidereal time after upper transit of the local meridian by the First Point of Aries, *i.e.* when L.S.T., which is measured clockwise from E. to W. from Upper Transit to the First Point of Aries, is 1 *h.* 30 *m.*

From the *Nautical Almanac,* the R.A.M.S. is 10 *h.* 45 *m.* and this is measured counter-clockwise from W. to E. from the First Point of Aries as shown.

Local Mean Time is seen from the figure to be 1 *h.* 30 *m.* + 12 *h.* 0 *m.* − 10 *h.* 45 *m.* = 2 *h.* 45 *m.*, this being measured clockwise W. to E. from the *lower* transit of the sun, *i.e.* from midnight to the Sun's position at S.

In the *Abridged Nautical Almanac,* R is given as 22 *h.* 45 *m.* (*i.e.* R.A.M.S. + 12 *h.*), therefore

$$\begin{aligned} \text{L.M.T.} &= \text{L.S.T.} - \text{R.} \\ &= (24\ h. + 1\ h.\ 30\ m.) - 22\ h.\ 45\ m. \\ &= 2\ h.\ 45\ m. \end{aligned}$$

The addition of 24 *h.* to L.S.T. is made because 1 *h.* 30 *m.* on one day is equivalent to 25 *h.* 30 *m.* on the previous (sidereal) day.

Incidentally, the declination of the sun at the time assumed, is about N. 7°-51′ so that its position would be where shown on the diagram, *i.e.* below the horizon and north of the equator.

The diagram is conventional in that the scale of declinations is divided equally from 0° to 90° inwards along the radius of the outer circle.

The astronomical date now coincides with the civil date, the day in each case covering the period between two successive midnights. The astronomical day is subdivided into 24 hours measured continuously from 0 to 24 from midnight to midnight, while the civil day is usually subdivided into two periods each of 12 hours.

Previous to 1925, the astronomical day was measured from noon to noon, so that 1 A.M. June 2 civil time was 13 o'clock June 1 astronomical time, and 1 P.M. June 2 civil time was 1 o'clock June 2 astronomical time.

The difference between the Right Ascension of the true sun and that of the mean sun is known as the *Equation of Time* : it is the difference therefore between the hour angles, *i.e.* between apparent and mean time. Originally the equation of time was considered to be the mean minus apparent time, but now, when mean time is most usually ascertained by conversion of sidereal time obtained from stellar observations or directly from wireless signals, the correction tabulated in the

Nautical Almanac and in *Whitaker's Almanack* is that to be applied to *mean* time to give *apparent* time : thus E = A.T. − M.T., *i.e.* A.T. = M.T. + E.

The equation of time is zero four times during the year, *i.e.* about April 15, June 14, September 1, and December 25 ; so that on these four occasions apparent time is identical with mean time.

A curve showing the variation in E. is given in Fig. 16.8, the maximum values in 1946 being about

M.	S.	
−14	21	on Feb. 12,
+ 3	47	on May 15,
− 6	23	on July 27,
+16	22	on Nov. 4.

Abbreviations.—The following abbreviations are in common use :

L.A.T. = Local Apparent Time.
G.A.T. = Greenwich Apparent Time.
L.A.N. = Local Apparent Noon.
G.A.N. = Greenwich Apparent Noon.
L.M.T. = Local Mean Time.
G.M.T. = Greenwich Mean Time.
L.M.N. = Local Mean Noon.
G.M.N. = Greenwich Mean Noon.
L.S.T. = Local Sidereal Time.
G.S.T. = Greenwich Sidereal Time.
E. = Equation of Time.
R.A. = Right Ascension.
R.O. = Referring Object.
♈ = First Point of Aries.
♎ = First Point of Libra.

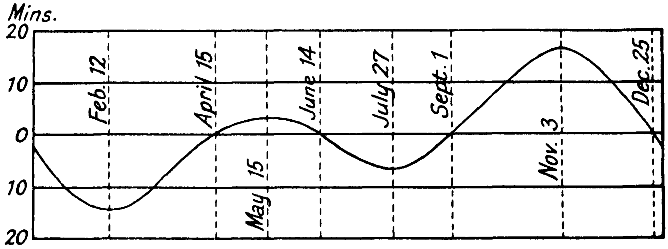


FIG. 16.8.—Variation of Equation of Time

The following symbols are also used in the following chapters :

a = altitude.
δ = declination.
l = latitude.
z = zenith distance.
G.M.M. = Greenwich Mean Midnight, *i.e.* 0 h.
L.M.M. = Local Mean Midnight, *i.e.* 0 h.

Longitude and Time.—As the time interval between two successive mean noons at any one point is 24 mean solar hours, or as the mean sun describes an arc of 360° in 24 hours, it follows that at B, situated in longitude $x°$ W. of A, mean noon will occur $\frac{x}{360} \times 24$ hours later than at A : if B is east of A, mean noon will occur earlier.

A difference of longitude of 1° (*i.e.* $x = 1$) thus corresponds with a difference of $\frac{1}{15}$ hour = 4 minutes in the local mean times, or

15°	longitude	converted	into time	= 1 h.
1°	,,	,,	,,	= 4 m.
15′	,,	,,	,,	= 1 m.
1′	,,	,,	,,	= 4 s.
15″	,,	,,	,,	= 1 s.

Example.—What would be the G.M.T. to correspond with 2 h. 5 m. 3 s. A.M. L.M.T. at B on June 2 in longitude 56°-30′-15″ E. ?

		H.	M.	S.
56° longitude expressed in time = 56 × 4 mins.	=	3	44	0
30′ longitude expressed in time . . .	=		2	0
15″ longitude expressed in time . . .	=			1
		3	46	1

But as the longitude is E., the clock at Greenwich is behind the clock at B,

		H.	M.	S.
Therefore the L.M.T. (by adding 24 h.) June 1 .	=	26	5	3
Deduct for longitude	=	3	46	1
		22	19	2

i.e. G.M.T. is 22 h. 19 m. 2 s. June 1, *i.e.* 10 h. 19 m. 2 s. P.M.

To convert Sidereal Time into Solar Time, and vice versa.—From the relationship given above, *i.e.*

$$366{\cdot}2422 \text{ sidereal days} = 365{\cdot}2422 \text{ mean solar days,}$$

it may be deduced that

$$1 \text{ hour S.T.} = 1 \text{ hour M.T.} - 9{\cdot}8295 \text{ s. M.T.}$$

and

$$1 \text{ hour M.T.} = 1 \text{ hour S.T.} + 9{\cdot}8565 \text{ s. S.T.}$$

From these equations an interval of S.T. may be easily converted into an interval of M.T., and *vice versa.*

To simplify the computations, however, tables are given in the *N.A.*, together with examples of their use.

Example.—To find L.S.T. at B in longitude 82°-4′-30″ W., at 10 h. 45 m. A.M. Dec. 30, 19— (L.M.T.).

		H.	M.	S.
10 h. 45 m. A.M. civil time converted to astronomical time		10	45	00
To convert longitude 82°-4′-30″ to time, H. M. S. 82° = 82 × 4 m. = 5 28 0 4′ = 4 × 4 s. = 16 30″ . . = 2	=	5	28	18
Therefore, since B is west of Greenwich, the G.M.T. .	=	16	13	18
To convert this to S.T. we add 9·86 seconds approximately for each hour. Correction for 16 h. = 157·76 s. 13 m. = 2·14 s. 18 s. = ·05 s. 159·95 s.		0	02	39·9
The interval in S.T. is therefore	=	16	15	57·9
But S.T. at G.M.M. from the *N.A.* (*i.e.* at 0 h.) . .	=	6	29	32·2
Therefore G.S.T. at the given instant	=	22	45	30·1
But the correction for longitude is		5	28	18
Therefore L.S.T. at B is		17	17	12·1

Example.—To find the L.M.T. of transit of Polaris (*i.e.* α Ursae Minoris) in longitude 20° E. on Dec. 30, 19—.

		H.	M.	S.
The S.T. of G.M.M. on Dec. 30 from the *N.A.*	=	6	29	32·23
But in longitude 20° E., L.M.M. occurs 1 h. 20 m. (solar mean time) before G.M.M.				
This interval expressed in S.T. = 1 h. 20 m. 13·15 s.				

Difference for 1 h. = 9·86 s.
20 m. = 3·29 s.
13·15 s.

		H.	M.	S.
Consequently the L.S.T. at L.M.M. in 20° E. longitude is less than the S.T. at G.M.M. by 13·15 s.				13·15
i.e. L.S.T. at L.M.M.	=	6	29	19·08
Again the R.A. of Polaris from the *N.A.*	=	1	32	27·40
And as L.S.T. at L.M.M.	=	6	29	19·08
The interval in S.T. between L.M.M. and the culmination of Polaris (add 24 to the R.A.)	=	19	03	08·32
To convert this interval to solar time, we deduct for each hour 9·83 s.				

19 h. S.T. = 186·77 s.
3 m. S.T. = ·49 s.
8·32 s. = ·02 s.
187·28 s.

		H.	M.	S.
			03	07·28
The interval in solar time is thus		19	00	01·04

That is, if the chronometer records L.M.T., the time of culmination of Polaris will be about 7 h. 0 m. 1 s. in the evening of Dec. 30, 19—.

Example.—To find the L.M.T. of elongation of Polaris on Dec. 30-31, 19—. in longitude 38°-15′-5″ E., latitude 60° N.

		H.	M.	S.
S.T. of G.M.M., *i.e.* 0 h. on Dec. 31	=	6	33	28·79
S.T. of G.M.N. on Dec. 30 (ddt. 12 h. 01 m. 58·28 s.)	=	18	31	30·51
To obtain L.S.T. at L.M.N., subtract 9·86 s. per hour of longitude.				

	H.	M.	S.
38° long. =	2	32	0
15′ ,, =		1	0
5″ ,, =			·33
	2	33	0·3

2 h. × 9·86 = 19·72
30 m. . = 4·91
3 m. . = ·49
25·12s.

		H.	M.	S.
				25·12
i.e. L.S.T. at L.M.N.	=	18	31	05·39

From the formula (10*a*), p. 437, for a right-angled spherical triangle (the angle at the star, *i.e.* ZSP = 90°),

$$\cos P = \cot PZ \,.\, \tan SP = \tan l \,.\, \tan p,$$

where P is the hour angle, l the latitude, and p the co-declination or polar distance.

The declination of Polaris on Dec. 31 from *N.A.* = 88°-53′-01″·36

Therefore the polar distance p . . . = 1°-06′-58″·64

log tan 60° . = ·2385606

log tan 1°-6′-58″·64 = $\bar{2}$·2897078

and log cos P . = $\bar{2}$·5282684

		H.	M.	S.
P in arc	= 88°-03′-57″·32			
P in sidereal time	=	5	52	15·82
But the R.A. of Polaris	=	1	32	27·40
Therefore L.S.T. of Western Elongation .	=	7	24	43·22
But L.S.T. at L.M.N.	=	18	31	05·39
Therefore interval in S.T. after L.M.N. . .	=	12	53	37·83
Correction (deduct 9·83 s. for each hour) .	=		02	06·74
Interval in solar time after L.M.N. . .	=	12	51	31·09

Therefore L.M.T. of Elongation = 0 h. 51 m. 31·09 s. A.M. on Dec. 31, 19—.

EXAMPLES

1. What would be the correction for dip when the eye is situated at a distance at a height of 25 ft. above sea-level ?

2. (I.C.E.) (*a*) A lies to the west of B and the meridian distance between them is 1 h. 30 m. The longitude of B is 62°-30′-40″ W. What is the longitude of A ?

(*b*) How would you determine the rate at which a chronometer is gaining or losing time by astronomical observations ?

3. (I.C.E.) On a certain date, the right ascension of α Draconis was 14 h. 2 m. 5 s. From the *Nautical Almanac* and the longitude of the place, the local sidereal time of local mean noon was found to be 6 h. 35 m. 44 s. The declination of the star was 64°-47′-33″ N. Find the local mean time of east elongation. (Assume the latitude to be 60° N.)

4. (I.C.E.) On June 5 a chronometer was fast on mean time at A 3 h. 29 m. 10 s. ; on June 12 the chronometer was fast on mean time at A 3 h. 29 m. 31 s. ; on June 20 the chronometer was fast on mean time at B 5 h. 31 m. 39 s. ; on June 26 the chronometer was fast on mean time at B 5 h. 32 m. 0 s. If the longitude of A is 37°-22′-57″ E., what is the longitude of B ?

5. (U. of L.) The Greenwich sidereal time at Greenwich mean midnight (0 h.) on a particular day is found from the *N.A.* to be 19 h. 18 m. 36·7 s.

An observation of a star is taken in longitude 2° west at local sidereal time 17 h. 30 m. 50 s., on the following evening.

The correction of S.T. for longitude is 9·86 s. per hour.

Find the local mean time at instant of observation, P.M.

366·2422 sidereal days = 365·2422 mean solar days.

NOTE.—The reader is referred to Appendix II, page 546, for particulars of recent changes in the *Nautical Almanac*, and for notes on the special *Star Almanac for Land Surveyors*, first introduced in 1951.

CHAPTER XVII

MERIDIAN, LONGITUDE, LATITUDE, TIME

MERIDIAN

Determination of the Meridian and Azimuth.—The following are the chief methods employed for the determination of the meridian at a place, or for the azimuth of a line :

(1) Direction of shadow at apparent noon.
(2) Magnetic compass.
(3) Circumpolar Stars on the same vertical.
(4) Observation of Polaris or other circumpolar star at culmination.

} Approximate methods.

(5) The two greatest elongations of a circumpolar star.
(6) Equal altitudes of a circumpolar star.
(7) Equal altitudes of the sun.
(8) Single observations of a circumpolar star at elongation.
(9) Extra-meridian observation of the sun.
(10) Extra-meridian observation of a circumpolar star or of a star near the Prime Vertical.

(1) **Direction of Shadow at Apparent Noon.**—The direction of the meridian may be found very approximately by noting the direction in which the shadow of a vertical pole falls at apparent noon.

Thus in Fig. 17.1, if O represents the position of a vertical pole or a plumb-line, and the end B of the shadow is noted at apparent noon, the line OB fixes approximately the direction of the meridian.

To find the instant of apparent noon it is necessary to know the equation of time (E), and also local mean time. Thus, suppose the observer is in longitude 3° W., and his watch records Greenwich mean time, while E as given in the *N.A.* is 7 minutes 32 seconds, *i.e.* apparent noon at Greenwich occurs at 7 minutes 32 seconds after 12 o'clock on the watch.

Local apparent noon occurs $\frac{3}{360} \times 24$ hours $= 12$ minutes later still on account of the difference in longitude.

The direction of the shadow should therefore be observed at $19\frac{1}{2}$ minutes past 12 o'clock on the watch, in order to determine the meridian.

If the *Nautical Almanac* is not available, the approximate time of apparent noon and the direction of the meridian may both be determined if a number of concentric rings are described upon the ground about the base of the pole, and the points marked upon these where they are intersected by the locus of the end of the shadow as the sun rises and sets in the heavens.

Thus if aa_1 be two points on the same circle, the meridian will lie through O, and the mid-point (a_2) of aa_1.

Oa_2 should coincide with the direction of the shadow when this has its least value.

The method is not sufficiently refined to allow of any allowance being made for the change in the sun's declination during the interval between the observations.

It is obvious that this method will yield only very rough results, and if it were attempted to deduce the azimuth of a line OA from a distance as short as Oa_2 necessarily is, further considerable error would result.

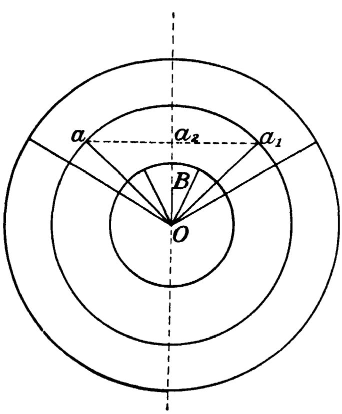


FIG. 17.1.—Shadows from Sun

(2) A **magnetic compass** may be employed to determine the magnetic meridian at any point; and the true meridian may then be approximately found by making due allowance for the magnetic declination of the place.

This method is, of course, only approximate, but it is sufficiently accurate to determine the position of the "N. point" upon maps of small surveys; it is, in fact, the method generally adopted in such cases.

The best results are obtained when the compass is attached to a theodolite or other instrument, and the magnetic azimuth of one of the survey lines or of some referring object is found as described in Chapter V.

With the magnetic compass fitted to an ordinary theodolite, the readings cannot be observed to less than about $\frac{1}{2}°$, and in most districts there is a considerable daily variation in the declination, as has already been mentioned on p. 62.

If the magnetic declination of the needle for the locality is unknown, or if the district is not free from local attraction, the result may be far from truth.

(3) **Circumpolar Stars on the same Vertical.**—If two stars have the same Right Ascension, or Right Ascensions which differ by 12 hours, then the great circle joining them passes through the celestial poles.

If they are "circumpolar" stars, they will be visible from the view-point of an observer, both at upper and lower transit. They will appear to the observer to describe circles about the pole, just as two points on the minute hand of a clock describe circles about the clock-face centre P, and just as the latter appear in a vertical plane when they cross the XII, P, VI line, so the stars appear in a vertical plane when they lie in the meridian, and conversely, when they appear in a vertical plane they lie in the meridian.

To be suitable for observation, the stars should have polar distances not too large, be sufficiently bright to be seen in a theodolite telescope, and be easily identified.

These conditions are not readily satisfied and pairs of stars need to be chosen where the R.A. of the one is not exactly equal to the R.A. (or R.A. + 12 h.) of the other, but nearly so.

The two stars will not then be in the meridian when they are on the same vertical, but be in a plane the azimuth of which can be calculated as explained below (Angle PZS, in Fig. 17.3).

The meridian can then be determined if the azimuth of the stars, when on the same vertical, is known. It can also be determined by an observation on one of the stars S_1, if previously calculations have been made to determine the time interval which elapses between the instant the stars are observed to be on the same vertical, and the time of transit of S_1. This is explained more fully below (Fig. 17.3).

The method of making the observations[1] with a theodolite is as follows: The instrument is set up at some convenient station O, and the telescope directed to a suitable referring object A. This object is bisected by the cross-hairs, and the readings of both verniers (or micrometers) of the horizontal circle are noted. The telescope is now directed to the stars which previous calculations or observations have shown to be approaching the desired positions. When it is observed that the cross-hairs intersect both stars as the telescope is slightly swung up and down in the vertical plane, either of two procedures may be adopted—or sometimes both may be combined.

(*a*) Note the readings of the two verniers, from which, later, knowing the azimuth of the stars at this instant, the direction of the meridian and the azimuth of OA can be determined.

(*b*) Note the time on a stop-watch, and, keeping the chosen star S_1 on the cross-hairs, follow its motion until the calculated interval of time has elapsed for it to have reached the meridian. At that instant the telescope is directed along the meridian, and the verniers can be read without hurry.

If the time interval is sufficiently long, the verniers may be read both when the stars are on the same vertical as (*a*), and later when S_1 is on the meridian.

After the verniers have been read, the telescope should be redirected to A to check that no " slip " has occurred, or if it has, to note its amount and decide whether the readings are sufficiently close to be averaged, or whether they must be rejected.

The main objection to this method, and one which precludes its adoption for accurate work, is that both stars have their greatest velocity in azimuth at the instant of observation, so that there is insufficient time to allow of both faces of the instrument being used.

[1] When using a theodolite for observations upon the stars, it is necessary to illuminate the cross-hairs. This may be done by means of a small lamp, the light from which is directed into the interior of the telescope through the horizontal or transverse axis, one side of this being made hollow for the purpose. The lamp is supported upon a bracket attached to the A frame. In the absence of such a contrivance, light may be reflected from a lamp, through the object-glass into the telescope, by means of a sheet of white paper or cardboard.

Hence, any lack of truth in the adjustments or levelling of the instrument may introduce errors of unknown magnitude in the results.

An approximate and necessarily rougher method, which does not involve the use of a theodolite, is as follows. A vertical pole (or preferably a long plumb-line) is fixed at a convenient point O, and when the two stars have revolved until they are seen to lie upon the vertical line thus set out, a second pole B is fixed in the direction of the line joining the first pole to the stars. This may be arranged by one observer, who can put himself and the pole in position on the opposite side of O to the stars. The line OB then marks out the approximate meridian through O. The method can be modified, if desired, to allow for the time interval between the instant the stars are on the same vertical and the transit of one of them, just as when a theodolite is used.

The following data are abstracted from the *N.A.* for 1946 :

	Mean R.A.			Mean Declination
	H.	M.	S.	
Polaris or α Ursae Minoris . . .	1	46	14·50	89°-00′-32″·2
,, δ Cassiopeiae . . .	1	22	15·78	59°-57′-19″·4
Mizar or ζ Ursae Majoris . .	13	21	45·26	55°-12′-24″·7
Benetnasch or η Ursae Majoris . .	13	45	24·86	49°-34′-56″·0

It will be seen from these figures that, in 1946, Polaris and η Ursae Majoris make one pair : the difference between the R.A.s is, however, decreasing from year to year. Mizar and δ Cassiopeiae make another pair ; and other combinations can be formed.

In choosing a pair of stars their suitability should be noted beforehand—particularly as to whether they will transit during the hours of darkness and at a convenient time. Their meridian altitudes should also be considered, as described on p. 456 and Fig. 16.7. From the figures quoted above, it will be seen that the meridian altitude at upper transit of η Ursae Majoris (Benetnasch) in latitude 50° N. is $90° - 25' - 04''.0$, *i.e.* slightly to the south of the zenith, and hence this transit is not very convenient for a theodolite, and quite useless for the plumb-line method.

At lower transit, the meridian altitude of η would be 9°-34′-56″, while Polaris would be at about 51° elevation.

Similarly, with δ Cassiopeiae or ζ Ursae Majoris (Mizar) at upper transit, one star of the pair would have a meridian altitude of 80° or more, and thence would not be very convenient for the rougher approximate method referred to above.

Fig. 17.2, which has been referred to on p. 458, represents a view of the celestial sphere looking downwards on the North Pole, along the polar axis. The whole diagram can be considered as revolving clockwise about P except the broken straight line representing the meridian, and the curve representing the horizon corresponding to the zenith position marked Z. Of the few selected stars shown, Polaris is nearing upper transit, between P and Z, while η Ursae Majoris, the end star of the handle of the Plough is nearing lower transit. Mizar, the middle star of the handle, has passed the meridian at lower transit, and δ Cassiopeiae has passed it at upper transit.

A diagram such as Fig. 17.2 is always helpful in deducing times of transit.

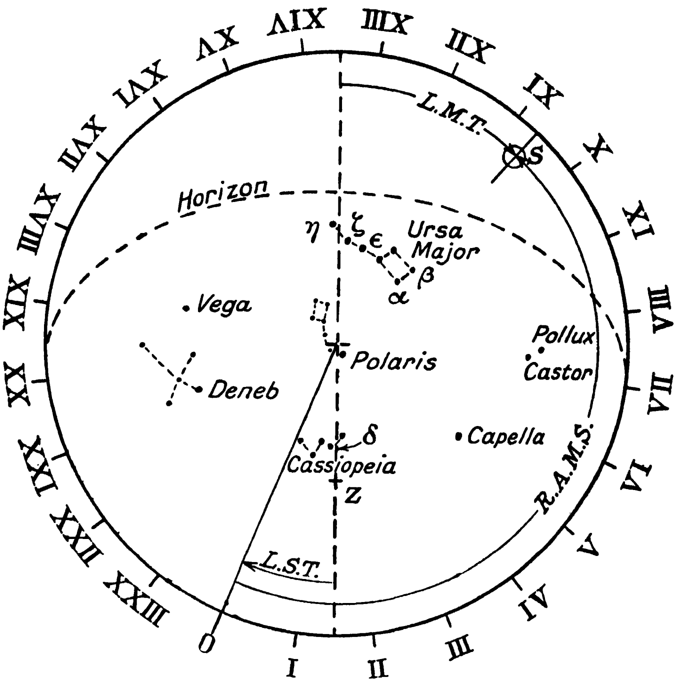


FIG. 17.2

Example.—The following example illustrates the method of determining the interval between the time two stars are on the same vertical, and the time of transit of one of them.

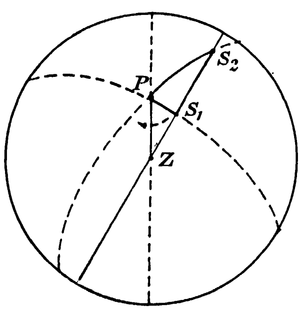


FIG. 17.3

In Fig. 17.3, which represents a view of the celestial sphere looking downwards on the zenith, vertical planes will be represented by straight lines. S_1 and S_2 thus lie on a straight line through Z. P represents the pole ; the straight line, PZ, the meridian. S_1 and S_2 lie on great circles passing through P, and these rotate about the polar axis. S_1 describes a small circle and S_2 a larger circle about P. The triangle PS_1S_2 retains its shape and rotates about P. The time interval to be calculated is represented by the angle ZPS_1, in the spherical triangle ZPS_1. In this example :

S_1 = Polaris whose R.A. = 1 h. 46 m. 14·50 s.
S_2 = Benetnasch whose R.A. = 13 h. 45 m. 24·86 s.
The polar distances = p_1 = 00°-59′-27″·8,
p_2 = 40°-25′-04″·0.

The latitude of the observer = 50° N.

In the spherical triangle PS_1S_2, we have :

the angle P = $RA_2 - RA_1$ = 11 h. 59 m. 10·36 s.
= 179°-47′-35″·4
and the sides $PS_1 = p_1$ = 00°-59′-27″·8,
$PS_2 = p_2$ = 40°-25′-04″.

By the application of formulae (7) and (8), p. 544,

$$\log . \tan \tfrac{1}{2}(S_1+S_2) = \log . \cot \tfrac{1}{2}P + \log \cos \tfrac{1}{2}(p_2-p_1) - \log \cos \tfrac{1}{2}(p_2+p_1),$$

and $$\log . \tan \tfrac{1}{2}(S_1-S_2) = \log . \cot \tfrac{1}{2}P + \log \sin \tfrac{1}{2}(p_2-p_1) - \log \sin \tfrac{1}{2}(p_2+p_1).$$

Substituting numerical values in these equations,

$$\tfrac{1}{2}(S_1+S_2) = 0°\text{-}06'\text{-}14''{\cdot}7,$$
$$\tfrac{1}{2}(S_1-S_2) = 0°\text{-}05'\text{-}55''{\cdot}2,$$
$$S_1 = 0°\text{-}12'\text{-}09''{\cdot}9,$$
$$S_2 = 0°\text{-}00'\text{-}19''{\cdot}5.$$

The sine formula may now be applied, *i.e.*

$$\log \sin Z = \log \sin S_1 + \log \sin PS_1 - \log \sin PZ,$$

where PZ is the colatitude $=40°$. Solving this equation, $Z = 00'\text{-}19''{\cdot}7$

This is the azimuth of the stars when they are on the same vertical, and knowing this, the meridian can be set out, or the azimuth of any referring object can be measured. We can, however, proceed to find the angle ZPS_1, by the application of formula (7) or (8), p. 544, to the spherical triangle ZPS_1, *i.e.*

$$\log \cot \tfrac{1}{2}P = \log \tan \tfrac{1}{2}(S_1+Z) + \log \cos \tfrac{1}{2}(PZ+p_1) - \log \cos \tfrac{1}{2}(PZ-p_1),$$

or $$\log \cot \tfrac{1}{2}P = \log \tan \tfrac{1}{2}(S_1-Z) + \log \sin \tfrac{1}{2}(PZ+p_1) - \log \sin \tfrac{1}{2}(PZ-p_1).$$

In these, equation $S_1 = 179°\text{-}47'\text{-}50''{\cdot}1$,

$Z = 0°\text{-}00'\text{-}19''{\cdot}7$,

and, by substitution in either, $\tfrac{1}{2}P = 05'\text{-}57''{\cdot}4$,

$P = 11'\text{-}54''{\cdot}8$.

Converting this angle into time, $P = 00$ m. 47·4 s.,

i.e. Polaris transits about 47 sidereal seconds after this star and Benetnasch appear on the same vertical.

If the data given above for the other stars are made use of, it will be found that Polaris crosses the meridian 23 m. 24 s. after this star and Mizar appear on the same vertical: and the upper transit of δ Cassiopeiae occurs 5·2 sidereal seconds after it and Mizar have appeared on the same vertical. The student can check these figures for himself, using the data on p. 466, or he can abstract other data from the *N.A.* for other dates.

(4) **Observation of a Circumpolar Star at Culmination.**—The chronometer time at which Polaris or other circumpolar star culminates is deduced from the *N.A.*, as explained in the example on p. 460. The horizontal angle between the referring object and the chosen star is observed at the exact instant of transit by means of a theodolite, and the azimuth of the referring object is deduced as in Method (3).

The objection to an observation at culmination is that the star at that period has its greatest relative velocity in azimuth—consequently, there is not sufficient time to use both faces of the instrument, and the results are liable to error, because it is practically impossible to ensure that the instrument is in perfect adjustment.

The nearer the star is to the pole, *i.e.* the greater its declination, the less is its apparent velocity, and hence the smaller is any error due to a given error in determining the L.M.T. of culmination (see p. 499).

As in other astronomical observations, it is advisable that the R.O.,[1] which must, of course, be luminous for night operations, shall be at some considerable distance from the instrument station, in order that it may be unnecessary to alter the focusing of the telescope, as otherwise errors may be introduced if the draw tube is not perfectly

[1] On the Survey of the Transvaal and Orange River Colony, a bull's-eye lantern shining through a circular aperture in an iron plate was employed.

true. Errors of bisection and centering may be considerable also if the R.O. is not sufficiently distant.

To obtain more reliable results, a number of observations may be made upon various stars, some at upper and some at lower culminations, and using " face right " and " face left " in alternate cases.

A comparison of the results will show what degree of accuracy is being obtained. The accuracy of a single observation can also be computed (see Chapter XIX).

This obviously depends upon a number of considerations, *e.g.* upon the particular instrument available. With a 6-in. theodolite reading to 20″, the probable error due to reading alone might be $\pm 3{\cdot}5$ seconds (see p. 120), but errors due to inaccurate adjustment might be much more appreciable, and several times as large as this.

An error in timing the observation may introduce a serious error in the result. If Polaris is being observed, this error is minimized, since the star is describing a circle in the heavens at less than 1° from the pole. If the observer is at latitude 50° then his colatitude is 40°. It will be seen from Fig. 17.3 that when S_1 representing Polaris, is about to cross the meridian $S_1Z : S_1P = 39$ or $41 : 1$,—say 40 to 1. The angle at Z is then altering at 1/40 the rate that the angle P is altering. P is altering 360° in 24 hours, *i.e.* 15″ of arc per second of sidereal time, so that its azimuth is changing 15/40″ per second, or 3″ of arc in 8 seconds of sidereal time.

Thus an error of 8 seconds in time in observing Polaris will produce about 3 seconds of error in the azimuth of the referring object, in latitude 50° N.

In the case of other circumpolar stars, obviously the error will be considerably larger, depending on the ratio $S_1P : S_1Z$ at culmination.

By the observation of two stars, one at the upper and one at the lower culmination, the effect of such an error in time tends to be eliminated, as the sign of the error would be different in the two cases.

On the Transvaal and Orange River Colony Survey [1] this method was adopted, a 10-in. Repsold micrometer instrument being used, and the final probable error in different examples varied from $\pm 0''{\cdot}13$ to $\pm 0''{\cdot}20$.

On the Orange River base, for instance, eight pairs of stars were observed on each of two nights, and six pairs on a third night, *i.e.* equivalent to twenty-two pairs in all. A comparison of the results showed a p.e. in the adopted azimuth of $\pm 0''{\cdot}13$, the greatest range of variation being $3''{\cdot}57$.

At other stations more observations were taken in some cases, *e.g.* on the South End Houts River Base thirty-eight pairs in all were observed, yielding a result with a p.e. of $\pm 0''{\cdot}18$, and a range of variation of $4''{\cdot}86$.

A single F.R. and F.L. observation of a single star with a 6-in. theodolite reading to 20″ might be expected to give a result within, say, 20″ of truth, if the time was known fairly accurately (see pp. 497-500);

[1] Geodetic Survey of the Transvaal and Orange River Colony.

or the mean of two observations at upper and lower culmination within about 10″, as the error due to time would be eliminated.

(5) **Two Greatest Elongations of a Circumpolar Star.**—As already mentioned, a circumpolar star appears to trace a circle in the heavens. Thus, in Fig. 17.4 let the circle represent the path of a circumpolar star S, and let O be the position of the observer and R any referring object.

At a little before the time of elongation, as determined from the *Nautical Almanac*, a theodolite is set up and levelled at O, the cross-hairs directed to R, and the readings of the two horizontal scale verniers noted.

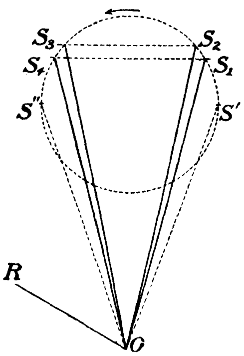


FIG. 17.4

The telescope is directed to the star a minute or so before it is at its greatest elongation to the east, say. The upper plates are then clamped, and, exact coincidence having been obtained with the slow-motion screw, the vernier readings are noted and the horizontal angle ROS′ computed.

As the motion of the star in azimuth is nil at the instant of elongation, and unappreciable for several minutes before and after the exact instant, there is sufficient time for the angle ROS′ to be re-measured on the opposite face of the instrument in order to eliminate errors due to lack of adjustment.

About twelve hours later the star elongates on the opposite side of the pole, and the angle ROS″ is similarly observed from a mean of four readings, *i.e.* two vernier readings "face right" and two "face left."

The mean of the mean values of the angle ROS′ and the angle ROS″ (*i.e.* the mean of the eight readings) gives the value of the angle ROP where P is the celestial pole, and therefore of the azimuth of the line OR.

The objection to this method is that one observation must be made in daylight, and this is seldom possible with an ordinary instrument. A further objection is the long interval which elapses between the two observations, during which time refraction possibly alters very considerably, or the sky may become overcast.

With suitable instruments, however, very accurate results can be obtained. The mean of the results of observations upon several stars may be adopted if necessary.

The time of elongation may be calculated as explained in the example on p. 461.

(6) **Equal Altitudes of a Circumpolar Star** (Fig. 17.4).—To obviate the necessity of taking one of the observations in daylight, and to lessen the long interval between the two observations of Method (5), the method of Equal Altitudes may be employed.

The time interval between the two sets of observations should be as long as possible, so that the star shall have a considerable velocity in altitude, *i.e.* shall not be very near its culminating points where the

curve appears very flat, and a small change in altitude causes considerable displacement in azimuth.

The instrument is set up at O, the cross-wires directed to R, and the readings of the two horizontal scale verniers recorded. The telescope is then directed to the star, which is in a position S_1, say, and exact coincidence obtained. The vertical angle α_1 is recorded, and also the values of the horizontal angle ROS_1 from both verniers.

The " face " of the instrument is then changed, the telescope redirected to R, both verniers read, and after an interval of, say, 15 to 20 minutes, the same star is again bisected with the cross-hairs, and the horizontal angle ROS_2 and the vertical angle α_2 recorded.

[An alternative method of procedure, which is perhaps preferable, is—instead of bisecting the star, and adopting whatever reading of α_1 results—to set the vernier of the vertical circle at a suitable definite value for the angle α_1, *e.g.* at an even minute, and to follow the star with the telescope until it makes its own contact with the horizontal web. The instrument is then clamped, the intersection of the cross-wires moved laterally to bisect the star accurately, and the horizontal angles read. Similarly, the star would be allowed to make its own contact at a definite angle for α_2.]

After culmination, the path of the star is again observed in the telescope, until it is found that at S_3 it has returned to the same altitude α_2. The approximate time can easily be predicted. The angle ROS_3 is then observed on both verniers.

The face of the instrument is changed to its original face, R intersected, the vertical circle vernier adjusted to read α_1, and the star observed in the telescope until it coincides exactly with the intersection of the cross-wires, when the angle ROS_4 is recorded.

The values of the vertical angles, α_1 and α_2, are not used further, but the azimuth of R, *i.e.* the angle ROP, is computed as the mean of the eight readings of the horizontal angles to S_1, S_2, S_3, and S_4.

In Fig. 17.4 the star is observed when on the same side of the pole as the zenith, but if more convenient observations may be taken to a star upon the lower portion of its orbit.

Unless unavoidable, the altitudes α_1 and α_2 should, however, not be very small, as refraction is then uncertain.

(7) **Equal Altitudes of the Sun.**—The method of procedure is very similar to that described above, except that the observations are carried out during the day instead of at night, and, as the position of the centre of the sun's disc cannot be directly estimated with any degree of certainty, the east and west limbs must be observed in azimuth, and the upper or the lower in altitude ; also a dark glass must be employed.

In the measurement of the horizontal angles, if one limb is observed in the forenoon and the opposite limb in the afternoon, the correction for semi-diameter is eliminated.

The time interval between the two observations must be recorded, in order that allowance may be made in the calculations for any change in the sun's declination.

Thus in the spherical triangle PZS_1 (Figs. 16.3 and 17.5), where S_1 is the position of the sun in the forenoon,

$PS_1 = 90 - \left\{\delta - \frac{\delta(\delta)}{2}\right\}$ nearly, where $\delta(\delta)$ is the total change in declination between the two observations, and δ is the mean declination, both $\delta(\delta)$ and δ being obtained from the *N.A.*

$PZ = 90 - l$, where l is the latitude.

$ZS_1 = 90 - a$, where a is the altitude.

Therefore, by equation (2), p. 544,

$$\cos PS_1 = \cos PZ \cos ZS_1 + \sin PZ \sin ZS_1 \cos \angle Z_1,$$

where $\angle Z_1$ is the azimuth $\angle PZS_1$, *i.e.*

$$\sin\left\{\delta - \frac{\delta(\delta)}{2}\right\} = \sin l \sin a + \cos l \cos a \cos Z_1, \quad . \quad (1)$$

and similarly in the spherical triangle PZS_2, where S_2 is the position of the sun in the afternoon (Fig. 17.5),

$$\sin\left\{\delta + \frac{\delta(\delta)}{2}\right\} = \sin l \sin a + \cos l \cos a \cos Z_2. \quad . \quad (2)$$

From (1) and (2) we have by subtraction

$$\sin\left\{\delta + \frac{\delta(\delta)}{2}\right\} - \sin\left\{\delta - \frac{\delta(\delta)}{2}\right\} = \cos l \,.\, \cos a \,(\cos Z_2 - \cos Z_1),$$

i.e. $$2 \cos \delta \sin \frac{\delta(\delta)}{2} = \cos l \,.\, \cos a \; 2 \sin \frac{Z_2 + Z_1}{2} \sin \frac{Z_1 - Z_2}{2},$$

from equations (11) and (13), p. 543.

But $\delta(\delta)$ is very small, as is $(Z_1 - Z_2)$ and $\frac{Z_2 + Z_1}{2}$ is approximately equal to Z_2, say, consequently we may write the equation as

$$\delta(\delta) \,.\, \cos \delta = \cos l \,.\, \cos a \,.\, \sin Z_2 \,.\, (Z_1 - Z_2),$$

i.e. $$Z_1 - Z_2 = \frac{\delta(\delta) \,.\, \cos \delta}{\cos l \,.\, \cos a \,.\, \sin Z_2}, \quad . \quad . \quad (3)$$

or as by the sine rule, if $ZPS_2 = \frac{t}{2}$ nearly, *i.e.* half the observed interval between the two observations,

$$\frac{\sin \frac{t}{2}}{\sin S_2Z} = \frac{\sin Z_2}{\sin S_2P},$$

i.e. $$\frac{\sin \frac{t}{2}}{\cos a} = \frac{\sin Z_2}{\cos \delta} \text{ nearly},$$

or $$\frac{1}{\sin \frac{t}{2}} = \frac{\cos \delta}{\cos a \sin Z_2},$$

therefore equation (3) [1] may be written

$$(Z_1 - Z_2) = \frac{\delta(\delta)}{\cos l \,.\, \sin \frac{t}{2}}, \qquad (4)$$

From equation (3) or (4) we may therefore find the difference between the azimuth of the sun in the two positions S_2 and S_1.

Thus in Fig. 17.5, if θ_1 and θ_2 are the bearings of S_1 and S_2 respectively, from the referring object R, the bearing of the centre line of the angle S_1ZS_2 is $\frac{\theta_1 + \theta_2}{2}$.

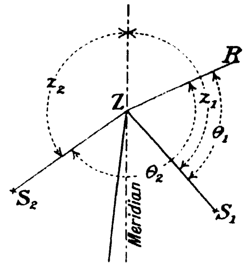


FIG. 17.5

If the declination is changing to the north, *i.e.* increasing, $Z_1 > Z_2$, and the centre line lies west of the meridian by an amount $\frac{Z_1 - Z_2}{2}$, consequently the bearing of the meridian from R is

$$\frac{\theta_1 + \theta_2}{2} - \frac{Z_1 - Z_2}{2},$$

i.e.

$$\frac{\theta_1 + \theta_2}{2} - \tfrac{1}{2} \,.\, \frac{\delta(\delta)}{\cos l \,.\, \sin \frac{t}{2}}. \qquad (5)$$

Similarly, if the declination is changing to the south, *i.e.* decreasing, the bearing of the meridian from R is

$$\frac{\theta_1 + \theta_2}{2} + \tfrac{1}{2} \,.\, \frac{\delta(\delta)}{\cos l \,.\, \sin \frac{t}{2}}. \qquad (6)$$

These equations refer to the Northern Hemisphere. For the Southern Hemisphere the terms north and south are transposed.

From the results of these equations the true azimuth of R can be easily computed.

(8) **Single Observation of a Circumpolar Star at Elongation.**—As in Method (5), the horizontal angle which the line joining the instrument station O to the referring object R makes with the direction OS—the line joining O to a circumpolar star at its greatest elongation, west or east—is observed on both faces of the instrument.

From this single observation, the azimuth of OR can be calculated if the latitude of the observer is known. A small error in the assumed latitude makes a much smaller error in the computed direction of the meridian, so that it is usually sufficiently accurate to measure the latitude from a map.

If a map is not available, the latitude may be determined by one of the methods enumerated later.

[1] See Chauvenet, *Spherical and Practical Astronomy.*

In the spherical triangle PZS (Fig. 17.6) we have the following data:

PS = co-declination or polar distance of S (from *N.A.*),
PZ = co-latitude,
$\angle$PSZ = 90° because the star is at elongation, therefore by the application of the sine formula (1), p. 544.

$$\sin PZS = \frac{\sin PSZ \,.\, \sin PS}{\sin PZ}$$

$$= \frac{\sin PS}{\sin PZ} \left(\text{or } \frac{\cos \delta}{\cos l}\right) \text{ because } \sin PSZ = 1. \qquad (7)$$

But, by observation, the horizontal projection of the angle ROS or the angle RZS is known, therefore the azimuth of R is obtained by adding or subtracting the angle PZS as the case may require.

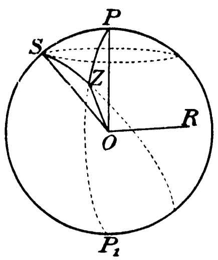


FIG. 17.6

An analysis of the errors in azimuth which result from a given error in latitude or declination or the hour angle is given on pp. 497-500.

Very accurate results should be obtained by this method, especially if a number of stars be observed and the azimuth computed separately from each. In this way the probable error of the observations can be computed, and by taking a sufficiently large number the p.e. of the result can be made as small as is required.

If the declination and latitude are known accurately, and as a small error in time has very little effect, the errors are chiefly due to instrumental defects and personal causes. Thus with a 6-in. theodolite reading to 20″, the maximum error of a single reading should be 10″. Taking F.R. and F.L. observations on two verniers, this error would be reduced to 5″. But as readings are taken both on the star and on R, the error would become $5\sqrt{2} = 7''$, say, due to reading alone. Other errors not eliminated by F.R. and F.L. observations (*e.g.* dislevelment, bisection, etc.) would increase this to 15″ to 20″, say, *i.e.* a single F.R. and F.L. observation on Polaris might be expected to yield a result within 15″ to 20″ of truth.

Example.—Polaris was observed at its western elongation on Aug. 30, 1919, and the horizontal angles measured from the referring object (which was to the west of the star) were :

	°	′	″
Face Right (mean of 2 verniers)	34	16	20
Face Left (mean of 2 verniers) .	34	16	40
Average . .	34	16	30

From the *N.A.* $\delta = 88°\text{-}52'\text{-}22''\cdot73$ and the polar distance PS = $1°\text{-}7'\cdot37''\cdot27$.
From the map the latitude was 50° N. and the co-latitude PZ = 40°.

$$\log \sin PS = \bar{2}\cdot2937810$$
$$\log \sin PZ = \bar{1}\cdot8080675$$

$$\log \sin \angle Z = \bar{2}\cdot4857135$$
$$\therefore \angle Z = 1°\text{-}45'\text{-}12''\cdot6 \text{ nearly}$$

The azimuth of R west of north is therefore $\angle Z + 34°\text{-}16'\text{-}30''$, *i.e.* $36°\text{-}1'\text{-}43''$, or the azimuth, considering due north as zero, $= 323°\text{-}58'\text{-}17''$.

The time of elongation may be predicted as in the example on p. 461.

If observations are made at elongation on two stars which elongate about the same time, it is not necessary to know the latitude, as that term may then be cancelled.

Thus if Z_1 and Z_2 are the azimuths at elongation of two stars whose declinations are δ_1 and δ_2, and z is the horizontal angle between them, then as $\sin Z_1 = \frac{\cos \delta_1}{\cos l}$, and $\sin Z_2 = \frac{\cos \delta_2}{\cos l}$,

$$\therefore \frac{\sin Z_1}{\sin Z_2} = \frac{\cos \delta_1}{\cos \delta_2},$$

i.e.

$$\frac{\sin (z \pm Z_2)}{\sin Z_2} = \frac{\cos \delta_1}{\cos \delta_2},$$

where the angle between the stars, $z = Z_1 - Z_2$ when the stars are in conjunction, or $z = Z_1 + Z_2$ when in opposition ; *i.e.*

$$\frac{\sin z \cos Z_2 \pm \cos z \sin Z_2}{\sin Z_2} = \frac{\cos \delta_1}{\cos \delta_2},$$

i.e.

$$\cot Z_2 = \frac{\cos \delta_1}{\cos \delta_2} \operatorname{cosec} z \mp \cot z,$$

or similarly,

$$\cot Z_1 = \cot z \mp \frac{\cos \delta_2}{\cos \delta_1} \operatorname{cosec} z,$$

the $-$ sign being taken when the stars are in conjunction and the $+$ sign when they are in opposition.

As before, the azimuth of a referring object R may be determined if from it the bearings ROS_1 and ROS_2 of two stars of known declination are measured, and the value of z computed.

(9) **Single Observation upon the Sun when out of the Meridian.**—If the bearing of one limb of the sun with reference to some referring object R be observed exactly at apparent noon, and the result corrected for semi-diameter, the direction of the meridian may be found approximately.

As in the case of observations upon a star at culmination (Method (4)), only one face of the instrument can be used.

More accurate results may be obtained by observations to the sun when the latter is near the Prime Vertical, its motion in azimuth being then much less rapid.

The bearing of the centre of the sun from OR is found in the usual way, and the altitude is noted at the same instant : corrections for parallax, refraction, dip, and semi-diameter are applied where required.

In the spherical triangle PZS (Fig. 16.3) the following data are known, therefore :

PZ = co-latitude—obtained from a map or observed independently.
PS = co-declination (from *N.A.*).
ZS = zenith distance = (90 − altitude).

By the application of formulae (11) or (9) or (2), p. 544, the value

of the angle PZS, *i.e.* the azimuth of the sun at the moment of observation, may be computed, *i.e.*

$$\cos^2 \frac{Z}{2} = \sin s \,.\, \sin (s - PS) \operatorname{cosec} PZ \operatorname{cosec} ZS,$$

or $$\tan^2 \frac{Z}{2} = \operatorname{cosec} s \,.\, \sin (s - PZ) \sin (s - ZS) \operatorname{cosec} (s - PS),$$

or $$\sin \delta = \sin l \sin a + \cos l \cos a \cos Z,$$

where $Z = \angle PZS$, and $s = \frac{PZ + ZS + PS}{2}$.

To obtain the declination from the *N.A.* it is necessary to know the G.M.T. of the observation ; or the L.M.T. and the longitude, as the rate of change of declination of the sun is considerable, especially at the equinoxes, when it may be as much as 59 seconds or more per hour.

At the solstices the rate is zero.

If possible, both faces of the instrument should be employed, and the mean of the two results adopted.

If the hour angle SPZ and the altitude and declination are known, the direction of the meridian may be found by the application of the sine formula (1), p. 544, *i.e.*

$$\sin SZP = \frac{\sin PS}{\sin SZ} \,.\, \sin SPZ. \quad . \quad . \quad (8)$$

The hour angle is deduced from the difference between L.M.T. of observation and that of culmination.

Similarly, the triangle may be solved when the hour angle, and either the latitude and altitude, or the latitude and declination are known.[1]

The effect upon the result of a small error in one of the quantities may be ascertained by differentiation, as shown on p. 499.

With a 6-in. theodolite, and single F.R. and F.L. readings, the result should not be more than 30″ to 1′ out of truth. If two or three independent sets of F.R. and F.L. observations are taken, a much greater degree of accuracy can be obtained ; and if the results be compared, the probable error of a single observation by a certain observer with a certain instrument may be roughly ascertained.

Example.—On March 24, 19—, at 10 A.M., in latitude 22°-35′-0″, and longitude 117°-30′-15″ W., the average apparent mean altitude of the sun when out of the meridian was 52°-16′-9″·06. The average bearing of the sun from a referring object R was 30°-14′-16″. Find the azimuth of OR.

	′	″		°	′	″
Mean apparent altitude			=	52	16	09·06
Correction for refraction[2] . . .			=			44
				52	15	25·06
Correction for semi-diameter[3]	16	3·90				
Correction for parallax[4] (8″·83 × cos 52°-15′)		5·40			16	9·30
True altitude . .				52	31	34·36

[1] See Example 5, p. 442.
[2] Chambers' *Mathematical Tables.*
[3] *Nautical Almanac.*
[4] *Ibid.*

	H.	M.	S.	
L.M.T. of observation (approx.) . .	10	0	0	(Mar. 24)
Allow for longitude [1]	7	50	1	
G.M.T. of observation	17	50	1	(Mar. 24)

	°	′	″
Declination N at 0 h. Mar. 25 . .	1	18	34·5
Variation [2] $= 59''\cdot09$ per hour increasing, ∴ change in δ in 6 h. 9 m. 59 s., to be deducted since time of observation is before 0 h. Mar. 25 . . . =		6	04·4
Declination at time of observation . =	1	12	30·1
and Polar distance (PS) . . . =	88	47	29·9

In the spherical triangle PZS we have

	°	′	″
PZ = co-lat. =	67	25	00
PS = co-declin. =	88	47	29·9
ZS = co-alt. =	37	28	25·64
2)	193	40	55·54
s =	96	50	27·77
s − PZ =	29	25	27·77
s − PS =	8	02	57·87
s − ZS =	59	22	02·13

By the application of formula (9), p. 544,

$$\begin{aligned}
\log \operatorname{cosec} s \quad . &= \cdot0031030\\
\log \sin (s-\mathrm{PZ}) \ . &= \bar{1}\cdot6913242\\
\log \sin (s-\mathrm{ZS}) \ . &= \bar{1}\cdot9347262\\
\log \operatorname{cosec} (s-\mathrm{PS}) &= \cdot8537882\\
2)\ &\ \cdot4829416\\
\log \tan \tfrac{1}{2}\mathrm{Z} &= \cdot2414708\\
\tfrac{1}{2}\mathrm{Z} &= 60^\circ\text{-}9'\text{-}57''\cdot5\\
\mathrm{Z} &= 120^\circ\text{-}19'\text{-}55''
\end{aligned}$$

By the application of formula (11), p. 544,

$$\begin{aligned}
\log \sin s \quad . &= \bar{1}\cdot9968970\\
\log \sin (s-\mathrm{PS}) &= \bar{1}\cdot1462118\\
\log \operatorname{cosec} \mathrm{PZ} \ . &= \cdot0346468\\
\log \operatorname{cosec} \mathrm{ZS} \ . &= \cdot2158119\\
2)\ &\ \bar{1}\cdot3935675\\
\log \cos \tfrac{1}{2}\mathrm{Z} &= \bar{1}\cdot6967838\\
\tfrac{1}{2}\mathrm{Z} &= 60^\circ\text{-}9'\text{-}57''\cdot5\\
\mathrm{Z} &= 120^\circ\text{-}19'\text{-}55'' \text{ as before}
\end{aligned}$$

But the bearing of the sun from R . . . . . $=30^\circ\text{-}14'\text{-}16''$
Therefore the azimuth of R measured clockwise from the north $=90^\circ\text{-}5'\text{-}39''$

(10) **Extra-Meridian Observation of a Circumpolar Star, or of a Star near the Prime Vertical.**—The procedure in this case is similar to that described in Method 9 for an extra-meridian observation of the sun, with the exceptions that—

(1) No corrections are necessary for semi-diameter or parallax.
(2) The declination varies very slowly, so that it is not necessary to know accurately the L.M.T. or G.M.T. in order to obtain this quantity from the *N.A.* (See Question 9, p. 503.)

[1] See Example on p. 460. [2] *Ibid.*

In the case of the circumpolar star, the apparent velocity is much less than that of the sun, hence there is more opportunity of taking F.R. and F.L. observations without undue haste.

Method 10, of which Method 8 is a particular case, is therefore rather more reliable than Method 9, though, as the work must be done at night, the sun is rather more convenient for many purposes.

Example.—At a certain time on November 2 the bearing of α Andromedae from a referring object R was 84°-16′-0″. The mean altitude corrected for refraction was 48°-14′-27″. Latitude, 45°-18′-20″ N. Declination from *N.A.* 28°-36′-7″·3. Find the azimuth of R.

	°	′	″	°	′	″
Latitude . . . =	45	18	20			
Therefore co-lat. = PZ =	44	41	40	44	41	40
Altitude . . . =	48	14	27			
Therefore co-alt. = ZS =	41	45	33	41	45	33
Declination N. . =	28	36	07·3			
Co-declin. = PS . =	61	23	52·7	61	23	52·7
			2)	147	51	05·7
			s =	73	55	32·9
			s − PS =	12	31	40·2

By the application of formula (11), p. 544,

$$\begin{aligned}
\log \sin s \quad . &= \bar{1}\cdot9826800\\
\log \sin (s - \mathrm{PS}) &= \bar{1}\cdot3362872\\
\log \operatorname{cosec} \mathrm{PZ} &= \cdot1528434\\
\log \operatorname{cosec} \mathrm{ZS}\ . &= \cdot1765251\\
&2)\bar{1}\cdot6483357\\
\log \cos \frac{Z}{2} &= \bar{1}\cdot8241678\\
\frac{Z}{2} &= 48°\text{-}9'\text{-}32''\cdot74\\
Z &= 96°\text{-}19'\text{-}5''\cdot5
\end{aligned}$$

If the star is in the west at the time of observation, a sketch will show that the bearing of A from the north is 360° − (84°-16′-0″ + 96°-19′-5″·5), *i.e.* 179°-24′-54″·5 nearly.

If the hour angle is known, *i.e.* the M.T. interval since culmination converted into L.S.T. and then into arc, ∠Z may be found by the sine formula (1), p. 544, *i.e.* if ∠P = 48°-56′-6″,

$$\begin{aligned}
\log \sin \mathrm{P} &= \bar{1}\cdot8773511\\
\log \sin \mathrm{PS} &= \bar{1}\cdot9434777\\
\log \operatorname{cosec} \mathrm{ZS} &= \cdot1765251\\
\log \sin Z &= \bar{1}\cdot9973539\\
Z &= 96°\text{-}19'\text{-}6''\cdot4
\end{aligned}$$

Longitude

Longitude is calculated by the comparison of local time with the time at the standard meridian. *E.g.* if the meridian at Greenwich is adopted as the standard or zero meridian, the longitude of a place A may be found by noting the difference between Greenwich time and the local time at A.

Thus if mean noon at A occurs at 12.35 P.M. G.M.T., the longitude of A will be 8°-45′ W. (see Chapter XVI).

The determination of longitude thus involves two problems :

(1) The determination of local time.
(2) The determination of Greenwich time.

Local Time

The chief methods adopted for the direct determination of local time are :

(1) Shadow from sun (very rough).
(2) Meridian transit of the sun or a star.
(3) Equal altitudes of the sun or a star.
(4) Extra-meridian observation of the sun or a star.

(1) **Shadow from Sun.**—Local apparent noon may be very roughly ascertained by noting the time at which the shadow of a vertical pole thrown by the sun is of minimum length (see p. 463), or more easily by taking the mean of the times at which the end of the shadow cuts concentric circles described about the foot of the pole or plumb-line.

Local mean noon may be deduced from this by adding or subtracting the equation of time found from the *Nautical Almanac.*

(2) **Meridian Transit of the Sun or of a Star.**—If the direction of the meridian is accurately known, the transit of the sun or a star may be observed with a theodolite, and the chronometer time noted.

When the sun is observed, the mean time of transit of the two limbs is noted.

This determines local apparent noon, to which is added or subtracted the equation of time to compute local mean noon.

When a star is observed, the local mean time of transit is calculated from the *Nautical Almanac* (as in the example on p. 461) and compared with the time recorded on the watch of the observer.

(3) **Equal Altitudes of the Sun or a Star.**—When the direction of the meridian is not accurately known, the time of transit, as recorded on a watch, may yet be ascertained by noting the times at which the sun or a star has an equal altitude before and after culmination, and taking a mean of the values (cf. Fig. 17.4).

By this means several sources of error are avoided, *e.g.*

(*a*) The actual altitude is not required, so that errors of graduation, index error, etc., are eliminated.
(*b*) Refraction will probably have about the same value for each observation, so that no correction need be applied.
(*c*) In observations upon the sun the semi-diameter correction is unnecessary, if the same limb is observed upon each occasion.

Consequently very accurate results may be expected when a suitable star, *i.e.* one with a large polar distance, is observed.

(4) **Extra-Meridian Observation of the Sun or a Star.**—By means of an extra-meridian observation on the sun or a star, the hour angle may be calculated if the latitude and declination are known, and if

the altitude is observed, *i.e.* by the application of formula (10), p. 544,

$$\sin\frac{P}{2}=\sqrt{\sin(s-SP)\sin(s-ZP)\operatorname{cosec}.SP.\operatorname{cosec}ZP},$$

or from the fundamental formula (2), p. 544,

$$\sin a=\sin l\sin\delta+\cos l\cos\delta\cos P,$$

or if the azimuth of the star (or the sun) from the meridian is observed, in addition to the altitude, the sine formula may be applied, *i.e.*

$$\sin P=\frac{\sin Z.\sin SZ}{\sin SP}. \quad . \quad . \quad (9)$$

Then if the time of culmination is known from the *Nautical Almanac* (see p. 461), and the hour angle computed from one of the above formulae, the local mean time of observation may be deduced, and compared with the time recorded on the watch of the observer.

The best results may be expected when the sun or star is near the prime vertical (see p. 449).

A single F.R. and F.L. observation should yield a result within, at most, one second of time (see example, p. 501), but with several observations of course much greater accuracy can be obtained.

Example.—On March 24, 19—, in latitude 22°-35′-0″, and longitude 117°-30′-15″ W., the mean apparent altitude of the sun's lower limb was 52°-16′-9″·06. The chronometer read 10 h. 0 m. 0 s. A.M. What was the chronometer error on L.M.T.?

				°	′	″
Mean apparent altitude			=	52	16	09·06
Correction for refraction from tables [1] .			=			44
				52	15	25·06
Correction for semi-diameter [2]	16′	3″·90				
Correction for parallax [2] .		5·40			16	09·30
True altitude . .				52	31	34·36
				H.	M.	S.
Chronometer time (A.M.) approximately			=	10	00	00
Correction for longitude . . .			=	7	50	01
Therefore G.M.T. of observation .				17	50	01 (Mar. 24)
				°	′	″
Declination [2] N at 0 h. Mar. 25 . .			=	1	18	34·5
Change in δ (59″·09 per hour) [2]						
(Deduct. See example, p. 477) . .			=		6	04·4
Declination at time of observation .			=	1	12	30·1
and Polar distance (PS) . . .			=	88	47	29·9

We have then in the spherical triangle PZS

		°	′	″
ZS = co-alt.	=	37	28	25·64
PS = co-declin.	=	88	47	29·90
PZ = co-lat.	=	67	25	00·00
	2)	193	40	55·54
s	=	96	50	27·77
s − PS	=	8	02	57·87
s − PZ	=	29	25	27·77

[1] Chambers' *Mathematical Tables*. [2] *Nautical Almanac*.

Applying formula (10), p. 544,

$$\begin{aligned}
\log\sin 8°\text{-}2'\text{-}57''{\cdot}87 &= \bar{1}{\cdot}1462118\\
\log\sin 29°\text{-}25'\text{-}27''{\cdot}77 &= 1{\cdot}6913242\\
\log\text{cosec}\ 88°\text{-}47'\text{-}29''{\cdot}9 &= {\cdot}0000966\\
\log\text{cosec}\ 67°\text{-}25'\text{-}0''{\cdot}0 &= {\cdot}0346468\\
&\ 2)\bar{2}{\cdot}8722794\\
\log\sin\frac{P}{2} &= 1{\cdot}4361397\\
\frac{P}{2} &= 15°\text{-}50'\text{-}31''{\cdot}22\\
P &= 31°\text{-}41'\text{-}02''{\cdot}44
\end{aligned}$$

			H.	M.	S.	
P in time			=2	6	44·16	
i.e. L.A.T. of observation . .			=9	53	15·84	A.M.
	M.	S.				
Equation of time (25 d. 0 h.)	6	34·11				
6 h. 10 m. × 0·760 s. per hour		4·68		6	38·79	
Therefore L.M.T. of observation .			=9	59	54·63	A.M.
Error of chronometer on L.M.T. .			=		5·37	fast

Similarly, if the azimuth were known 120°-19′-55″ from the north (cf. Example on p. 447), then applying the sine formula

$$\begin{aligned}
\log\sin 120°\text{-}19'\text{-}55'' &= \bar{1}{\cdot}9360683\\
\log\sin 37°\text{-}28'\text{-}25''{\cdot}64 &= \bar{1}{\cdot}7841881\\
\log\text{cosec}\ 88°\text{-}47'\text{-}29''{\cdot}90 &= {\cdot}0000966\\
\log\sin P &= \bar{1}{\cdot}7203530\\
P &= 31°\text{-}41'\text{-}02''{\cdot}44
\end{aligned}$$

from which the error of the chronometer is calculated as before.

Example.—On November 2 of a certain year in latitude 45°-18′-20″ an observation upon the star α Andromedæ gave an altitude (corrected for refraction) of 48°-14′-27″. The declination from the *N.A.* = 28°-36′-07″·3 N. What was the L.S.T. of the observation ?

	°	′	″
PZ = co-latitude =	44	41	40
ZS = co-altitude =	41	45	33
PS = co-declination =	61	23	52·7
2)	147	51	05·7
s =	73	55	32·9
s − PS =	12	31	40·2
s − PZ =	29	13	52·9

By formula (10), p. 544,

$$\begin{aligned}
\log\sin(s-PS) &= \bar{1}{\cdot}3362872\\
\log\sin(s-PZ) &= \bar{1}{\cdot}6887200\\
\log\text{cosec}\ PS &= {\cdot}0565223\\
\log\text{cosec}\ PZ &= {\cdot}1528434\\
&\ 2)\bar{1}{\cdot}2343729\\
\log\sin\frac{P}{2} &= \bar{1}{\cdot}6171865\\
\frac{P}{2} &= 24°\text{-}28'\text{-}03''\\
P &= 48°\text{-}56'\text{-}06''
\end{aligned}$$

To convert arc into time (1° = 4 m.),

	H.	M.	S.
48° =	3	12	0
56′ =		3	44
6″ =			·4

	H.	M.	S.
The interval in S.T. since culmination thus =	3	15	44·4
But the R.A. of the star from the *N.A.* =	0	03	60·2
Therefore L.S.T. of observation . . =	3	19	44·6

If the longitude is known, and the sidereal time at G.M.M. (*i.e.* 2 h. 41 m. 41·45 s.), the L.M.T. can be calculated as in the previous example.

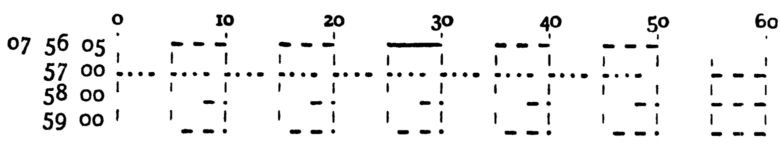
Greenwich Time

The chief method of finding Greenwich Mean Time, nowadays, is by means of wireless time signals which are sent out regularly from a large number of stations situated in various parts of the world.

(1) **Wireless Time Signals.**—The *Admiralty List of Radio Signals*, Vol. II, issued annually, gives full particulars of the various time signals sent out by different stations, and the current volume should be consulted if further information is required. In this publication the systems are classified as—

(1) International ONOGO.
(2) ONOGO Modified. 1924. New International.
(3) United States. Old.
(4) United States. New.
(5) Modified Rhythmic.
(6) Russian ordinary system.
(7) Other systems.

The *Old International* (*ONOGO*) type of signal is arranged as shown in the diagram below :

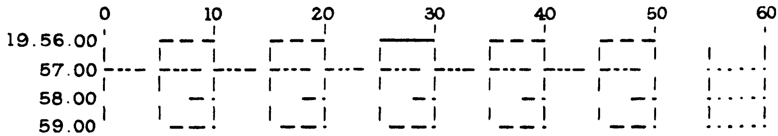


The actual signal, in some cases, is considered to be the end of the final dash in the letter O (— — —), which represents an even minute. In other cases the final dots of the N (— ·), and the final dots of the G (— — ·) groups are specified, and these give the even 10 seconds. Generally it is sufficiently accurate to take any of these as the correct signal.

The three minute intervals terminate, it will be noticed, in the letters O, NO, GO respectively, from which the name of the system is derived. This system is still employed by Spain, India, Ceylon, Java, Western Australia, and South Australia.

In the *New International*, or *modified ONOGO*, system, instead of

each minute terminating in three dashes, each of 1 second duration, six dots are substituted, at the 55, 56, 57, 58, 59, and 60 seconds of the three minute intervals as shown below :



This system has been adopted by France, Siberia, South Africa, Portuguese East Africa, Victoria, New Zealand, and Argentina.

The *United States New Signal* consists of a dot at each second over a period of 5 minutes, with certain dots omitted. The dot omissions

in the 1st minute are	29,	51,				56,	57,	58,	59
,, 2nd ,,	29,		52,			56,	57,	58,	59
,, 3rd ,,	29,			53,		56,	57,	58,	59
,, 4th ,,	29,				54,	56,	57,	58,	59
,, 5th ,,	29,	51,	52,	53,	54,	56,	57,	58,	59

The 60th second is, like the remainder, indicated by a dot, except at the end of the 5th minute, when a 1-second dash is transmitted, the beginning of the dash being the time signal.

Time signals consisting of the automatic transmission by the standard clock at Greenwich Observatory, of six dots at 1-second intervals, are sent out on various wave-lengths and at various times on the B.B.C. Home, Light, and Overseas programmes. The commencement of the final dot is the actual signal and is normally accurate to 0·1 second. On the B.B.C. Home programme, for instance, the transmissions are at 07 *h.* 00 *m.* (except Sundays), 08 *h.* 00 *m.*, 13 *h.* 00 *m.*, 18 *h.* 00 *m.*, 23 *h.* 00 *m.*, on eight wave-lengths from 203·5 *m.* to 514·6 *m.* A similar signal of six " pips " is sent out from Cleethorpes every hour excepting 10 *h.* and 18 *h.*, and this is normally correct to 0·05 seconds.

The *Rhythmic* system employs the vernier principle in that 306 beats, equally spaced, are transmitted in 300 seconds. Of these beats, the following are dashes, 1, 62, 123, 184, 245, and 306, while the remainder are dots. The method employed is either (1) to count the number of dots from each dash in turn until coincidence occurs between the beat of the signal and the tick of the chronometer, or (2) to note the chronometer time of each dash and the chronometer time of each coincidence, when the desired results can be interpolated and averaged. For many purposes it is sufficient to note only the commencement of the dashes, indicating the exact minutes, and disregard the dots ; while for extreme accuracy the vernier principle should be employed as described, and allowances made for any corrections to the signal, published later in " Notices to Mariners ". The Rhythmic signal is employed by Great Britain, France, Russia, French Indo-China, Japan, etc.

The various time signals transmitted are too numerous to be listed

here, but full particulars may be obtained, both as regards the nature of the call signal and the time signal, in the *Admiralty List of Radio Signals.*

As examples, mention may, however, be made of the following, the G.M.T. values signifying the completion of International, and the commencement of Rhythmic signals :

G.M.T.			Station	Signal	Kcs. and Wave-length	
H.	M.	S.				
10	00	00	Rugby (G.B.R.)	Rhythmic	16·00	18750
18	00	00	,, ,,	,,	,,	,,
09	30	00	Eiffel Tower (FLE)	New Inter.	113·2	2650
09	31	00	,, ,,	Rhythmic	,,	,,
22	30	00	,, (FLD)	New Inter.	13950	21·5
22	31	00	,, ,,	Rhythmic	,,	,,
08	30	00	Culcutta (VWC)	ONOGO	150	2000
16	30	00	,, ,,	,,	,,	,,

(2) **Signals.**—Greenwich or Standard Time may be transmitted by ordinary telegraphy instead of by radio ; and heliograph and other signals have been used under some conditions. An example of a somewhat similar use of signals occurs when the difference in longitude between two stations A and B is required. Here it is advisable that a number of telegraphic signals be sent in each direction and the mean result adopted. This procedure eliminates errors due to the fact that the transmission of the signal is not instantaneous, but occupies a definite period of time. It also enables the readings at certain instants and the rates of the chronometer at A and B to be compared. If, then, at each station the time of transit of a chosen star is observed, the interval between the transits at A and B may be recorded, and from this the difference in longitude can be computed.

For accurate work the observations are conducted on several nights, and a number of stars observed ; where practicable, also, the two operators are interchanged between the stations, in order to eliminate the " personal equation ".

(3) The **Chronometer**, at one time, was largely relied upon to give Greenwich Mean Time, particularly for nautical purposes : and it is still a valuable piece of equipment used in conjunction with radio and other signals. Much care requires to be taken in its transport, and it is not very convenient for the general surveyor.

The " rate " of the chronometer, *i.e.* the rate at which it gains and loses, must be accurately determined by frequent comparisons with standard time, so that the error at the time of any observation may be calculated.

It is, of course, not necessary that the chronometer shall always record Greenwich Mean Time directly, provided that (*a*) the error at some specific instant is known and (*b*) the " rate " of the chronometer is uniform and also known.

(4) **Lunar Observations.**—The moon moves comparatively quickly,

relatively to the fixed stars, so that if the lunar distances subtended between the bright limb of the moon and certain stars are measured, the G.M.T. of the observation can be deduced from published tables. Similarly the times of immersion or emersion of certain stars when eclipsed by the moon (" lunar occultations ") afford data from which the G.M.T. can be computed.

Another method is to observe the transit across the meridian of a star which has approximately the same declination and R.A. as the moon, and note the interval in sidereal time between that and the transit of the moon's bright limb over the meridian. Knowing the R.A. of the star, the R.A. of the moon's bright limb is deduced and the corresponding G.M.T. calculated from the *Nautical Almanac.*

Other methods involve observations of the meridian altitude of the moon. None of these methods are of much service to the surveyor, and are not required now that radio signals are so readily available.

(5) Another method, again not likely to be adopted by a surveyor, involves observations of the eclipses of Jupiter's satellites (see *Nautical Almanac*).

LATITUDE

The following are a few of the chief methods employed for the determination of latitude :

(1) Altitudes of Polaris at upper and lower culminations.
(2) Meridian altitude of the sun or a star.
(3) Meridian altitude of one or more " pairs " of stars.
(4) Circum-meridian altitudes.
(5) Extra-meridian altitudes.
(6) Extra-meridian altitudes of Polaris.

(1) **Altitude of Circumpolar Star at Upper and Lower Culminations.**—As already mentioned, and as may be seen from Fig. 17.7, the latitude of a place (*i.e.* the arc EZ) is equal to the altitude of the Pole (*i.e.* the arc H_1P). Consequently as the altitude of the Pole is the mean of the altitudes of the Pole star (or any other circumpolar star) at its upper and lower culminations, the latitude may be determined from the measurement of these two quantities, after applying the usual corrections for refraction, etc.

It is not necessary in such a case that the declination of the star should be known, but the times of culmination must be ascertained (see the example on p. 461).

The two observations are necessarily twelve hours apart, and consequently, as in Method 5 for meridian, this method cannot often be resorted to.

(2) **Meridian Altitude of the Sun or a Star.**—If the declination of the sun or of the star is known, it is unnecessary to take observations at the two points of culmination : sufficient data are obtained if the altitude at either one or the other is observed.

It is always advisable that both faces of the instrument should be used : but as the observed body is not at rest, one at least of the

observations is necessarily made when the sun or star is slightly off the meridian, and consequently the result obtained is not as accurate as those derived by the methods to be described later.

In the case of a star near the Pole, *e.g.* Polaris, the error introduced in altitude is small, but it is more considerable in the case of the sun or of stars at some distance from the Pole.

The latitude is best deduced from the altitude—corrected for refraction, and, if necessary, for parallax, semi-diameter, and dip—by means of a sketch.

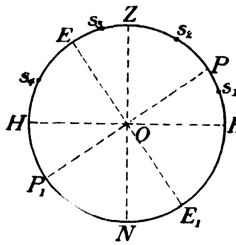


FIG. 17.7

Thus in Fig. 17.7, if the plane of the paper represents the plane of the meridian ZPNHE, the celestial body may lie upon any of the four arcs H_1P, PZ, ZE, or HE, where P represents one of the Poles, Z the zenith, E the Equator, and H or H_1 the horizon.

If upon any of the remaining arcs, the body would be below the horizon, and, as a rule, invisible.

The altitude in each of the four cases $= H_1S_1$, H_1S_2, HS_3, HS_4.

The declination in each of the four cases $= +E_1S_1$, $+ES_2$, $+ES_3$, $-ES_4$.

The latitude is therefore the arc EZ or H_1P, *i.e.*

(1) Latitude $= 90° - ($declin. $-$ alt.), *i.e.* alt. + co-declin. . (10)

(2) Latitude = declin. + alt. $- 90°$, *i.e.* alt. $-$ co-declin. . (11)

(3) Latitude $= 90° - ($alt. $-$ declin.), *i.e.* co-alt. + declin. . (12)

(4) Latitude $= 90° - ($alt. + declin.), *i.e.* co-alt. $-$ declin. . (13)

Example.—An observation for latitude was made on December 30, 19—, in longitude 82°-17′-30″ E., the meridian altitude of the sun's lower limb being then 40°-15′-13″. What was the approximate latitude of the place (*a*) the sun being south of the observer's zenith; (*b*) the sun being north of the observer's zenith?

	′	″		°	′	″
Apparent altitude			=	40	15	13
Correction for refraction [1] (allowing for temperature and barometer—from tables)			=		1	10
				40	14	03
Correction for semi-diameter [2] . .	16	17·5				
Correction for parallax [2] $= 8''{\cdot}95 \times \cos 40°\text{-}15'$		6·9			16	24·4
True altitude			=	40	30	27·4

	H.	M.	S.
To convert longitude to time 82° =	5	28	0
17′ =		1	8
30″ =			2
=	5	29	10
G.A.T. at the time of observation, *i.e.* at L.M.N. on Dec. 30 =	6	30	50

[1] Chambers' *Mathematical Tables.* [2] *Nautical Almanac.*

	°	′	″
Sun's declination at G.A.N.,[2] Dec. 30 (and it is decreasing at the rate of 9″·17 per hour)[1] .	=23	13	15·0 S.
Therefore at 6 h. 30 m. 50 s. Dec. 30, the increase was $9{\cdot}17 \times 5\frac{1}{2}$	=		50·4
Sun's declination (δ) at L.A.N. . .	=23	14	05·4 S.

By means of a sketch it may be seen that in case (a)

$$\begin{aligned}\text{Lat.} &= 90^\circ - (\alpha + \delta)\\ &= 90^\circ - 63^\circ\text{-}44'\text{-}32''{\cdot}8\\ &= 26^\circ\text{-}15'\text{-}27''{\cdot}2\ \text{N.}\end{aligned}$$

In case (b)

$$\begin{aligned}\text{Lat.} &= 90^\circ - (\alpha - \delta)\\ &= 90^\circ - 17^\circ\text{-}16'\text{-}22''{\cdot}0\\ &= 72^\circ\text{-}44'\text{-}38''{\cdot}0\ \text{S.}\end{aligned}$$

(3) **Meridian Altitudes of Pairs of Stars.**—A greater degree of accuracy may be obtained by observing the meridian altitudes of pairs of stars, one star of each pair being north and one south of the zenith.

The stars selected are, of course, not on the meridian at exactly the same instant, as the two meridian altitudes could not be observed simultaneously with the one instrument. The exact time of culmination of each of two stars, which have only a slight difference in R.A., is therefore deduced from the *Nautical Almanac*, and at the ascertained times a F.R. and a F.L. observation is taken to each.

The result is that the computed latitude is found from the *difference* in the altitudes, and from the sum or difference of the declinations. Hence any index error in the vertical circle, and some of the effects of refraction are eliminated, especially if the altitudes, one to the north and one to the south, are nearly equal and greater than about 45°.

Thus if in Fig. 17.7 P is considered the North Pole, the star, when in positions S_1 and S_2 would be north of the zenith, and when in positions S_3 and S_4 south of the zenith.

There are consequently four suitable cases of pairing :

S_1 and S_3 $\qquad$ S_2 and S_3

S_1 and S_4 $\qquad$ S_2 and S_4

Thus if α_1, α_2, α_3, and α_4 are the altitudes, and δ_1, δ_2, δ_3, and δ_4 the declinations in the four positions, then combining S_1 and S_3 from equations (10) and (12), p. 486,

$$\begin{aligned}l_1 &= \alpha_1 + (90 - \delta_1),\\ l_3 &= (90 - \alpha_3) + \delta_3,\end{aligned}$$

therefore the mean value of the latitude, *i.e.*

$$\frac{l_1 + l_3}{2} = \frac{\alpha_1 - \alpha_3}{2} - \frac{\delta_1 - \delta_3}{2} + 90.$$

Similarly, combining S_1 and S_4,

$$\text{lat.} = \frac{\alpha_1 - \alpha_4}{2} - \frac{\delta_1 + \delta_4}{2} + 90 :$$

[1] *Nautical Almanac.*

combining S_2 and S_3,

$$\text{lat.} = \frac{a_2 - a_3}{2} + \frac{\delta_2 + \delta_3}{2};$$

combining S_2 and S_4,

$$\text{lat.} = \frac{a_2 - a_4}{2} + \frac{\delta_2 - \delta_4}{2}.$$

The above method is due to Talcott, and for very precise work is generally carried out by means of a zenith telescope.

In this case, the stars chosen have very nearly the same altitude, so that on rotating the instrument from the star on one side of the zenith to view that upon the opposite side, the latter appears in the field of view, and the difference between the zenith distances (or altitudes) is then measured directly by the movement of a micrometer wire in the eye-piece of the instrument.

On the Transvaal and Orange River Survey, a 10-in. Repsold was used as a zenith instrument, and the following results were obtained :

Group (i), the probable error of a single observation $= \pm 0''\cdot 40$.
Group (ii), the probable error of a single observation $= \pm 0''\cdot 62$.

Or out of the thirty-five points in Groups (i) and (ii), the mean of the probable error of a single pair was $0''\cdot 33$,

Group (i) the probable error of the final latitude $= \pm 0''\cdot 13$.
Group (ii), the probable error of the final latitude $= \pm 0''\cdot 16$.

The number of pairs observed at a station varied from seven on two nights at Salt Lake, where the final result was computed to have a p.e. of $\pm 0''\cdot 30$, to sixteen at Blaauwberg, where the result was computed to have a p.e. of $\pm 0''\cdot 13$.

On the Geodetic Survey of Egypt,[1] the practice was to observe four pairs of stars on at least three nights, with a 10-in. Repsold theodolite.

From special observations carried out at Helwan Observatory, where the latitude was determined by the same method, at monthly intervals, it was found that the p.e. of a single night's observations, considered independently of other results, averaged about $\pm 0''\cdot 1$, but that the monthly means varied considerably, *i.e.* from $-0''\cdot 9$ to $+0''\cdot 9$ from the mean.

The p.e. of a monthly set of observations, as computed from the total series of eleven, was $\pm 0''\cdot 5$, *i.e.* this was apparently the p.e. of the latitude determined at any particular station in the field.

(4) **Circum-Meridian Altitudes of the Sun or a Star.**—The altitude in this case is observed when the sun or star is approximately upon the meridian, *i.e.* within 10 to 15 minutes before and after culmination.

This allows several F.R. and F.L. observations to be made.

The mean of several sets of observations made upon various stars should be adopted for accurate work.

The exact L.M.T. of culmination must be deduced from the *N.A.* and that of each observation must be booked.

[1] *B.A. Report*, Sect. E, 1908.

The meridian altitude, or the zenith distance z, may then be deduced by means of a formula [1] originally due to Delambre, *i.e.*

$$z = z_1 \mp (Am - Bn), \quad . \quad . \quad . \quad (14)$$

where z_1 is the observed zenith distance. $A = \frac{\cos l \cos \delta}{\sin z}$, $B = A^2 \cot z$,

$m = \frac{2 \sin^2 \frac{t}{2}}{\sin 1''}$, $n = \frac{2 \sin^4 \frac{t}{2}}{\sin 1''}$, l = latitude, δ = declination, t = hour angle.

Approximate values of l and z must be assumed in calculating A and B. Values of m and n for each value of t may be abstracted from tables.

Mean values of m and n (say m_0 and n_0) may then be deduced and substituted in the above formula, *i.e.*

$$z = z_0 \mp (Am_0 - Bn_0),$$

where z_0 is the arithmetical mean of all the observed zenith distances z_1.

The $-^{ve}$ sign is taken for an upper culmination, and the $+^{ve}$ sign for a lower.

The value of z, and hence that of the meridian altitude (*i.e.* $a = 90 - z$) being known, the equations given for Method 2 may be employed to determine the correct latitude.

(5) **Extra-Meridian Altitudes of the Sun or a Star.**—The altitude of the body and the L.M.T. of observation are taken at any convenient time, when the sun or the star is not on the meridian.

In the spherical triangle PZS the following data therefore are known, or can be deduced :

SZ = co-altitude or zenith distance,
SP = co-declination or polar distance,
SPZ = hour angle.

By the application of the sine formula (1), p. 544, the azimuth may be calculated as in Method 9 or 10, pp. 475-478, *i.e.*

$$\sin SZP = \frac{\sin SPZ \sin PS}{\sin ZS}.$$

Conversely, if the direction of the meridian is accurately known, the azimuth may be measured directly, and the hour angle t computed by means of the sine formula above, instead of noting the L.M.T. of the observation. In the case of an observation upon the sun the L.M.T. is necessary in order to find the declination.

The co-latitude may then be found by the application of the formula (6), p. 544, *i.e.*

$$\tan \frac{PZ}{2} = \frac{\sin \frac{1}{2}(Z+P)}{\sin \frac{1}{2}(Z-P)} \cdot \tan \tfrac{1}{2}(SP - SZ).$$

As an alternative, the value of PZ may be calculated directly from

[1] Chauvenet, *Spherical and Practical Astronomy*, vol. i.

the data by means of equation (2), p. 544, though this is not so convenient, *i.e.*

$$\cos ZS = \cos PZ \cos PS + \sin PZ \sin PS \cos SPZ,$$

i.e. $$\sin a = \sin l \,.\, \sin \delta + \cos l \,.\, \cos \delta \,.\, \cos P.$$

Example.—In longitude 120°-30′ W., on December 16, 19—, an observation for latitude was made on the sun's lower limb at 3·10 P.M. L.M.T., the altitude being 40°-28′-17″. What was the latitude of the place ? The sun was south of the observer.

	°	′	″
Apparent altitude . . . . . =	40	28	17
Correction for refraction [1] . . . . =		1	10·5
	40	27	06·5
Correction for semi-diameter [2] . . =		16	16·7
Correction for parallax [2] = 8·94 × cos 40°·27′ =			06·8
True altitude . . =	40	43	30
Co-altitude (ZS) . . =	49	16	30

	H.	M.	S.
L.M.T. of observation . . =	15	10	0
120°-30′ W. longitude . . =	8	2	0
G.M.T. of observation Dec. 16 . =	23	12	0
Declination [2] S. at 0 h. Dec. 17 . =	23	19′	36·6
Increasing to S. at 7″·26 per hour [2]			
Therefore correction for 0 h. 48 m. =		0	05·8
Declination at time of observation =	23	19	30·8
Co-declination . . . =	113	19	30·8

	H.	M.	S.
E. at 0 h. Dec. 17 . . =		4	19·9
Increase at 1·217 s. per hour =			01·0
E. at time of observation . =		4	20·9
Interval since L.M.N. . =	3	10	00
Interval since L.A.N. . =	3	14	20·9
Therefore ∠ P in arc . =	48°	35′	13″·5

In the spherical triangle PZS

$$\left.\begin{array}{rrrr} ZS = & 49^\circ & 16' & 30'' \\ PS = & 113 & 19 & 30{\cdot}8 \\ \angle P = & 48 & 35 & 13{\cdot}5 \end{array}\right\}$$

∠ Z may therefore be calculated by the sine formula

$$\begin{aligned} \log \sin P &= \bar{1}{\cdot}8750393 \\ \log \operatorname{cosec} ZS &= {\cdot}1204169 \\ \log \sin PS &= \bar{1}{\cdot}9629714 \\ \log \sin Z &= \bar{1}{\cdot}9584276 \\ \text{Therefore } Z &= 114^\circ\text{-}40'\text{-}18'' \end{aligned}$$

To solve the spherical triangle again for ZP,

$$\begin{aligned} &\tfrac{1}{2}(P+Z) = 81^\circ\text{-}37'\text{-}45''{\cdot}75 \;\therefore\; \log \sin \tfrac{1}{2}(P+Z) = 1{\cdot}9953487 \\ &\tfrac{1}{2}(Z-P) = 33^\circ\text{-}02'\text{-}32''{\cdot}25 \;\therefore\; \log \operatorname{cosec} \tfrac{1}{2}(Z-P) = {\cdot}2633980 \\ &\tfrac{1}{2}(PS-ZS) = 32^\circ\text{-}01'\text{-}30''{\cdot}4 \;\therefore\; \log \tan \tfrac{1}{2}(PS-ZS) = \bar{1}{\cdot}7962126 \\ &\qquad \log \tan \tfrac{1}{2} PZ = 0{\cdot}0549593 \\ &\qquad \tfrac{1}{2} PZ = 48^\circ\text{-}36'\text{-}56''{\cdot}55 \\ &\qquad PZ = 97^\circ\text{-}13'\text{-}53''{\cdot}1 \end{aligned}$$

The latitude of the place is therefore . . = 7°-13′-53″·1 S.

[1] Chambers' *Mathematical Tables.* [2] *Nautical Almanac.*

(6) **Extra - Meridian Observation of Polaris.**—This method is a special case of Method 5, but, owing to the proximity of Polaris to the Pole, a special formula is generally employed.

In the spherical triangle PZS the known data are

PS = co-declination (*i.e.* $90-\delta$) or polar distance $=p$ say, obtained from the *N.A.*

ZS = co-altitude or zenith distance $=90-a$, determined from the measured altitude.

ZPS = P = the hour angle of the star at the time of observation.

From equation (2), p. 544,

$$\cos ZS = \cos PZ \cos PS + \sin PZ \sin PS \cos P,$$

or

$$\sin a = \sin l \cos p + \cos l \sin p \cos P. \quad . \quad (15)$$

Writing $l=(a-x)$ say, and expanding by Taylor's Theorem,

$$\sin l = \sin (a-x) = \sin a - x \cos a - \frac{x^2}{2!} \sin a + \frac{x^2}{3!} \cos a + \ldots$$

$$\cos l = \cos (a-x) = \cos a + x \sin a - \frac{x^2}{2!} \cos a - \frac{x^2}{3!} \sin a + \ldots$$

$$\sin p = p - \frac{p^2}{3!} + \ldots$$

$$\cos p = 1 - \frac{p^2}{2!} + \ldots$$

Substituting in formula (15)

$$\sin a = \left[1-\frac{p^2}{2}\right]\left[\sin a - x \cos a - \frac{x^2}{2} \sin a\right] + \left[p - \frac{p^2}{6}\right][\cos P]\left[\cos a + x \sin a - \frac{x^2}{2} \cos a\right]$$

$$\sin a = \sin a - x \cos a + p \cos a \cos P - \tfrac{1}{2} \sin a\,(x^2 - 2xp \cos P + p^2) + \ldots$$

or

$$x = p \cos P - \tfrac{1}{2} \tan a\,(x^2 - 2xp \cos P + p^2) + \ldots$$

Writing $x=p \cos P$ as a first approximation, and substituting in the second term, and neglecting powers of p and x above the second,

$$x = p \cos P - \tfrac{1}{2}p^2 \tan a\,(1-\cos^2 P)$$
$$= p \cos P - \tfrac{1}{2}p^2 \tan a \sin^2 P.$$

But in the expansions p and x are in radian measure, and as p radians $=p'' \times \sin 1''$, we can write the equation as

$$x \sin 1'' = p \sin 1'' \cos P - \tfrac{1}{2}\, p^2 \sin^2 1'' \tan a \sin^2 P, \quad (16),$$

or

$$x = p \cos P - \tfrac{1}{2} \sin 1''\,(p \sin P)^2 \tan a,$$

where x and p are in seconds.

Thus the form of equation generally adopted is

$$l = a - p \cos P + \tfrac{1}{2} \sin 1'' \, (p \sin P)^2 \tan a. \quad . \quad (17)$$

For very accurate work still further terms in the expansions may be employed.

The next term in the expansion [1] is

$$\tfrac{1}{3} p^3 \sin^2 1'' \cos P \sin^2 P. \quad . \quad . \quad (18)$$

The maximum value of this may be found by differentiating with respect to P, *i.e.*

$$\frac{\delta e}{\delta P} = \cos P \,.\, 2 \sin P \cos P - \sin^3 P = 0,$$

i.e. $$\sin P \, (2 \cos^2 P - \sin^2 P) = 0.$$

For a maximum, $$2 \cos^2 P = \sin^2 P,$$

$$\tan^2 P = 2,$$

i.e. $$P = 54°\text{-}44' \text{ roughly}.$$

Substituting in (18) $\sin P = \frac{\sqrt{2}}{\sqrt{3}}$ and $\cos P = \frac{1}{\sqrt{3}}$, $\sin 1'' = 0{\cdot}000004848$, and $p = 59'\text{-}28''$ (in 1946) $= 3568''$.

$$e_3 \text{ (maximum)} = 0''{\cdot}137.$$

The following corrections vary in sign, and their values may be found in a similar manner.

Equation (17) may therefore be taken to give the value of the latitude to the nearest second at least.

The values of the 1st and 2nd corrections, together with a third correction, depending upon the true and assumed values of p and a, are given in a modified form in the *N.A.* and in the *Star Almanac for Land Surveyors*. See Appendix II, page 546.

Example.—On July 2, at 10 h. 15 m. P.M. in longitude 36°-14′ E., the observed altitude of Polaris was 48°-15′-10″, the barometer being 30″·1 and the temperature 58° F. What is the approximate latitude ?

		°	′	″
Apparent altitude . . .		48	15	10
Mean refraction for this alt.[2]	= −51″			
Correction for temperature	= + 1″			51
Correction for barometer	= − 1″			
True altitude . . .		=48	14	19

To find L.S.T. of observation,

	H.	M.	S.
L.M.T. of observation	=22	15	0
Correction for longitude	= 2	24	56
G.M.T. of observation	=19	50	4

[1] *Vide* Chauvenet, *Spherical and Practical Astronomy.*

[2] Chambers' *Mathematical Tables.*

		H.	M.	S.
S.T. of G.M.M.[1] (*i.e.* 0 h. July 2)	=	6	36	44
M.T. interval after 0 h.	=	19	50	04
S.T. interval (add 19 h. 50 m. × 9·86 s.)	=		03	16
G.S.T. of observation, July 3 (ddt. 24 h.)	=	2	30	04
Correction for longitude	=	2	24	56
L.S.T. of observation	=	4	55	00
The R.A. of Polaris[2]	=	1	28	26
Therefore the hour angle P	=	3	26	34
Therefore the angle P in arc	=	51°-	38′-	30″

The declination[3] $= 88°\text{-}50'\text{-}43''$ $\therefore$ $p = 90 - \delta = 1°\text{-}9'\text{-}17'' = 4157''$

The first correction $p \cos P = 4157'' \times \cos 51°\text{-}38'\text{-}30''$,

$$\begin{aligned} \log p \quad . \quad . &= 3{\cdot}61878 \\ \log \cos 51°\text{-}38'\text{-}30'' &= \bar{1}{\cdot}79280 \\ & 3{\cdot}41158 \\ \text{and } e_1 \quad . \quad . &= 2579{\cdot}8 \text{ secs. or } 43'\text{-}00'' \end{aligned}$$

The second correction $= \frac{1}{2}p^2 \tan a \sin^2 P \sin 1''$,

$$\begin{aligned} \log \tfrac{1}{2} \quad . \quad . &= \bar{1}{\cdot}69897 \\ \log p \quad . \quad . &= 3{\cdot}61878 \\ \log p \quad . \quad . &= 3{\cdot}61878 \\ \log \tan 48°\text{-}14'\text{-}19'' &= {\cdot}04920 \\ \log \sin 51°\text{-}38'\text{-}30'' &= \bar{1}{\cdot}89440 \\ \log \sin 51°\text{-}38'\text{-}30'' &= \bar{1}{\cdot}89440 \\ \log \sin 1'' \quad . \quad . &= \bar{6}{\cdot}68557 \\ \log e_2 &= 1{\cdot}46010 \text{ and } e_2 = 28{\cdot}8 \text{ secs.} \end{aligned}$$

The approximate latitude is therefore $a - e_1 + e_2$, *i.e.*

	°	′	″
$a =$	48	14	19
$e_1 =$		43	00
	47	31	19
$e_2 =$			29
$l =$	47	31	48

Special Instruments.—Two special instruments occasionally used for astronomical observations will now be described.

The **Solar Attachment** is fitted to an ordinary transit theodolite, and enables the Surveyor to determine the direction of the meridian, the latitude of the instrument station, and local time.

As shown in Fig. 17.8, which depicts the instrument as made by Stanley, the " polar axis " is fixed to the centre of the telescope tube and at right angles to the line of collimation. The " declination arc " revolves about the " polar axis ", and carries the " hour circle ", which is read by means of a fixed index mark.

To read the declination arc, a rotating arm which carries a vernier at its extremity is provided. In addition, this arm carries a lens at each end, and at the opposite ends two small silver plates, upon each of which two horizontal and two vertical lines are engraved. The former are known as " equatorial lines ", and the latter as " hour lines ".

To determine the direction of the meridian, and local time, the

[1] *Nautical Almanac* (or use S.T. of 0 h. July 3, and deduct 9·86 secs./hour × 4 h. 10 m.). [2] *Nautical Almanac.* [3] *Ibid.*

instrument is accurately levelled, the co-latitude of the place set off on the ordinary vertical arc of the theodolite, and the declination of the sun at the time of the observation—corrected for refraction—set off on the declination arc.

The horizontal scale plate of the theodolite having been clamped

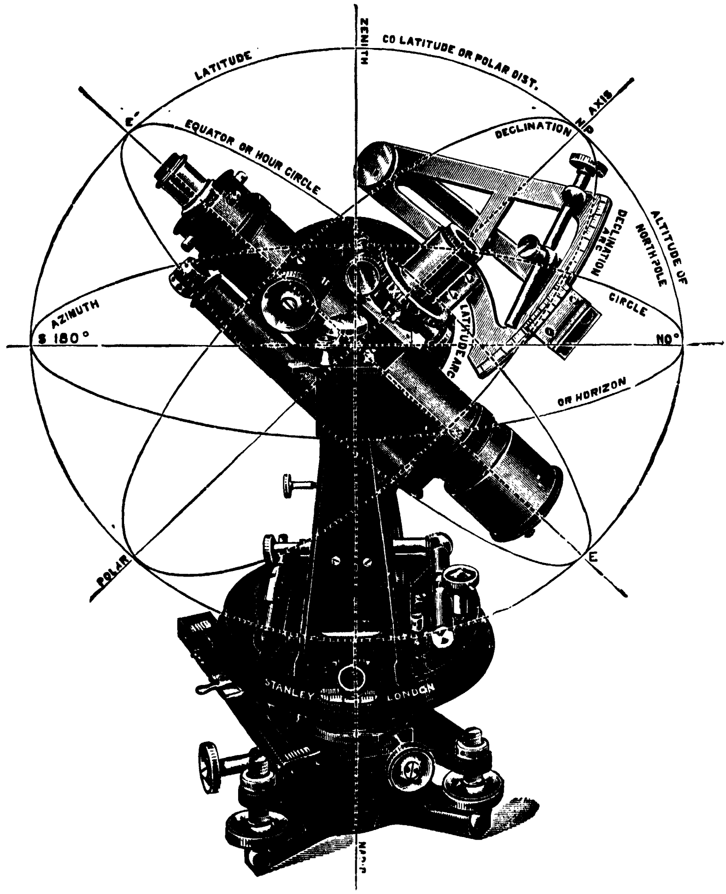


FIG. 17.8

at zero, the instrument is rotated about the ordinary outer vertical axis, until one of the lenses is directed towards the sun and throws the image of this upon the opposite plate.

The declination arc is then rotated slightly backwards and forwards about the polar axis, and the path of the image of the sun upon the silver plate noted. If the image travels horizontally in such a way that it keeps exactly between the equatorial lines, the lower clamp of the theodolite axis is tightened, and the telescope will be found to lie in the meridian.

If not, the body of the instrument is turned slightly about the outer vertical axis, until the path of the image does fall within the equatorial lines, when the declination arc is slightly rotated about the polar axis. The " polar axis " is then parallel to the polar axis of the earth.

The operation is completed more quickly if the approximate direction of the north is known by means of a magnetic compass.

It is claimed that if a magnifier is used to observe the sun's image upon the solar screen, an error of 20″ in the azimuth may be detected easily.

In addition to the direction of the meridian, the above procedure also furnishes the local apparent time at the instant that the image lies in the small square enclosed by the hour and equatorial lines. This is recorded upon the hour circle, which is attached to the declination arc.

To find the latitude of the observer, the theodolite is accurately levelled (see Fig. 17.8), and the value of the sun's declination at apparent noon corrected for refraction, set on the declination arc. Shortly before noon the telescope is directed due north, so that the image travels between the equatorial lines when the declination arc is slightly rotated. The instrument is then clamped. The declination arc is turned until it is parallel to the telescope, after which the image of the sun is kept within the small square of the solar screen, adjusting vertically by means of the ordinary vertical circle tangent screw, and horizontally by means of the tangent screw of the hour circle. When the image ceases to fall below the lower equatorial line, *i.e.* when the sun ceases to rise in the heavens, the index of the hour circle should indicate XII, and the latitude may be deduced from the reading of the vertical arc upon which the co-latitude is recorded.

The **Prismatic Astrolabe** is a French instrument, which has proved very successful for the determination of latitude and time by the observation of stars at a constant altitude of 60°. It has been used upon boundary surveys in the French colonies, upon the measurement of a meridian arc in Ecuador, and in other districts.

In front of the object-glass of the telescope is an equilateral prism (Fig. 17.9) having its edges horizontal and that face vertical which is nearest the telescope ; below this is an artificial horizon.

The direct rays from a star which has an altitude of 60°, entering the prism in a direction normal to the upper face, are refracted from the lower face and proceed horizontally and normal to the back vertical face of the prism into the telescope.

Similarly, the rays which meet the mercury surface of the artificial horizon and have an angle of incidence equal to 60° are reflected upwards at an angle of 60°, enter the lower face of the prism normally, and are reflected from the upper face horizontally into the telescope.

If the observed star is setting towards the right of the observer, the two images formed—one by the rays through the upper surface, and one by the reflected rays through the lower surface of the prism—will appear to be moving in the field of the telescope towards the left ; the upper

image will be that reflected from the mercury. At any given instant the two images will lie upon the same vertical line, and will gradually approach each other along paths inclined to the horizontal. When the altitude becomes exactly 60° the two images will coincide, and then, as the altitude decreases, the images will again separate, that from the mercury now being the lower in the field of view.

The rate at which the images appear to move relatively to each other is obviously twice that at which a star would appear to move through the field of an ordinary telescope of the same magnification. Hence the exact instant at which the altitude reaches 60° can be much more accurately determined than can the time of intersection of the star with the webs of an ordinary diaphragm.

For latitude observations stars on or near the meridian are the

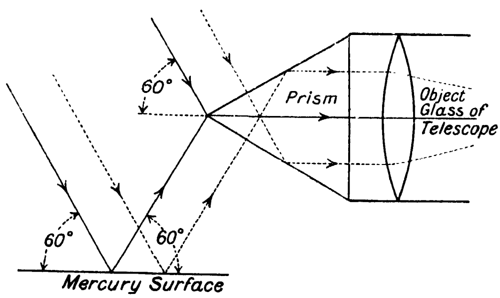


FIG. 17.9.—Principle of the Prismatic Astrolabe

most suitable, while for time observations those in the neighbourhood of the prime vertical are preferable.

In *A Handbook of the Prismatic Astrolabe*, by Ball and Knox-Shaw, it is recommended that at least four stars should be observed to form one set of observations, the chosen stars being in the four different quadrants N.W., N.E., S.W., and S.E. In this way, errors due to uncertain refraction and instrumental errors are largely eliminated. It is often possible, within an interval of about two hours, to observe eight or more stars forming complete sets.

Generally it may be noted that co-latitude and azimuth may both be determined from the spherical triangle PZS when the declination of the observed star, the hour angle at the time of observation, and the altitude (*i.e.* 60° corrected for refraction) are known. The hour angle, and hence local time, may be found when the declination, altitude, and either the latitude or the azimuth are known. Or the hour angle and the latitude may both be found if the declination, azimuth, and altitude are known.

Absolute determinations of longitude may be made by observing equal altitudes of stars and the moon's bright limb.

The methods of calculation are similar to those previously described in the present chapter, but additional computations are necessary to predict suitable stars.

To facilitate the choice of suitable stars and to enable the Surveyor

to estimate the approximate times at which they will reach an altitude of 60° in various latitudes, very convenient tables have been computed. The necessary calculations are explained in the *Handbook of the Prismatic Astrolabe* already referred to, and a graphical method is described of deducing the most probable values of the latitude and time, by means of position lines—following the method of " False Positions " due to Marcq St. Hilaire.

The treatise *Description et usage de l'astrolabe à prisme*, by A. Claude et L. Driencourt, can also be referred to for a full description of the instrument and its adjustments, for the calculations and graphical constructions involved in its use, and for a discussion of the probable errors of the observations and results.[1]

The 45° Prismatic Astrolabe, which is a later development, possesses certain advantages in that a greater choice of stars is available, and the calculations are simplified to some extent. It is described and illustrated in *Theodolite Design and Construction*, by Cooke, Troughton & Simms.

Accuracy.—A few notes upon the accuracy of the various methods of determining the direction of the meridian, latitude, and longitude have already been given, with the descriptions of the methods in the previous pages.

An analysis of the effect upon the result of an error in azimuth, altitude, latitude, declination, or hour angle may, however, be studied with advantage, as it is very difficult to say that such and such a method will give an accuracy of such and such an amount without knowing all the particular circumstances.

If, however, we know that using a 6-in. transit graduated to 20″ the probable error in elevation[2] with a single F.R. and F.L. observation is about ±5″, we can determine what effect this will produce in the result under the given circumstances, *i.e.* considering the magnitude of the other factors such as the hour angle, declination, etc. Similarly, with measurements in azimuth, time, and latitude the effects of probable errors may be studied.

Considering the fundamental equation (2), p. 544,

$$\cos PS = \cos PZ \,.\, \cos ZS + \sin PZ \sin ZS \,.\, \cos Z,$$

or

$$\sin \delta = \sin l \,.\, \sin \alpha + \cos l \,.\, \cos \alpha \,.\, \cos Z. \qquad . \quad (19)$$

Differentiating with respect to δ and Z,

$$\cos \delta \,.\, \delta(\delta) = -\cos l \,.\, \cos \alpha \,.\, \sin Z \,.\, \delta Z,$$

i.e.

$$\frac{\delta Z}{\delta(\delta)} = -\frac{\cos \delta}{\cos l \,.\, \cos \alpha \,.\, \sin Z} = -\frac{1}{\cos l \,.\, \sin P},$$

or

$$\frac{\delta Z}{\delta(\delta)} = -\sec l \,.\, \operatorname{cosec} P. \qquad . \quad . \quad (20)$$

[1] See also " The Design of a Prismatic Astrolabe ", by T. Y. Baker, *Journal of Scientific Instruments*, Dec. 1923; " The 45° Prismatic Astrolabe ", by T. Y. Baker, *The Geographical Journal*, May 1931; and " A Combined Theodolite and Prismatic Astrolabe ", by E. A. Reeve, *The Geographical Journal*, vol. lxi. p. 41.

[2] See p. 120.

By differentiating equation (19) with respect to l and Z,

$$\sin \delta = \sin l \sin \alpha + \cos l \cos \alpha \cos Z,$$

$$0 = \cos l \,.\, \sin \alpha \,.\, \delta l - \sin l \,.\, \cos \alpha \,.\, \cos Z \,.\, \delta l - \cos l \,.\, \cos \alpha \,.\, \sin Z \,.\, \delta Z,$$

$$\therefore \frac{\delta Z}{\delta l} = \frac{\cos l \,.\, \sin \alpha - \sin l \,.\, \cos \alpha \,.\, \cos Z}{\cos l \,.\, \cos \alpha \,.\, \sin Z}$$

$$= \frac{1}{\cos l}\left\{\frac{\cos l \sin \alpha}{\cos \alpha \sin Z} - \frac{\sin l}{\sin Z}\left(\frac{\sin \delta - \sin l \sin \alpha}{\cos l \cos \alpha}\right)\right\} \text{ from (19)}$$

$$= \frac{1}{\cos l}\left\{\frac{\sin \alpha \cos^2 l + \sin \alpha \sin^2 l - \sin l \sin \delta}{\cos l \,.\, \cos \alpha \,.\, \sin Z}\right\}$$

$$= \frac{1}{\cos l}\left\{\frac{\sin \alpha - \sin l \sin \delta}{\cos l \,.\, \cos \alpha \,.\, \sin Z}\right\}.$$

Substituting cos P for the similar expression to (19), *i.e.* $\dfrac{\sin \alpha - \sin l \sin \delta}{\cos l \cos \delta}$,

$$\frac{\delta Z}{\delta l} = \frac{\cos \delta \cos P}{\cos l \cos \alpha \sin Z} = \frac{\cos P}{\cos l \sin P},$$

i.e.

$$\frac{\delta Z}{\delta l} = \sec l \,.\, \cot P, \qquad (21)$$

or

$$\frac{\delta l}{\delta Z} = \cos l \,.\, \tan P. \qquad (22)$$

Similarly, as equation (19) is symmetrical in l and α,

$$\frac{\delta Z}{\delta \alpha} = \sec \alpha \cot S. \qquad (23)$$

To find the effect of an error in the hour angle (equation (8), p. 476)

$$\frac{\sin P}{\cos \alpha} = \frac{\sin Z}{\cos \delta},$$

$$\therefore \frac{\cos P \,.\, \delta P}{\cos \alpha} = \frac{\cos Z \,.\, \delta Z}{\cos \delta},$$

$$\frac{\delta Z}{\delta P} = \frac{\cos P \,.\, \cos \delta}{\cos Z \,.\, \cos \alpha} = \cot P \,.\, \tan Z. \qquad (24)$$

By differentiating the equation similar to (19), viz.

$$\sin \alpha = \sin l \sin \delta + \cos l \cos \delta \cos P, \qquad (25)$$

$$\cos \alpha \; \delta \alpha = -\cos l \cos \delta \sin P \,.\, \delta P,$$

$$\frac{\delta P}{\delta \alpha} = -\frac{\cos \alpha}{\cos l \cos \delta \sin P}$$

$$= \frac{1}{\cos l \,.\, \sin Z}. \qquad (26)$$

By differentiating and solving as for (21) and (23), or writing down the results by comparison, observing the cyclic order of the notation,

$$\frac{\delta P}{\delta l} = \sec l \cot Z, \qquad (27)$$

and

$$\frac{\delta P}{\delta(\delta)} = \sec \delta \cot S \qquad (28)$$

$$= \frac{\sec \delta \,.\, \sqrt{\dfrac{\cos^2 \delta}{\cos^2 l} - \sin^2 Z}}{\sin Z}. \qquad (29)$$

By the differentiation of (25),

$$\cos l \,.\, \delta l \,.\, \sin \delta + \cos \delta \,.\, \delta(\delta) \,.\, \sin l - \sin l \,.\, \delta l \,.\, \cos \delta \cos P - \sin \delta \,.\, \delta(\delta) \,.\, \cos l \cos P = 0.$$

Substituting for cos P and simplifying,

$$\delta l \left(\frac{\sin \delta - \sin \alpha \sin l}{\cos l}\right) = -\delta(\delta) \,.\, \left\{\frac{\sin l - \sin \alpha \sin \delta}{\cos \delta}\right\},$$

$$\delta l \cos Z = -\delta(\delta) \,.\, \cos S,$$

$$\frac{\delta l}{\delta(\delta)} = -\frac{\cos S}{\cos Z} = -\frac{\sqrt{1 - \dfrac{\cos^2 l}{\cos^2 \delta} \,.\, \sin^2 Z}}{\cos Z}. \qquad (30)$$

Differentiating the same equation with respect to α,

$$\cos \alpha \,.\, \delta\alpha = \cos l \,.\, \delta l \,.\, \sin \delta - \sin l \,.\, \delta l \,.\, \cos \delta \,.\, \cos P$$

$$= \delta l \,.\, \left\{\cos l \,.\, \sin \delta - \sin l \,.\, \frac{(\sin \alpha - \sin l \sin \delta)}{\cos l}\right\}$$

$$= \delta l \left\{\frac{\sin \delta - \sin l \sin \alpha}{\cos l}\right\},$$

$$\therefore \frac{\delta l}{\delta \alpha} = \frac{1}{\cos Z} = \sec Z. \qquad (31)$$

Useful deductions as to the most and least favourable conditions for the observations relating to any particular method may be drawn from these and similar results by considering the effects of probable errors in the ascertained data. Care must be taken not to generalize too widely—the conclusions are not necessarily valid when the data used in the computations are other than those used in the particular formula under consideration.

Azimuth.—Equation (20) indicates that in the determination of azimuth from the values of α, l, and δ substituted in equation (19), the error in the result due to a small error in the assumed value of the declination decreases as the product $\sec l \operatorname{cosec} P$ decreases.

Thus, in so far as this particular error is concerned, the most favour-

able position of the observed star (or sun) is when cosec P is least, *i.e.* when the hour angle is 90° or 270°, and the error is of less importance for observations made near the equator than for those made in higher latitudes.

It will be noted that when P approaches 0° or 180°, *i.e.* when S approaches the meridian ZP, the resulting error in azimuth may become relatively large. It is not justifiable to apply the formula quantitatively under such conditions (or at the limit when S lies on the meridian PZ) by substituting such values of $\delta(\delta)$ and cosec P as would made δZ unreasonably large. The assumption made in deriving equation (20) is that ∂Z and $\partial(\delta)$ are infinitesimals, and it is only a reasonable approximation to substitute finite values of δZ and $\delta(\delta)$ provided they are both very small.

Equation (21) similarly indicates that when using equation (19) the error in azimuth resulting from a small error in the assumed latitude decreases as the product $\sec l \cot P$ decreases, and the best position again is when the hour angle is near 90° or 270°.

Equation (23) refers to the same given data substituted in equation (19) and indicates that the error in the calculated azimuth resulting from a small error in the assumed altitude decreases as the product $\sec a \cot S$ decreases, where S is the parallactic angle PSZ.

The least numerical value is when $\cot S = 0$, *i.e.* when $S = 90°$, and this condition is satisfied when a circumpolar star is being observed at elongation.

It can be shown that cot S can equal zero, only when the star is such that it transits between the zenith and the elevated pole. Thus, from the sine formula, $\sin PZ / \sin S = \sin PS / \sin Z$, and if $S = 90°$, $\sin S = 1$ and $\sin PZ = \sin PS / \sin Z$. Hence, as sin Z cannot exceed 1, sin PZ cannot be less than sin PS, *i.e.* $PZ \nless PS$, which is the limitation stated.

The condition for $\delta Z/\delta a$ to be a minimum under other circumstances is not obvious by inspection of the formulae, but from

$$\delta Z/\delta a = (\sin l - \sin a \sin \delta)/\cos l \cos^2 a \sin Z$$

it is seen that $\delta Z/\delta a$ is a maximum when $\sin Z = 0$, *i.e.* when the observed body is on the meridian, and this is therefore the worst position.

Equation (24) is deduced from equation (8), p. 476, and the data are thus different from those in equation (19). Here they are the hour angle P, the altitude a, and the declination δ.

The error in the azimuth determination due to a small error in the assumed time, *i.e.* in the hour angle P, decreases as the product $\cot P . \tan Z$ decreases. The minimum value of $\delta Z/\delta P$ occurs when $\cot P = 0$, *i.e.* when the hour angle is 90° or 270°.

The solution that $\tan Z = 0$ indicates that the star (or sun) is on the meridian, so that both cos P and $\cos Z = \pm 1$, and $\delta Z/\delta P = \pm \cos \delta/\cos a$

which can readily be evaluated if the declination and approximate latitude are known : cot P, it may be noted, becomes infinite when tan Z becomes zero.

The condition for $\delta Z/\delta P$ to be a minimum in a general case is complicated and not obvious by inspection. It is seen, however, that an observation on a star near the pole, and at or near elongation, should give low values for both cot P and tan Z and thus be favourable. For an observation on a body near the prime vertical, tan Z would be large, and cot P generally not small, so that the conditions would be unfavourable.

Latitude.—If latitude is to be determined from values of altitude, declination, and hour angle, equations (31), (30), and (27) apply, and it is readily seen that the error in latitude resulting from an error in α, δ, or P decreases as sec Z or tan Z decreases, *i.e.* the most favourable position of the observed star (or sun) is on or near the meridian.

Time.—In the determination of time, *i.e.* of the hour angle, by method 4, p. 479, the three quantities, latitude, declination, and altitude, are used, and equations (27), (28), (29), and (26) apply. The best conditions are when tan Z, sin Z and sin Z, respectively, are greatest, *i.e.* when the observed body is on or near the prime vertical, provided that this does not entail the possibility of a large error in a low-altitude measurement due to the uncertainty of refraction.

The latitude and time equations may be studied in more detail, as was done with the azimuth equations, and the student is recommended to do this, and to solve the problems at the end of the chapter.

Example.—The p.e. in a particular example may be calculated on the following lines, approximate values of the data being used if desired for the calculation of the p.e., though not, of course, for the main determinations.

$\alpha = 52° \; 31' \; 30''$	nearly :	assumed p.e. $= \pm 10''$,	say.
$\delta = 1° \; 12' \; 30''$	,,	,, ,, $= \pm 1''$	,,
$l = 22° \; 35' \; 00''$	,,	,, ,, $= \pm 3''$	
$P = 31° \; 41' \; 08''$	,,	calculated	
$Z = 120° \; 19' \; 50''$	,,	,,	

By the application of formulae (26), (29) and (27), respectively,

the p.e. due to the p.e. in $\alpha = \pm \delta P_1 = \pm 12{\cdot}55''$

,, ,, $\delta = \pm \delta P_2 = \pm 0{\cdot}76''$

,, ,, $l = \pm \delta P_3 = \pm 1{\cdot}89''$

Therefore δP, the p.e. in the hour angle,

$$= \pm \sqrt{(\delta P_1)^2 + (\delta P_2)^2 + (\delta P_3)^2}$$
$$= \pm \; 12{\cdot}7'', \text{ say}$$
$$= \pm \; 0{\cdot}85 \text{ second of time.}$$

Similarly, δZ, the p.e. in the azimuth determination, calculated from equations (20), (21) and (23), is $\pm 13{\cdot}7''$.

EXAMPLES[1]

1. (U. of L.) From the following notes find the azimuth of the mark A :

Observations for azimuth on the sun's lower limb, May 9, 19—.

(a) Latitude of station	52°-27′ N.
(b) Mean of G.M.T.'s of two observations .	11 h. 30 m. A.M.
(c) Sun's declination at 0 h. at Greenwich, May 9 (variation for one hour +39″·2) .	17°-03′-48″·4 N.
(d) Mean observed altitude . . .	50°-24′-50″
Correction for parallax . .	0°-0′-05″·6
Correction for semi-diameter .	0°-15′-51″·8
Correction for refraction . .	0°-0′-47″·4
(e) Mean bearing of the sun from Mark A .	32°-25′

2. (U. of L.) At a point in lat. N. 55°-46′-12″ the altitude of the sun's centre was found to be 23°-17′-32″ at 5 h. 17 m. P.M. (Greenwich mean time). The theodolite was first pointed to a reference mark, the vernier reading being 0°-00′ ; the horizontal angle between the sun's centre and the reference mark at the time of observation was found to be 68°-24′-30″. Find the geographical azimuth of the reference mark from the centre of the instrument.

Data :	°	′	″
Sun's declination at Greenwich apparent noon on day of observation . . .	17	46	52 N.
Variation of declination per hour . .			−38
Refraction for altitude of 23°-20′ . .		2	12
Parallax in altitude		0	8
Equation of time (app. − mean) . .		6 m.	6 s.

3. Find the effect upon the azimuth of 1″ error in (i) latitude, (ii) declination at 1 P.M. and at 6 P.M. in latitudes 30° and 50°.

In Question 1 what would be the effect upon the azimuth of an error of (*a*) 1′ in latitude, (*b*) 10 mins. in mean time of observation, (*c*) 10″ in altitude ?

4. (U. of L.) On the afternoon of May 12 (5 P.M. by Greenwich mean time) at a place A, where the latitude was 51°-30′-20″ N., the altitude of the sun's centre was found to be 23°-5′-20″, whilst the horizontal angle between the fixed line AB and the direction to the sun's centre was 18°-20′, the sun having crossed the line over an hour before.

Determine the azimuth of the sun from the south at the time of the observation, and the azimuth of the line AB, having given :

Correction for refraction, 57″ × tangent zenith distance.
Declination of sun at 0 h. May 13, 18°-07′-07″·0 N.
Increase of declination per hour at transit, May 12, 38″·19.
Increase of declination per hour at 0 h. May 13, 38″·01.
Sun's horizontal parallax = 8″·71.

5. (I.C.E.) Find the latitude of the observer from the following data: Meridian altitude of α Pavonis corrected for instrument, 34°-15′-0″. Thermometer, 44° F. Barometer, 28·56 in. Star south of observer. Declination of α Pavonis, 57°-4′-30″ S.

6. (U. of B.) The declination of the sun on June 1 at G.M.M. was N. 21°-58′-47″·2. The observed meridian altitude of the sun's centre at a place longitude 75° E. of Greenwich was 71°-36′-5″. Find the latitude of the place.

Given that the correction for refraction is 19″·30.
Given that the correction for parallax is 2″·90.
Given that the change in declination per hour is 20″·49.

Explain clearly how the time of transit across the meridian can be calculated in the case of (*a*) the sun, (*b*) a star.

[1] Data have been modified in some of the problems, to agree with more recent arrangement of *Nautical Almanac*.

7. (U. of B.) Two observed meridian altitudes of the sun on July 17 were 70°-8′-15″ and 70°-8′-35″ at a place whose longitude is 60° E. of Greenwich. Find the latitude of the place.

The correction for parallax is 3″·0.
The correction for refraction is 21″·1.

Level Correction

E.	O.
20	10
19	11

Value of one division on spirit level = 5″, also the declination of the sun at G.A.N. on July 17 was N. 21°-13′-57″·8, with an hourly variation of 25″·36.

No correction for sun's semi-diameter is to be made.

8. (I.C.E.) At a certain place on July 1, 1913, observations were made before noon on the sun for longitude. From the following figures calculate the longitude:

	°	′	″
Altitude	53	41	24
Latitude	52	44	0 N.
Declination	23	9	13 N.

Equation of time, 3 m. 30 s., to be added to apparent time.
Greenwich mean time, 10 h. 19 m. 45·5 s.

9. If the direction of the meridian is known with a p.e. of ±5″, what would be the p.e. in time (Method 2) assuming that the time of transit could be accurately estimated (*a*) $\delta = 88°\text{-}50'$ N., (*b*) $\delta = 60°$ N., and the latitude 50° N. ?

10. Assuming the data given for four stars on p. 466, calculate, for each of the three pairs mentioned, the azimuth of the two stars when they are on the same vertical, and the sidereal time interval from then to the *lower* transit of Polaris or δ Cassiopeiae (instead of to the upper transit as on p. 467). Assume latitude 52° N.

11. (U. of B.) An observation on the moon was made on the night of May 15-16 to determine G.M.T.

The moon's bright limb was observed to transit the meridian of the place at 12 h. 30 m. 15 s. on the watch, and star ω Ophiuchi crossed 13 m. 33½ s. later. The moon's centre crossed 65·66 sidereal seconds earlier than the bright limb.

From the *Nautical Almanac* it was found that the R.A. of ω Ophiuchi was 16 h. 27 m. 24 s., and that R.A. of the moon at 01 hours on May 16 was 16 h. 12 m. 16·09 s. with an increase of 21·21 s. per 10 m.

How much slow was the watch on G.M.T. ?

12. The reader is recommended to study the *Nautical Almanac* and make up problems such as the following for himself.

The following data are abstracted from the tables for the sun, in the *Nautical Almanac* for July 1, 1932:

	H.	M.	S.
(*a*) Apparent Right Ascension (at 0 h.)	=06	38	51·34
(*b*) Variation per hour	=		10·348
(*c*) Apparent R.A. at transit	=06	40	56·11
(*d*) Equation of time (at 0 h.)	=	−3	31·45
(*e*) Variation per hour	=		−0·492
(*f*) Sidereal Time (at 0 h.)	=18	35	19·89
(*g*) Transit of First Point of Aries	=05	23	46·92

Show by sketches the relative positions of the Mean Sun, Real Sun, and First Point of Aries, and using the tables for the conversion of Mean Solar Time to Sidereal Time intervals and vice versa:

(1) Derive (*f*) from (*g*) and (*g*) from (*f*)
(2) ,, (*a*) ,, (*d*) (*f*)
(3) ,, (*b*) ,, (*e*)
(4) ,, (*c*) ,, (*d*) (*e*) (*f*) and from (*a*) (*b*)
(5) Calculate Sidereal Time at 6 h. and 12 h. G.M.T. and G.A.T. from (*f*) (*d*) (*e*) and (*a*) (*b*) (*d*) and (*c*) (*d*), etc.

CHAPTER XVIII

PHOTOGRAPHIC AND AERIAL SURVEYING

Photographic Surveying or Photogrammetry is the art of producing plans or maps from photographs.

The method was first applied by a French engineer named Laussedat [1] in 1861, and has been largely used since that date by the French, Russian, German, and Austrian engineers. It has also been very successfully employed on the Canadian Survey [2] and in the United States.

As will be seen later, the principle is very similar to that of Plane Table surveying, with the difference that most of the work, which with the latter instrument is executed in the field, is here done in the office.

The accuracy attained is perhaps hardly as great as with a plane table—though opinions differ upon this point—but, under favourable conditions, the cost generally appears to work out at a much lower rate, and the field work is not so dependent upon the weather conditions.

High winds affect both systems, as does mist or fog, but, with the camera, advantage may be taken of brief periods of clear weather—during which only a very few points could be located with a plane table—to obtain sufficient photographs to plot a considerable area.

Photogrammetry is particularly suitable for topographical or preliminary survey work, though it has been largely used for the surveys of buildings, such as, for example, those of ancient Greece. The method is, however, unsuitable for well-wooded country, owing to the difficulty of identifying points upon pairs of photographs taken from different camera stations, and it is practically impossible in very flat districts, unless, as in Fig. 18.6, the plains are commanded by high ground. Mountainous and hilly districts with few trees are usually very satisfactorily surveyed, as good views of one side of a valley are generally obtainable from the high ground opposite.

The office work is long and tedious, especially if many contours are required to be shown upon the finished map.

The Photo-Theodolite.—Some form of photo-theodolite is generally employed for photographic surveying, and a brief description of the Bridges-Lee instrument (Fig. 18.1), as made by Messrs. Casella, is here given.

It consists essentially of a camera box A, which is fitted with a

[1] A. Laussedat, *Recherches sur les instruments, les méthodes et le dessin topographiques.*
[2] E. Deville, *Photographic Surveying.*

lens capable of giving good definition without distortion, and forming a camera of the fixed focus type. Inside this box is a hollow rectangular frame I, fitted on the rear face with two hair-lines K and K′,

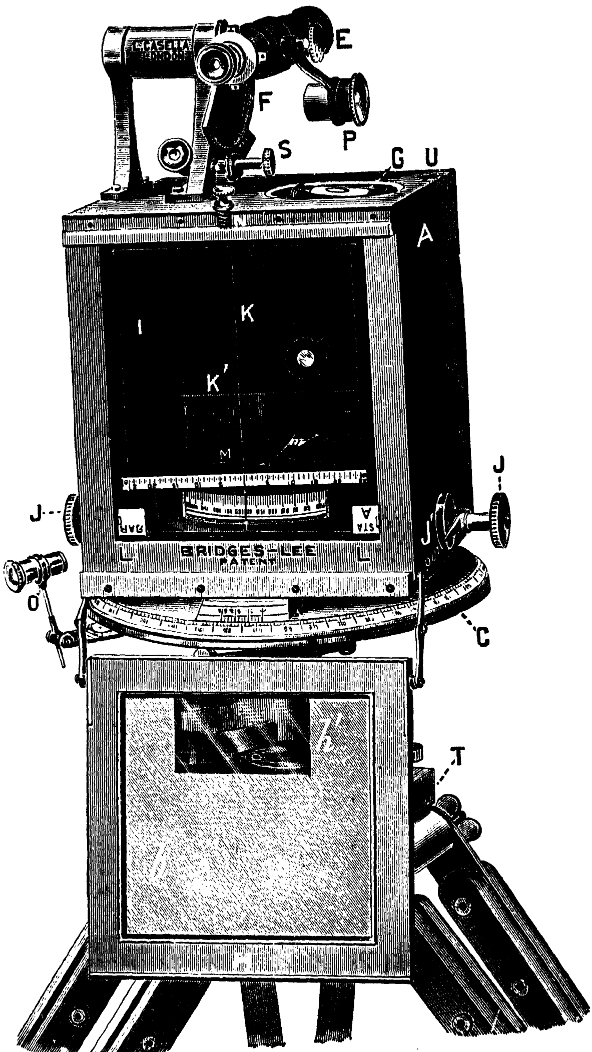


FIG. 18.1—Bridges-Lee Photo-Theodolite (Back View)

the one being vertical and the other horizontal. The intersection of these two hair-lines should be exactly opposite to the optical centre of the lens. Across the rear of the frame is also carried a straight transparent celluloid tangent scale, as shown in the figure, and upon

the base of the frame is pivoted a magnetic needle carrying a cylindrical transparent scale M.

Fitting into grooves in the lower corners of the frame I are two small celluloid strips. These may be easily removed, and any description of the view written upon them in red ink. (See Figs. 18.4 and 18.5.)

The back H*h* of the camera box A is hinged at the bottom corners, so that it may be folded downwards to permit a dark slide carrying the sensitised photographic plate to be fitted in its place, in which it is held by the spring N. When the front of the slide is withdrawn to expose the plate, and before the lens is uncapped, the frame I is moved backwards by means of either of the screws marked J in the figure, until the hair-lines and the tangent scale are in contact with the plate, the needle being simultaneously set free to swing on its pivot. When the lens is uncapped, sufficient time having been previously allowed for the needle to come to rest, the photographs of the hair-lines, the tangent scale, and the circular scale of the needle are imprinted on the negative, and are shown on the finished print as in Figs. 18.4 and 18.5.

Before replacing the front of the dark slide, the frame I must be moved forward, *i.e.* withdrawn from contact with the plate, by means of the screws J ; the needle is at the same time raised from its pivot.

The instrument is, in other details, very similar to a theodolite, being supported on a tripod, and furnished with—

(1) An inner and an outer axis, each of which is fitted with a clamp and fine adjustment screw.
(2) Parallel plates with three levelling screws, and
(3) A scale plate and verniers reading to single minutes.

Upon the top of the camera box is a small telescope E fitted with a vertical arc (F), a vernier, and a clamp (S), and tangent screw adjustment.

There is also one, or often two long sensitive bubbles (G) for levelling purposes.

The **Principle** of this method may be seen from the following brief description :

Let O (Fig. 18.2) be the optical centre of the camera lens, and k' the intersection of the cross-hairs, so that k'O may be termed the line of collimation of the camera. Then if A be any distant object which is within the field of view of the camera, rays from A will focus at a point a' on the plate $c'd'e'f'$, where, consequently, an inverted image of A will be formed. Also, as has been shown in Chapter II., AOa' will be in one straight line.

Let a_1' be the foot of the perpendicular from a' on to the horizontal hair-line, and let A_1 be the point vertically below A in the horizontal plane which contains the hair-line and the line of collimation. Also let K be the foot of the perpendicular from A on to the line of collimation k'OK.

It is now evident that the triangles $Ok'a'$, $Ok'a_1'$, $Oa'a_1'$ are similar to the triangles OKA, OKA_1, and OAA_1 respectively.

Hence if the negative *cdef*, shown by full lines in Fig. 18.2—or a positive print from the negative—is placed in a vertical erect position at a distance Ok, equal to Ok' in front of the point O, and at right angles to the line of collimation, k'OK, then the point a corresponding to a' will fall upon the ray OA, and the point a_1 corresponding to a_1' will fall on the ray OA_1.

Now if the position of the instrument station P is represented by the point p on the paper, the direction of the line of collimation pk may be drawn ; because, if the cylindrical scale M (Fig. 18.1) carried

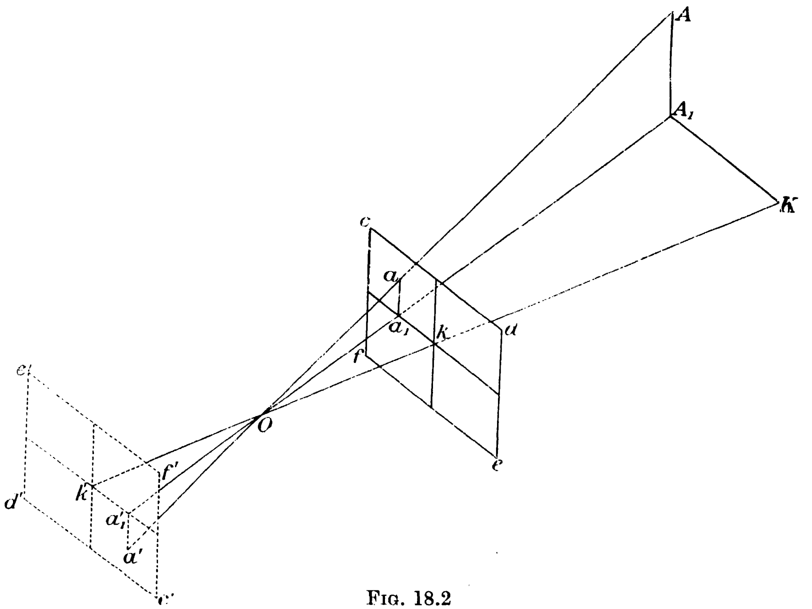


FIG. 18.2

by the magnetic needle has been allowed to come to rest before the lens is uncovered, the magnetic bearing of this line is indicated in the photograph ; *e.g.* in Fig. 18.4, the magnetic bearing of the line of collimation is 96°.

Along this line set off a distance pk (Fig. 18.3) equal to Ok (Fig. 18.2), *i.e.* equal to the focal length f of the camera lens, and through k draw a line ckd perpendicular to pk. This line may be taken to represent the horizontal projection of the photograph *cdef* in Fig. 18.2 : it is known as the picture trace.

To plot on the plan, the horizontal projection of a point A, shown by a in the photograph (Fig. 18.4), the horizontal distance of a from the vertical hair-line is measured with a scale or dividers, or on the edge of a strip of paper, and marked off along kd to a_1. The line joining p and a_1 then represents the direction of A from P.

If a second view is taken to include the point A from a second station Q, a similar line can be drawn from q, and the intersection of the two direction lines will locate A upon the plan.

If the horizontal distance D from A to P is now scaled from the plan, the altitude of A relatively to P may be calculated.

Thus if the horizontal angle A_1OK (Fig. 18.2) $=\theta$, and if the vertical distance of a above or below the horizontal hair-line on the picture

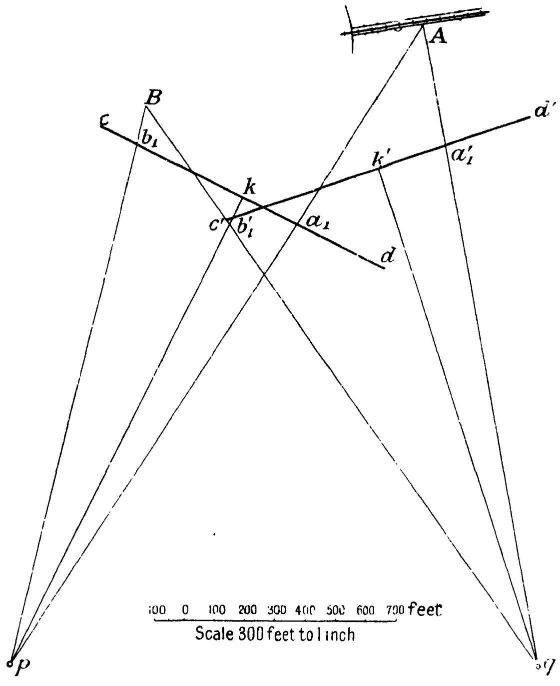


FIG. 18.3

is γ inches, while the focal length of the camera lens is f inches, then the angle of elevation AOA_1

$$=\tan^{-1}\frac{\gamma}{Oa_1}=\tan^{-1}\frac{\gamma}{f.\sec\theta}$$

$$=\tan^{-1}\frac{\gamma}{f}\cos\theta, \quad . \quad . \quad . \quad . \quad (1)$$

and the elevation of A relatively to P is

$$D.\frac{\gamma}{f}.\cos\theta. \quad . \quad . \quad . \quad (2)$$

θ may be calculated if the horizontal distance ka_1 is measured, *i.e.*

$$\theta=\tan^{-1}\frac{ka_1}{f},$$

or it may be obtained directly from the scale at the top of the print or negative, this being divided proportionally to the tangents of the angles and graduated in degrees and fractions.

If the prints are enlarged, f must be enlarged in the same ratio in the above formulae.

Method of Procedure.—The first operation in a photographic survey, as in other methods, is a careful reconnaissance of the area to be mapped, for the purpose of choosing the most suitable instrument stations. These must be so situated (1) that all objects which are to be represented on the final plan shall be clearly distinguishable on at least two photographs taken from different points, and (2) that the two direction lines which locate any particular point shall intersect at an angle which is neither too acute nor too obtuse (see below).

If a sketch plan is made, and the area roughly shaded which is likely to be covered by satisfactory intersections from each pair of stations, there is less likelihood of any portion being insufficiently surveyed than when the photographs are taken in a more haphazard fashion.

It is very desirable that important points should be included upon three photographs from different stations, so that the third direction line shall check the intersection of the first two. In fact, it is always preferable to take too many rather than too few views.

The various camera stations must be connected by a triangulation system. This may be executed by the usual methods, using an ordinary theodolite and working from a more or less accurately measured base, the degree of precision being determined by the extent of the survey and the scale to which the plan is to be plotted.

The camera stations will generally, though not always, be triangulation stations also, but the triangulation stations are not necessarily all camera stations.

The choice of camera stations is thus largely influenced by the nature of the triangulation which can be arranged between them.

In lieu of using a separate instrument for the triangulation, the angles may be observed with the photo theodolite at the same time that the photographs are being taken.

The number of plates to be exposed at any particular station varies with the special conditions of the case, *i.e.* upon the area to be mapped, and upon the field of view of the camera. The time of exposure is best calculated with an "exposure meter", and if the lighting will permit without undue loss of time, a small stop should be used. Orthochromatic plates with a colour screen are preferable for landscape work.

The views may be oriented :

(*a*) By sighting directly to another camera or triangulation station with the vernier reading zero, and then taking the bearing of the photograph axis from that point on the horizontal scale of the instrument. Under such circumstances it is very usual to make the exposures without waiting for the needle to come to rest, and so to neglect the

magnetic bearing altogether. If desired, however, readings may be taken as a check at a few stations.

If the views are taken at 36° intervals (*i.e.* 10 to a complete circuit at a station), the photographs will overlap slightly, and enable the pictures to be picked out very readily; though the negatives are easily identified if a description is written in red ink on the celluloid strips at the lower corner of the frame I (Fig. 18.1). *E.g.* Q – 72° R of P would indicate that the photograph was taken from station Q in a direction making an angle of 72° to the right of the line joining Q to another station P.

(*b*) A photograph may be taken in any direction to include the objects required, without reference to any other station, when the magnetic bearing of the line of collimation may be read on the curved scale at the top of the photograph.

If the bearing of any particular point shown on the picture is required, it may be obtained by measuring the horizontal distance of the object from the vertical hair-line on the tangent scale. Thus in Fig. 18.4 the bearing of A is 96° on the curved scale + 6°-50′ on the tangent scale, making a complete bearing of 102°-50′.

Plotting the Survey.—The plates having been developed by a competent photographer, the survey may be plotted from the normal prints—usually 5″ × 4″—made upon a glossy paper, employing a magnifying glass to pick out corresponding points on pairs of prints.

It is preferable, however, to enlarge the photographs to two or three times the original size, or sometimes even more.

A pair of prints covering the same area are then picked out, and corresponding points on each dotted and numbered in coloured ink.

One method of plotting may be illustrated in Fig. 18.3 by reference to the photographs in Figs. 18.4 and 18.5, which are from a photographic survey executed by the students of the University of Birmingham, under Professor S. M. Dixon.

Fig. 18.6 is reduced from a portion of the finished plan which was plotted to a scale of 300 ft. to 1 in.

Two points only are taken here: (1) the pier of the bridge, marked *a*, and (2) the point marked *b* on the hills above the town.

The positions of the camera stations P and Q were found by triangulation, and the view in Fig. 18.5 from Q was 72° to the right of P, while the view in Fig. 18.4 from P was 216° from another station R.

The magnetic bearings are also given on the curved scales at the top of the prints. A line *pk*, 5·66 in. in length, equal to the focal length of the lens is drawn in the correct direction from *p*, and a similar line *qk′* from *q*. At right angles to these are drawn the picture planes *cd* and *c′d′*. The horizontal distance of the point *a* from the vertical hair-line of the first photograph is marked off (ka_1) along *cd* to the right of *k*. The line pa_1 then gives the direction of *a* from *p*.

Similarly, the horizontal distance of *a* from the vertical line in the

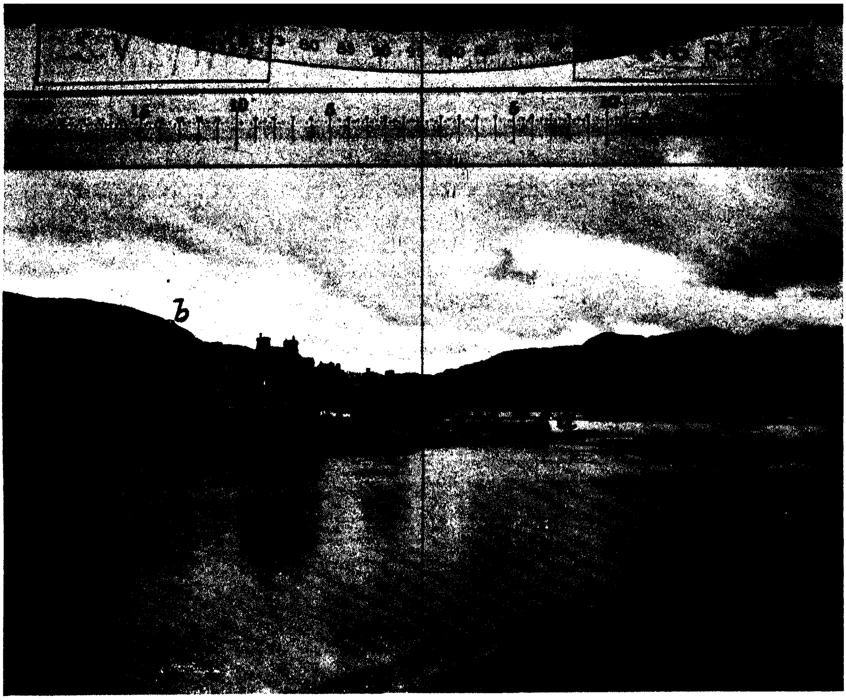


FIG. 18.4

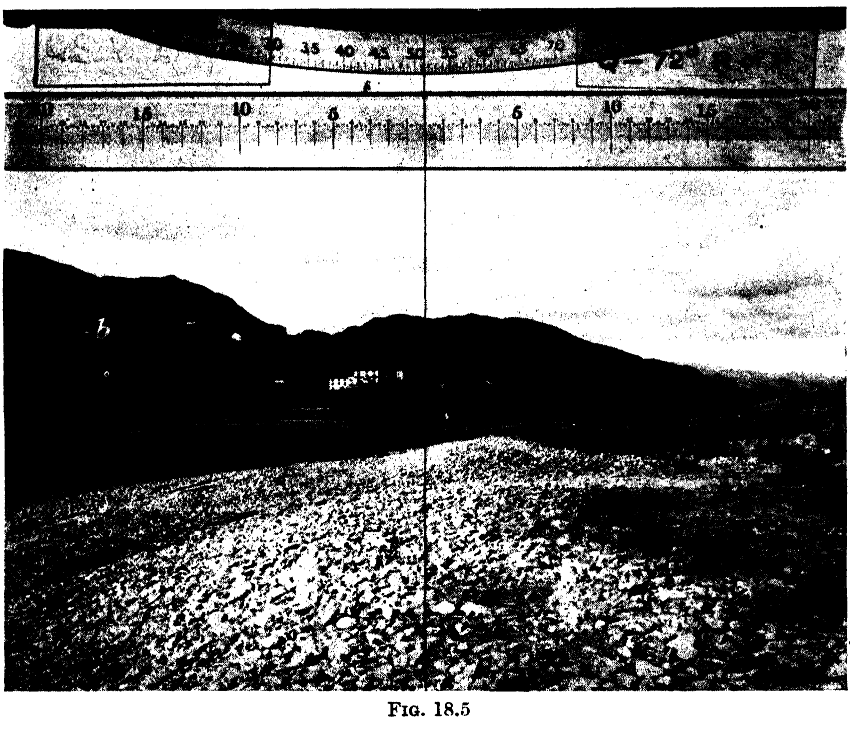


FIG. 18.5

second photograph is set off $k'a_1'$ along the line $c'd'$, so that the line qa_1' gives the direction of a from q.

The intersection at A of the two lines pa_1 and qa_1' locates the representation of the pier A upon the plan.

Similarly, the intersection of pb_1 and qb_1' locates B, the representation of the point B on the plan.

For a moderately small survey the points may be transferred from the photograph to the picture plane *cd* or *c'd'* by means of an ordinary plotting scale, or a pair of dividers, or by marking off the distances along the edge of a piece of paper.

For a large plan, especially if the prints have been considerably enlarged so that the length *pk* is 18 to 36 in., say, a convenient method is to take a long strip of paper and mark on this a transverse line to represent the central vertical hair-line; then to place this over the print so that the transverse line coincides with the vertical line, and to mark off the various dotted numbered points on the edge of the paper, each point being carefully numbered. The paper is then transferred to the plan so that the transverse line is along *pk*, while the edge of the paper lies along the trace *cd*. A long black thread, connected by a piece of thin elastic to a paper weight, is then attached to a pin placed upright in the plan at *p*. This thread is then stretched taut over a point—22, say—on the trace *cd*, while a similar thread attached to a pin at *q* is stretched over the corresponding point, 22, on the trace *c'd'*.

The intersection of the two threads then locates the point 22 on the plan, and this can be lightly marked as it is unnecessary to draw along pencil rays.

Professor Dixon [1] says: "The most expeditious method of transferring the horizontal distances from the prints to the picture lines is first of all to prick through the points desired, and then to cover the prints with the thinnest and most transparent tracing paper obtainable, on which are ruled hair-lines to correspond with the vertical lines through the centre of each picture. The points that have been marked on the prints are pricked and numbered on the tracing paper, which is then transferred to the drawing board for the survey to be plotted. By keeping the vertical hair-line on the tracing paper coincident with the line of direction of the picture under consideration, the points are easily marked along the picture line. With care this method will be found to be generally as accurate as is required by the scale of the map being plotted."

Levels.—The altitude of some or all of the camera stations may be determined by direct or trigonometrical levelling, as was done in the survey shown in Fig. 18.6, especially if the photographic work is for the purpose of filling in the topographical detail of a large scheme, or it may be approximately deduced if the reduced level of some point (*e.g.* the railway line on the bridge in Figs. 18.4 and 18.5) shown in one of the photographs is known.

To illustrate this, suppose the water-level at the pier A at the

[1] *Proc. Inst. C.E.*, "Surveying with a Camera", vol. clxxxiv.

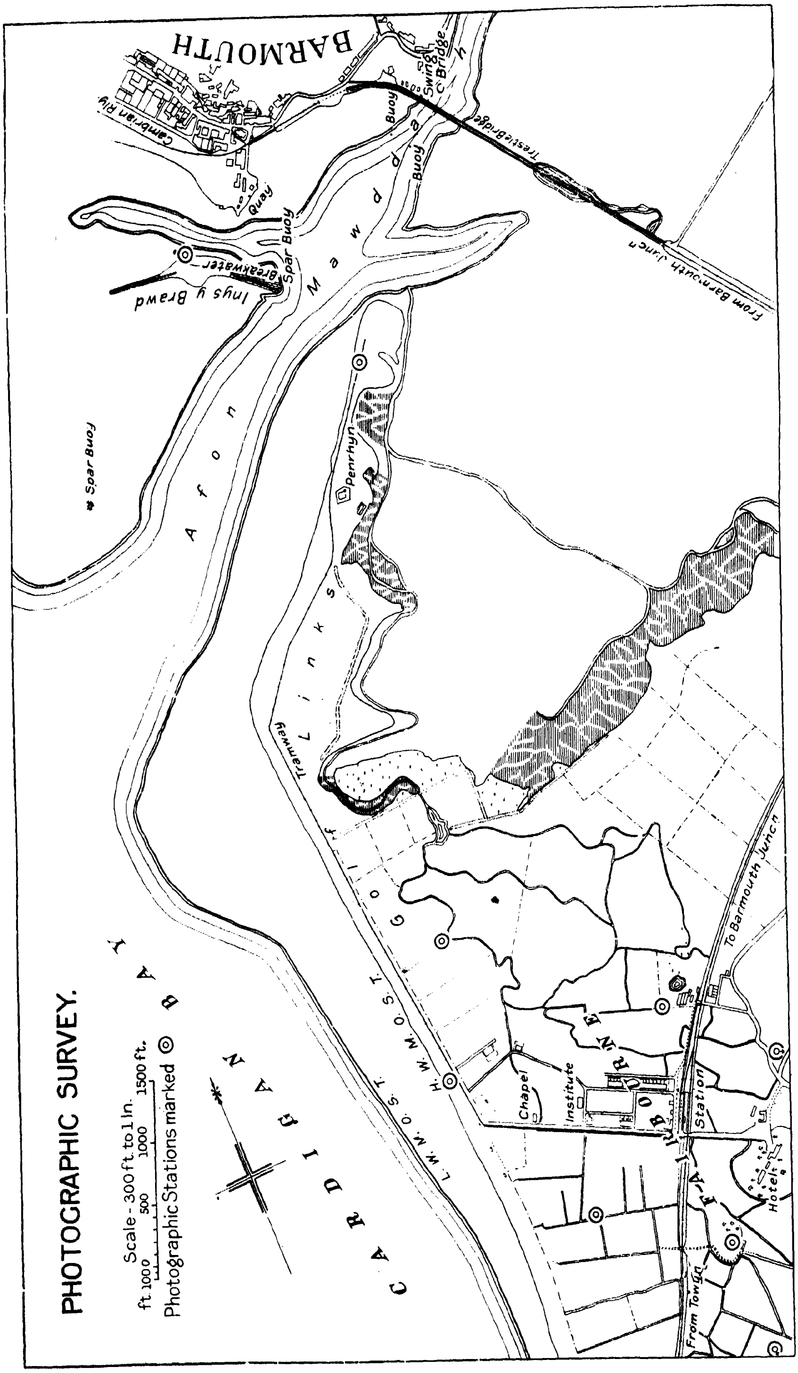


FIG. 18.6.—Photographic Survey

instant that the photograph was taken is considered to be datum level. The distance from P to A is scaled from the plan as 2475 ft., and the distance of the water surface at A below the horizontal hair-line of the first photograph is ·05 in., while the horizontal distance from the vertical hair-line is ·65 in.

By the formula deduced above, the reduced level of the camera axis at P is $D \cdot \frac{\gamma}{f \sec \theta}$, *i.e.*

$$2475 \times \frac{\cdot 05}{\sqrt{(5{\cdot}66)^2 + (\cdot 65)^2}} = 2475 \times \frac{\cdot 05}{5{\cdot}69} = 22 \text{ ft. nearly.}$$

Similarly, if the water-level is unchanged in the interval, the altitude of q

$$= 2110 \times \frac{\cdot 065}{5{\cdot}71} = 24 \text{ ft. nearly.}$$

The altitude of the point B calculated from p is therefore

$$22 + 1875 \times \frac{\cdot 47}{5{\cdot}80} = 174 \text{ ft. above datum,}$$

while from station q the altitude is

$$24 + 2240 \times \frac{\cdot 44}{5{\cdot}93} = 190 \text{ ft. above datum.}$$

The average value is therefore 182 ft. above datum, and the variation of either value from the mean about 8 feet. This discrepancy is rather excessive, but the error includes errors in plotting the two stations, and in scaling four distances from the prints and four from the plan. The water-level is assumed to be identical on both prints, and the line of collimation of the camera to be truly horizontal. Much closer results can be expected, particularly if the photographs are enlarged, and if the reduced levels of the camera stations are found by a direct method.

The reader may, as an exercise, determine for himself the reduced levels of a few points shown on the two photographs in Figs. 18.4 and 18.5, taking the rail level on the bridge as 26·38 ft. O.D.

Contours may be interpolated if the altitudes of a number of points are found in this way. It may be noted that all points on the first photograph intersected by the horizontal hair-line lie on a contour of approximately 22 ft. altitude, while all points on the second photograph intersected by that hair-line lie on a contour of approximately 24 ft. altitude.

Similarly, views from other stations fix other contours, and hence the altitudes of a large number of stations, between which the required contours can be easily interpolated.

To Determine the Focal Length of the Lens.—Usually the value of the focal length f is accurately determined and stated by the makers of the instrument. If not, the value may be deduced experimentally as follows:

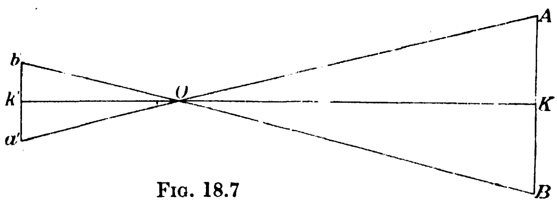


FIG. 18.7

(1) The angle AOB (Fig. 18.7) subtended at the instrument O by two distant objects A and B is accurately measured on the graduated horizontal limb of the theodolite. A plate is exposed to show A and B, and the distances $a'k'$ and $b'k'$ are scaled off the negative.

Now $$\angle AOB = a'Ob' = a'Ok' + b'Ok',$$

and $$a'Ok' = \tan^{-1}\frac{a'k'}{f},$$

and $$b'Ok' = \tan^{-1}\frac{b'k'}{f},$$

$$\therefore \angle AOB = \tan^{-1}\frac{a'k'}{f} + \tan^{-1}\frac{b'k'}{f}.$$

The value of f can therefore be calculated if the value of tan $\angle$AOB is abstracted from a table of tangents, for

$$\tan\angle AOB = \left(\frac{a'k'}{f} + \frac{b'k'}{f}\right) \Big/ \left(1 - \frac{a'k'}{f} \cdot \frac{b'k'}{f}\right).$$

(2) If A and B are two poles equidistant from the camera O, and the distances AB and KO are measured on the ground, and the distance $a'b'$ scaled from the negative, then f may be calculated as

$$f = \frac{a'b'}{AB} \cdot KO.$$

Photographic Surveying in Canada.—The Canadian equipment designed by the Surveyor-General, Dr. E. Deville, is much less elaborate than that usually employed in Europe. It consists essentially of a plain fixed-focus box camera provided with two spirit levels, and a 3-in. transit theodolite reading to minutes. The tripod is interchangeable between the two instruments. The photographic plates are $6\frac{1}{2}'' \times 4\frac{3}{4}''$, and bromide enlargements about $14'' \times 10''$ are used for plotting.

The first photographic survey was that of the main range of the Rocky Mountains adjacent to the Canadian Pacific Railway, executed in 1886–92, and covering an area of about 2500 square miles.

The following are a few particulars of the survey of a portion of the Selkirk Mountains range[1] in British Columbia, made in 1901–3 by A. O. Wheeler.

[1] "Photographic Methods employed by the Canadian Photographic Survey", by A. O. Wheeler, a paper read before the 8th International Geographical Congress, Washington, 1904.

The area surveyed was 1113 square miles, and the map which was contoured at 100 ft. vertical intervals was produced to a scale of 1/60,000. The field work occupied seven months, the party numbering six. One hundred and twenty camera stations were occupied, and 765 plates exposed. The distance between stations varied from 1 to 10 miles, being generally about 5 miles. The office work occupied 14½ months, three men being engaged upon it. The cost was deduced as follows :

Cost of field work	5,073·74 dollars
Cost of office work	4,395·60
Cost of material	120·00
Total	9,589·34
Cost per square mile	8·61 dollars
Cost per acre	1·34 cents

In the Survey of the Crownest Forest Reserve in Alberta,[1] made by M. P. Bridgland in 1913–14, and covering an area of 1500 square miles, the cost worked out at $9·50 per square mile.

Errors in Photogrammetry.—The following are a few of the chief sources of error in photographic surveying, apart from those due to any inaccuracy in the triangulation, these errors having been previously mentioned.[2]

1. Errors in orienting the photographs :

(*a*) When the directions are fixed by the observation of angles measured on the horizontal scale of the instrument, those directions are liable to such errors as are mentioned (with reference to the theodolite) in Chapter IV.

(*b*) These directions are subject to an additional constant error when the line of collimation of the camera (*i.e.* the line joining the intersection of the cross-hairs to the optical centre of the lens) does not lie in the same vertical plane as the line of collimation of the telescope by means of which the bearings are taken.

(*c*) When the directions are fixed by the bearings read on the curved scale at the top of the print the error may be very appreciable, due to local magnetic attraction, diurnal variations of the compass, and to the fact that the scale is not so finely divided as the main scale of the instrument. The p.e. may be, say, ±15 minutes.

2. Errors may be introduced when the instrument is not accurately levelled, *e.g.* through lack of adjustment of the bubbles. This source of error has been discussed in Chapter IV on the theodolite, and in Chapter IX on the plane table.

3. Errors in level, if the line of collimation of the camera is not exactly horizontal when the vertical axis of the instrument is truly vertical, and when the bubble is in the centre of its run. This error

[1] "Photographic Surveying in Canada", by M. P. Bridgland, *The Geographical Review*, July 1916.

[2] See "Sources of Error in Photographic Surveying", by W. N. Thomas, *Proc. I.C.E.* vol. ccx. p. 327.

may easily be of considerable magnitude, as it is difficult to test the accuracy of this adjustment, except by finding the altitude of a point at a known distance from the instrument by an examination of the photographs, and then comparing the results with those obtained by independent means with a theodolite or level. The telescope on top of the instrument may be used for this purpose if desired.

It will be seen from the example given above that a very small error in the position of the horizontal hair-line, due to sag, for instance, will produce a considerable error in altitude and in the position of contour lines. Consequently, the hair-lines should be examined occasionally to see that they are quite taut.

4. Errors caused by the displacement of the image on the photograph from its true position on account of—

(1) Distortion by the camera or enlargement lens.
(2) Distortion due to the plate not being truly vertical in the camera.
(3) Shrinkage or warping of the print during the development, or toning, fixing, washing, and drying processes.

5. Errors in measuring the distances from the cross-hairs to the various points on the prints.

6. Errors in transferring the measured distances to the picture trace on the drawing paper.

7. Errors in connecting—and producing, when necessary—the direction ray between the station point and the marked point on the picture trace to locate a point on the drawing.

8. Errors due to the difficulty of identifying exactly points on pairs of prints. For instance, a hedge or a road may be seen quite clearly on two photographs, and yet furnish no very distinct points that may be identified on two different views.

9. Mistakes of various kinds, in reading scales and transferring distances, confusion of points, etc.

The displacement on the plan of a point A, which is located by means of angles θ and ϕ at p and q respectively, as in Fig. 18.8, when these angles are subject to probable errors of $\pm\delta\theta$ and $\pm\delta\phi$ respectively, may be considered as follows :

Let the co-ordinates of A be X and Y with reference to p as origin and pq as the axis of X, and let $pq = \mathrm{D}$.

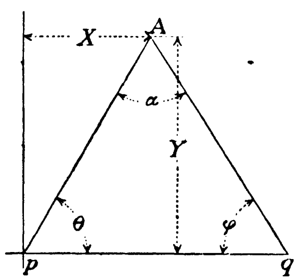


FIG. 18.8

Then
$$\frac{\mathrm{Y}}{\mathrm{X}} = \tan\theta$$
and
$$\frac{\mathrm{Y}}{\mathrm{D}-\mathrm{X}} = \tan\phi,$$
$$\therefore \mathrm{X} = \frac{\mathrm{D}\tan\phi}{\tan\theta + \tan\phi} \quad . \quad (3)$$
and
$$\mathrm{Y} = \frac{\mathrm{D}\tan\theta\tan\phi}{\tan\theta + \tan\phi} . \quad . \quad (4)$$

Then if $\pm\delta X_1$ and $\pm\delta Y_1$ be the probable errors in X and Y due to a probable error of $\pm\delta\theta$, and if $\pm\delta X_2$ and $\pm\delta Y_2$ be the probable errors in X and Y due to a probable error of $\pm\delta\phi$, then by differentiation,

$$\delta X_1 = \pm\frac{D \tan\phi \sec^2\theta}{(\tan\theta + \tan\phi)^2} \cdot \delta\theta, \qquad (5)$$

$$\delta X_2 = \pm\frac{D \tan\theta \sec^2\phi}{(\tan\theta + \tan\phi)^2} \cdot \delta\phi, \qquad (6)$$

$$\delta Y_1 = \pm\frac{D \tan^2\phi \sec^2\theta}{(\tan\theta + \tan\phi)^2} \cdot \delta\theta, \qquad (7)$$

$$\delta Y_2 = \pm\frac{D \tan^2 \sec^2\phi}{(\tan\theta + \tan\phi)^2} \cdot \delta\phi. \qquad (8)$$

The probable displacement of A, therefore, due to probable errors of $\pm\delta\theta$ and $\pm\delta\phi$ is

$$\pm\sqrt{(\delta X_1)^2 + (\delta X_2)^2 + (\delta Y_1)^2 + (\delta Y_2)^2},$$

i.e. $$\pm\frac{D \sec\theta \sec\phi}{\tan\theta + \tan\phi} \cdot \delta\theta, \text{ when } \pm\delta\theta = \pm\delta\phi,$$

i.e. $$\pm D \operatorname{cosec}(\theta + \phi) \cdot \delta\theta,$$

i.e. $$\pm D \operatorname{cosec} \alpha \cdot \delta\theta, \qquad (9)$$

where α is the angle included between the two rays.

The displacement is therefore a minimum when cosec α is a minimum, *i.e.* when α is 90°.

If the rays in the example above were fixed by means of magnetic bearings, and the total p.e. in each direction angle is taken as ± 15 minutes, then the p.e. in the location of A is $\pm D \operatorname{cosec} 42° \times 0{\cdot}00436 = \pm 1740 \times 1{\cdot}4945 \times 0{\cdot}00436 = \pm 11{\cdot}3$ ft., represented by 0·038 in. on the paper.

If the direction angles were measured on the horizontal scale, and the total probable error due to inaccurate centering, bisection etc., is taken as $\pm 1'$, while the probable displacement of a_1 is $\pm\frac{1}{200}$th of an inch, say, corresponding to $\pm 3'$, the total p.e. in θ is $\pm 3{\cdot}2$ minutes.

The probable displacement of A is therefore less than ·01 in. on the paper.

Had the prints been enlarged so that the picture trace was on the opposite side of A to p, the p.e. in θ would have been less, as the displacement of a_1 through $\frac{1}{200}$ in. would produce a p.e. of only $\frac{1}{2}$, $\frac{1}{3}$, or even less the amount given above. Consequently, the displacement of a point on the plan should be exceedingly small, and there should be no appreciable triangle of error formed when three rays are employed to locate a point.

Stereophotogrammetry is a further development of the art of photographic surveying. Two views of the same district are taken, one from each end of a measured base, and the negatives are examined by means of a special instrument known as a stereocomparator. Through this the two pictures are seen stereoscopically, and a view in relief is obtained.

Thus if A and B in Fig. 18.9 represent two objects, e_1 and e_2 the two eyes at a distance d apart, Ae_1, Ae_2, Be_1, Be_2 the rays from A and B to e_1 and e_2 respectively, the angle $Ae_1B = Ae_2B = \rho$, is known as the parallax of the objects. With the naked eye values of ρ as small as 30″ can be distinguished. In the stereocomparator the distance d is multiplied by means of mirrors, and owing to this and to the magnification of the image parallactic angles of less than ½ second can be distinguished and measured.

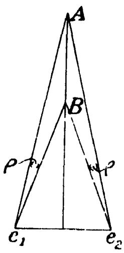


FIG. 18.9

From the co-ordinates of a point on the pictures—and these can be measured very accurately with the instrument—and from the value of the parallactic angle very precise results can be obtained.

As the two negatives are seen together stereoscopically, points are more easily identified than by the usual methods, and the field work also is much reduced.

The method has been largely used by the engineers of Russia, Germany, and Austria for preliminary railway and other surveys, and it is claimed that the results are quite as accurate as those of a good stadia survey. With a photo-theodolite having a focal length of 241·5 mm., using a base-line 50 m. long, points at a distance of 1000 m. were found to be uncertain within 0·8 metre. And at average distances of 200 m. to 400 m. with base-lines of 25 m. to 50 m., the accuracy has been found to be within 0·2 or 0·3 metre for distances, and 0·07 to 0·12 metre for elevations.

A further improvement known as the stereoautograph, used in conjunction with the stereocomparator, enables the plan to be plotted mechanically with contours complete, and effects a great saving in time and labour.

For a further account of these instruments the reader is referred to an article by Mr. Otto Lemberger on Stereophotographic Surveying, in the *Engineering News*, vol. lxix. No. 13, or to the short bibliography on p. 526.

AERIAL SURVEYING

It was during the Great War of 1914–18 that air photographs were first used to any appreciable extent for mapping purposes—in particular, they were used for the revision of detail such as enemy field works on existing maps. The methods used were crude, but the experience gained was invaluable and led to the appointment by the Government, in 1921, of the Air Survey Committee to investigate problems of aerial surveying. Its reports contain much valuable information.

To make a reliable map of a district by aerial survey methods, it is necessary—

(*a*) To furnish "ground control"—*i.e.* to fix points identifiable on photographs, by triangulation, traversing, or other ground survey methods.

(*b*) To procure suitable air photographs covering adequately the required area.
(*c*) To plot the survey from the photographs and adjust it to the ground control framework.
(*d*) To verify and fill in detail, if required, on the ground.

The photographing (*b*) is done by specialists, using aircraft and cameras suitable for the purpose. The plotting (*c*) may be done by specialist firms employing intricate and costly machines, or the surveyor may plot the survey from the photographs, using slower but simpler techniques. The two procedures may be combined, the preliminary work being done by the aid of the mechanical and optical devices, and the filling-in of the detail by graphical methods.

The extent of the ground control (*a*), and the ground survey of detail (*d*), depend upon the nature of the country, the scale of the map, and the accuracy demanded, etc.

The preparation of reliable maps requires an appreciable amount of ground survey work to control the plotting of the photographic detail, though mosaics prepared from vertical photographs of land inaccessible to ground surveyors have their uses.

Air-Photography.—The photographs may be either " verticals " taken with the camera pointing vertically or nearly vertically downwards, or " obliques " taken with the camera axis inclined to the vertical. At first sight it appears a simple matter to prepare plans from vertical photographs, the scale being f/H where f is the focal length of the camera and H the flying height above ground. Actually, there are many difficulties which are being gradually overcome by the design and development of suitable devices to aid both the " field " and the " office " work.

Sufficient photographs must be taken to cover the whole area, without leaving any gaps, and usually a considerable overlap must be allowed to ensure this. In each strip it is desirable that the overlap should be about 60 per cent : this allows all parts of the area to be depicted in at least two photographs, so that stereoscopic methods of examination may be employed. This amount of overlap also allows some detail to be shown on each three consecutive prints (see p. 523). The lateral overlap between adjoining strips of photographs may be 25 to 30 per cent. It is necessary, therefore, that the aircraft should fly in a straight course, in the specified direction, and at constant height : tilt of the machine either fore and aft or laterally should be avoided as far as possible.

If the aircraft is tilted laterally there is a danger that a gap will be left unphotographed between adjacent strips, whereas if the fore-and-aft inclination of the airplane changes during the flight there is danger of gaps being left between the photographs which form a strip, or of insufficient overlap being obtained for plotting purposes. Further possible causes of gaps are (1) the " drift " of the aircraft from its course, when a proper wind correction is not applied, and (2) the " crabbing " of the machine when such a wind correction is applied,

and the longitudinal axis of the machine is inclined (at the "crabbing angle") to the direction of flight. Under such circumstances, unless the camera is turned on its axis to allow for this, the photographs will not form a continuous strip as , but they will be in echelon as  and so gaps may be caused. A good example of a survey[1] made from vertical photographs is that of the forests of the Irrawaddy Delta. Here a ground control was provided consisting partly of a chain of triangulation along the main rivers, and partly of theodolite traverses. The average height of the camera was 9400 feet and the scale of the final map 3″ to 1 mile. The cost was 60 per cent of an estimated ground survey and it was considerably quicker, but these figures are not necessarily applicable to other conditions. Incidentally, useful information was given by the photographs concerning the distribution of different types of forest—but difficulty was found occasionally in interpreting the data.

For the preparation of very small-scale maps, particularly if contoured, oblique photographs may be preferred. They have been extensively used in Canada, and reference may be made to a Report by E. L. M. Burns.[2] The French Military Survey of Morocco in 1919 may be mentioned as an example of their use on this side of the Atlantic.

As an alternative to photographs taken with a single-lens camera, multi-lens cameras may be employed. An example of such a camera is the Barr and Stroud 7-lens camera which has a central lens about which are symmetrically placed 6 other lenses. The central lens takes a vertical photograph and the others obliques. Frequently the plotting is based on the combined use of the vertical and oblique prints, but with the 7-lens camera referred to, the "obliques" are "rectified" in the printing process, so that the 7 prints fit together to form one single vertical photograph, corresponding to that which would be taken with a single wide-angle lens of 120°.

Another important camera is the nine-lens camera of the United States Coast and Geodetic Survey.

Definitions and Geometrical Principles.—In Fig. 18.10 let O represent the camera lens, or perspective centre, and pOP the optical axis of the camera inclined at an angle θ to the vertical. The photographic plate or film lies in a plane at right angles to pOP, and hence is inclined at θ to the horizontal, as in the figure. If the value of θ is small, a "low oblique" results; if θ is large so that the visible horizon is included in the photograph, the result is a "high oblique". For vertical photographs, ideally $\theta = $ O, and the photographic film is horizontal.

The point vertically below O on the ground is the *ground plumb*

[1] "Aero-photo Survey and Mapping of the Forests of the Irrawaddy Delta", *Burma Forest Bulletin*, No. 11, Miscellaneous Series, No. 1, 1925.

[2] "Air Survey Technique in Canada—Present Position", E. L. M. Burns. Report of Conference of Empire Survey Officers, London, 1935. H.M. Stationery Office, London, 1936.

point, V, and the corresponding point on the photograph, vertically above O, is the *photo plumb point*, *v*. The point *p* where the optical axis cuts the photo-plane is the *photo principal point*, O*p* being perpendicular to the photograph and equal to *f*, the focal length of the camera lens. P is the *ground principal point*. Pairs of points such as V, *v* and P, *p* are known as homologous points, the line joining them passing through O.

The vertical *principal plane* contains *p*OP and *v*OV, and cuts the ground plane in PV*d*, the photo-plane in *vpd*, and the trace d_1d_2 in *d*. d_1d_2 will be perpendicular to this plane. *dpv* and *d*VP are the *photo* and *ground principal lines*. The axis of tilt is a line through O parallel to d_1dd_2 and the *direction of tilt* is along *d*VP. This tilt may be at any

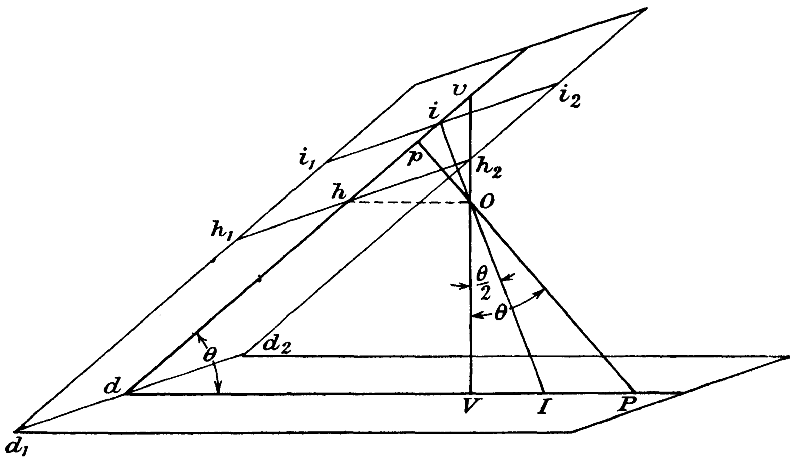


FIG. 18.10

angle to the direction of flight and is not necessarily in a fore-and-aft direction.

The horizontal plane through O cuts the photo plane in the *horizontal trace* h_1hh_2 and O*h* is the intersection of this horizontal plane and the principal plane OVP. The visible horizon would be shown on the photograph a little nearer *p* than h_1hh_2.

The line IO*i* in the principal plane, bisecting the angle VOP and equally inclined to the ground and photo planes, is the *isometric axis*, and the points I and *i* are the *ground* and *photo isocentres* respectively. There are two of each of these points, on the bisectors of the interior and exterior angles at O respectively—the former only are shown in Fig. 18.10.

Lines in the photograph, parallel to h_1hh_2, are *photo* or *plate parallels*, and that through *i* is called the *isometric parallel*. It is only along this line that there is no distortion on the print due to tilt, and there will be no distortion in the angles measured on the photograph at *i* provided that the corresponding angles on the ground are those between objects

at the same level. Any distortion due to tilt is thus along radial lines from the isocentre. The scale of the photograph along a plate-parallel through p will be given by the representative fraction Op/OP, *i.e.* $f/OP = f/H \sec \theta$; along the plate-parallel through i it will be Oi/OI, *i.e.* $f \sec \frac{1}{2}\theta / H \sec \frac{1}{2}\theta$, *i.e.* f/H, the same as for a " vertical " photograph —*i.e.* one taken with the optical axis vertical.

For any other parallel the scale can be calculated in the same way; it decreases at distances on the principal point side of the isocentre and increases at distances on the opposite side.

Thus at v it is $f \sec \theta/H$; and lines which are parallel to dVIP on the ground converge on the photo, the vanishing point coinciding with h.

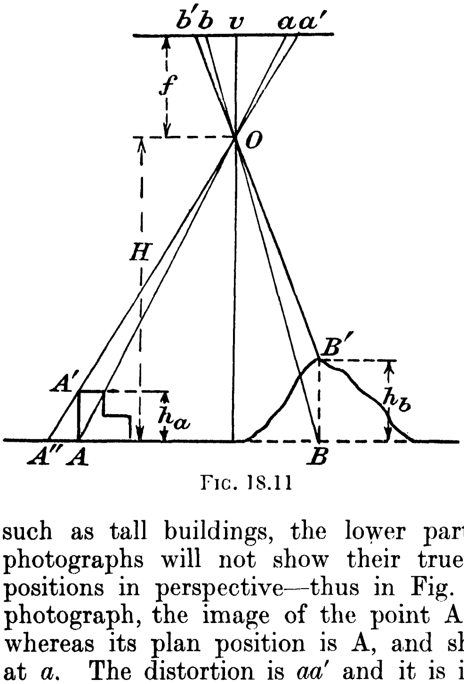


FIG. 18.11

In addition to distortion due to tilt, there may be distortion due to differences in the level of objects photographed—*i.e.* their images may be displaced on the prints from their true plan positions. Thus if on the ground there are natural features such as hills and valleys, or artificial features such as tall buildings, the lower parts of which are hidden, the photographs will not show their true positions in plan, but their positions in perspective—thus in Fig. 18.11, representing a vertical photograph, the image of the point A′ appears on the print at a', whereas its plan position is A, and should be shown on the print at a. The distortion is aa' and it is in a radial direction from the photo plumb point v, not from the isocentre as is the distortion due to tilt.

The amount of the displacement aa' can be calculated. Thus, $aa' = AA'' . f/H = h_a \tan \alpha . f/H$, and $\tan \alpha$ is obtained by scaling va' on the print and dividing by f, *i.e.* $aa' = va' . h_a/H$. Similarly $bb' = vb' . h_b/H$.

The *airbase* for a pair of air photographs is the distance between the two relevant camera stations; ideally it should be horizontal. Fig. 18.12 illustrates this for four successive camera positions, taking vertical photographs, the air bases for the successive pairs of photographs being O_1O_2, O_2O_3, O_3O_4, etc.

To determine the principal point of each photograph, in British practice it has been usual to fit the camera with an optically plane pressure plate, on which, near the edges, are engraved collimating marks. At the moment of exposure the film is pressed against the plate, and the marks are recorded on it. From them, the principal

point can be determined (see Fig. 18.13) by intersecting lines, or it may be marked directly on the film from an engraved mark on the pressure plate. The apparatus needs very accurate calibration to ensure that errors are negligibly small, and that the principal point so determined coincides with the " optical axis " of the camera. Other methods of determining the principal point, and which dispense with the pressure plate in front of the film, have been employed on the Continent. Pressure plates, engraved with a fine *reseau* or grid of 1 to 2 cm. squares, the lines being 0·001 inch wide, have been used. The device recorded on the negative enables distortion at various positions on the photograph to be estimated.

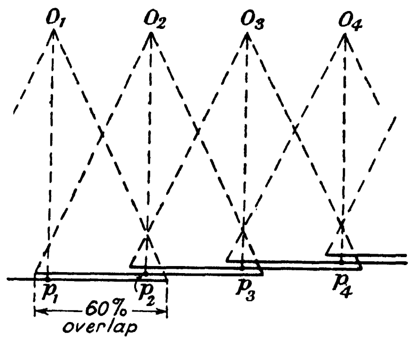


Fig. 18.12

Plotting the Survey from Vertical Photographs.—As already mentioned, the principal point on each photograph is sometimes marked directly on the photograph : if not, it can be located by the intersection of fine lines drawn from the collimating marks near the edges or corners of the photographs. The points thus located are transferred to the adjoining prints, by identification of detail, etc., so that each print has marked on it three principal points, *e.g.* on the third photo, p_3 directly from the pressure plate markings, and 2 and 4 from the adjoining photos.

Lines are drawn on the photographs joining these points and forming what corresponds to a traverse line, approximately along the centre line of a strip of photographs.

In the Arundel method of radial plotting, minor control points, *e.g.* *c*, *d*, etc., are chosen in the common overlap of three photographs in turn, and rays are drawn through them from the principal points of these photographs (actually short lengths only of these rays need be drawn).

It is assumed, as an approximation, that the true position of *c* will lie on each of these radial lines from the *principal points* p_2, p_3, p_4, though, as already explained, the displacements due to tilt are along radial lines from the *isocentre*, and displacements due to differences of level along radial lines from the *plumb points*. The positions of the isocentres and plumb points are, however, unknown unless the tilt is known.

At the same time, rays are drawn from any principal point or points to known ground control stations, *e.g.* A, B, etc.

These stations are, of course, already plotted on the map which is being constructed, forming a skeleton framework within which the photographic detail is to be fitted. The scale of the photograph must now be estimated, for instance, by comparing the distances apart of

these points on the photographs with the distances determined in the ground survey. Other estimates of scale can be made, using altimeter readings, etc. The position of p_2 (say) is marked on Kodatrace or other transparent material, and the ray p_2A is drawn. Along the latter the distance to A is scaled off to the appropriate scale for the finished map.

Similarly, the points 1 and 3 are plotted at the scaled distances from p_2 in the traced directions towards points identified as representing p_1 and p_3, and the ray from p_2 to c is traced in direction.

The tracing is now transferred to the next photograph, and placed

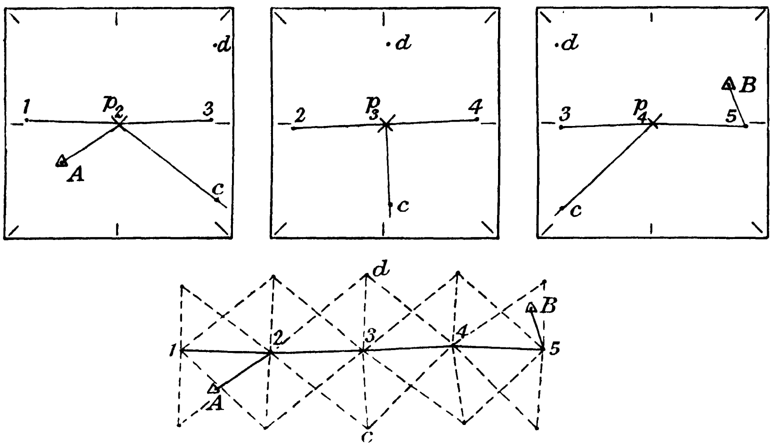


FIG. 18.13

with the point 3 above p_3 and the line 32 above the line 23 already drawn.

The ray from p_3 to c is traced, intersecting the ray previously drawn from p_2. The ray from p_3 towards p_4 is drawn and the length 34 scaled off. The tracing is then transferred to the next photograph, and correctly oriented, the 4 coinciding with p_4, when the ray from p_4 towards c should pass through the intersection of the two previous rays. If not, there will be a small triangle of error, to be eliminated by a trial adjustment.

When this procedure has extended sufficiently far to include other ground control points, *e.g.* B, a check is afforded on the work, and, exactly as in plane-table traversing, adjustments must be made, to distribute any small errors that have become apparent.

The filling-in of detail is a comparatively simple matter once the framework has been adjusted.

Application of Radar Techniques to Air Survey.—The term " Radar " is derived from the phrase " Radio Detection and Ranging ". It applies to those methods, developed during the late war, of locating distant objects by observations involving the use of radio pulses and their reflections. As is well known, one of the earliest uses of radar was

the detection and location of hostile aircraft approaching the coast. Later it was used to increase the bombing accuracy of our aircraft over Germany.

Some systems locate the aircraft (or other " target ") by means of range determinations with respect to two accurately located ground stations : other systems rely upon one range and a bearing.

A description of these methods would be out of place in this book, but it is relevant to note that radar methods have already been applied to air survey. The main application, so far, has been to the mapping of enemy-occupied country where ground control was unavailable—but the methods are not necessarily confined to military maps, and there are likely to be many developments in the near future.

At the present time, an aircraft taking air photographs can be located by radar from ground stations as far distant as 250 miles, with a random range error of about ± 20 metres to ± 50 metres, depending upon the technique employed. A map to a scale of say 1/25,000 can be plotted from the photographs without other ground control, heights being determined from altimeter readings to the nearest 100 metres, say, though relative heights should be accurate to perhaps ± 10 metres.

This is not quite as high a standard of accuracy as that to be expected from air surveys with normal ground control, where the maximum error in position on a map of 1/25,000 scale might be limited to 15 metres, and 2·5 to 5 metres in height, but the degree of accuracy attainable enables a useful map to be prepared and there is promise in future developments. There is also the possibility of combining radar control with a limited amount of ground control, and the employment of radar for navigational control to ensure complete coverage of an area at the one time—or to fill in gaps, if necessary, where these have been caused by the intervention of clouds or for other reasons.

In another branch of surveying it is claimed that radar methods can be applied to the measurement of very long lines on the earth's surface, up to 500 or 600 miles long. By present experimental radar methods such lines can be measured with an accuracy approaching 1/30,000, and no doubt this figure will be improved upon as the techniques are developed and improved.

Two papers by C. A. Hart, upon which the above notes are largely based, may be referred to for further information on the application of radar to air and other surveys. These papers, which were read before the Royal Society Empire Scientific Conference in Oxford, 1946, are : " Surveying from Air Photographs fixed by Remote Radar Control " and " Modern Mapping—including the Employment of Radio Techniques, and with special reference to Economic Developments ".

References.—The subject of aerial surveying is such a wide one that many of its aspects have necessarily had to be omitted from this book. Reference may be made to such specialized books and papers as have already been indicated above, or are listed below. This

list is obviously not exhaustive—some are of a general character and cover most aspects of the subject—others, as the titles indicate, are of a more restricted scope.

The extent to which aerial surveying may be used for the revision of Ordnance Survey plans is a matter of interest to British surveyors, and they may consult references Nos. 9–13.

Specialized equipment—whether used for the taking of the photographs or for the plotting of aerial surveys—is described in general books such as reference Nos. 2, 3, and 4, or in the Professional Papers of the Air Survey Committee, etc. Some special references are also given below (*e.g.* Nos. 16–18).

1. Professional Papers of the Air Survey Committee published by H.M. Stationery Office, 1923 onwards.
2. *Air Photography applied to Surveying.* C. A. Hart.
3. *Surveying from Air Photographs.* M. Hotine.
4. *Photogrammetry.* O. von Gruber, translated by G. T. McCaw and F. Z. Cazalet.
5. *Generalized Linear Perspective.* J. W. Gordon.
6. " Plotting from Oblique Aeroplane Photographs ", E. Deville (*Report of the Canadian Air Board*).
7. *Aerial Surveying by Rapid Methods.* B. Melville Jones and J. C. Griffiths.
8. Reports of the Topographical and Air Survey Bureau. Canada.
9. *Report on the Experimental Revision of the 1/2500 Ordnance Survey Plans with the Aid of Photographs taken from the Air.* Ordnance Survey, H.M.S.O. 1927.
10. *Report No. 2 on the Experimental Revision of 1/2500 Plans with the Aid of Photographs taken from the Air.* Ordnance Survey, H.M.S.O., 1930.
11. *Final Report of Departmental Committee on the Ordnance Survey.* H.M.S.O., 1938.
12. Revision by Air Survey. Professional Paper No. 16 of the Ordnance Survey. H.M.S.O., 1935.
13. Revision of Ordnance Survey 25-inch plans from Air Photographs. Lt.-Col. R. Ll. Brown. *Journal Chartered Surveyors Inst.*, vol. cvi. 1937.
14. " Some Recent Developments in British Surveying Instruments." M. N. McLeod. *Physical Society,* March 1939.
15. " The Rectification of Air-Photographs." H. G. Fourcade. *Empire Survey Review*, vol. iii. p. 272, 1935.
16. " The Radial Stereo-Plotter." E. L. M. Burns and R. H. Field. *Canadian Journal of Research*, A. 15, 1937.
17. " The Autocartograph." M. N. McLeod. *Geographical Journal*, vol. lix.
18. " The Stereoautograph." A. R. Hinks. *Geographical Journal*, vol. lix.

CHAPTER XIX

ERRORS IN SURVEYING: ADJUSTMENT OF OBSERVATIONS

THE discrepancy between the observed and the true value of any measured quantity in surveying is the result of

(*a*) Mistakes.
(*b*) Cumulative or systematic errors.
(*c*) Compensating or periodic errors.

Mistakes are generally avoidable and cannot be classed under any law of probability.

A few mistakes which occasionally occur in linear measurements are mentioned on p. 40, Chapter I; and in angular measurements on p. 118, Chapter IV. Numerous examples in connection with other branches of surveying will readily occur to the reader, *e.g.* the mis-counting of the revolutions of a current meter, miscalculation in arithmetical work, confusion of the rays in plane table operations, etc.

Cumulative and Compensating Errors may be due to

(*a*) Natural causes, such as temperature, barometric or magnetic changes, humidity, wind, curvature and refraction, etc.
(*b*) Defects in the construction or adjustment of instruments.
(*c*) Personal defects in vision, etc.

Cumulative Errors are those which tend always in the same direction, *i.e.* either to make the apparent measurement always too large or always too small.

Examples in linear measurements are given on p. 41; and in angular measurements on pp. 106-118; in levelling on pp. 207 *et seq.*, while other instances are mentioned throughout the volume.

Cumulative errors are directly proportional to the number of observations; *e.g.* if the error in the length of a chain is d links, the total error due to this cause in a distance of N chains is $N \times d$ links; or if a box sextant has an index error of d minutes, the total error due to this in the measurement of the angles of a polygon of N sides is $N \times d$ minutes.

Cumulative errors do not tend to be eliminated by adopting a mean of several independent measurements made under the same conditions; *e.g.* if the length of N chains is measured M times, the total error will be $M \times N \times d$ links, and the error in the arithmetic mean will be $\frac{M \cdot N \cdot d}{M} = N \cdot d$ links as before.

Compensating Errors are those which tend sometimes in one direction and sometimes in the other, *i.e.* they are equally likely to make the apparent result too large or too small.

Examples which occur in linear measurements are given on p. 41; in angular measurements on pp. 106-118, and in levelling operations on pp. 207 *et seq.* Other instances are mentioned throughout the volume.

It will now be apparent that if a number of measurements of the same quantity are made with the same instruments, and under the same conditions, it would be possible to obtain consistent though inaccurate values, if all the errors were cumulative. For instance, if a chain is 0·1 link too short, and all other sources of error were eliminated, the measurements of the length of a line might agree among themselves, and yet the result be 0·1 per cent in error.

Compensating errors, on the other hand, cause the variations between a number of observations, variations which occur to some extent, however carefully the quantity is measured, and it is to this class of error only that the Theory of Probability can be applied.

Theory of Errors.—It has been noted above that compensating errors tend sometimes in one direction and sometimes in the other. Two other features of compensating errors which usually occur are (i) small errors tend to be more frequent than large ones, and (ii) equal positive and negative errors are equally likely to occur. The theory of probability describes these features by saying that the relative frequencies of errors of different extents can be represented by a curve as in Fig. 19.1. Errors are read off on the horizontal axis XOX, a point on the left of O representing a negative error, *i.e.* an observation short of its correct value, and a point on the right of O a positive error. The ordinate of the curve at any point then represents the relative frequency or probability of the corresponding error. The curve is known as the curve of error or frequency curve.

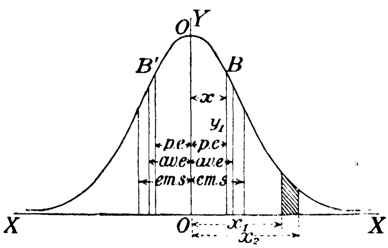


FIG. 19.1.—Probability Curve

It will be seen that any curve which (i) falls away from the centre, and (ii) is symmetrical about the centre, will have the features mentioned above. There is one curve which is found to be particularly applicable to this type of error, and it is known as the Normal curve of error (or Gaussian distribution). The equation of this curve is

$$y = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}, \qquad . \quad . \quad . \quad (1)$$

where e is the base of Napierian logarithms, *i.e.* 2·71828 and σ is a constant length. It can readily be verified that the ordinate y is a maximum for $x=0$ and decreases as x increases either in a positive or negative direction. The curve in Fig. 19.1 is, in fact, the normal curve.

The first point to be noticed about this equation is that any value for the error, however large positively or negatively, may be substituted in the equation to give a probability which is not equal to zero, corresponding to the fact that the curve approaches very near to the horizontal axis without ever reaching it. On the other hand, practical experience tells the surveyor that errors beyond certain limits will never occur. Actually there is no real inconsistency here; owing to the special form of the equation, the relative probabilities eventually fall off so rapidly that they can for all practical purposes be ignored, and the curve tells one that the chances of these large errors are effectively zero.

The second point is the importance of the quantity σ (the Greek letter sigma, σ). If we consider a set of curves corresponding to a set of values of σ (which is measured in the same units as the error, *i.e.* inches, seconds of angle, etc.), it will be seen that as σ increases, the curve tends to be depressed at the centre and to extend for a greater distance on either side of the centre before becoming effectively in contact with the axis. Thus, for large σ there is a greater relative probability of large errors. Conversely, if σ becomes small the curve becomes high at the centre and is confined to a narrow band, indicating that only small errors are likely to occur. The quantity σ does, in fact, give us a numerical measure of the *range* or *extent* which the errors may occupy, and for this reason it is known as the *standard error* or *standard deviation.* It is also known as the *mean square error* (*e.m.s.*) or the *root mean square* (*r.m.s.*) for the reason that if one takes a large number n of observations, with errors $x_1, x_2, \ldots, x_n$, then it can be shown that

$$\sigma=\sqrt{\Sigma x^2/n}, \qquad (2)$$

so that σ^2 is the average, over a large number, of the square of the error.

We have so far only described the frequency curve as giving relative probabilities or frequencies, whereas in practice we are concerned with estimating absolute probabilities. These can readily be obtained from the probability curve. Thus, in Fig. 19.1, the probability of obtaining a positive error lying between x_1 and x_2 in magnitude is equal to the shaded area, *i.e.* the area between the ordinates at x_1 and x_2, the horizontal axis and the curve, divided by the whole area under the curve. Or, taking the area under the curve as the unit, the probability is equal to the shaded area measured in that unit. We can express this mathematically in the case of the normal curve by saying, probability of obtaining error between x_1 and x_2

$$=\frac{1}{\sqrt{2\pi}\,\sigma}\int_{x_1}^{x_2} e^{-x^2/2\sigma^2}dx, \qquad (3)$$

since the total area under the curve is

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}dx=1. \qquad (4)$$

In surveying, the quantity known as the *probable error* (p.e.) is most commonly employed ; this is defined by saying that errors are equally likely to be numerically greater or less than the p.e., so we must have

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\text{p.e.}}^{+\text{p.e.}} e^{-x^2/2\sigma^2}\,dx = \tfrac{1}{2}. \qquad . \qquad . \qquad (5)$$

The integral in equation (3) cannot be evaluated in terms of simple functions, but ample tables of the integral exist, so that we can solve an equation such as (5) quite readily. We find, in fact that

$$\text{p.e.} = 0{\cdot}6745\sigma, \qquad . \qquad . \qquad . \qquad . \qquad (6)$$

or, in other words, the probable error is about two-thirds of the standard error.

The term " probable error " is, perhaps, apt to be misleading. It does not signify that this error is more likely to occur than any other, for an error of zero is more likely to occur than any other. The title is, in fact, only a conventional one.

It is of interest to quote the probabilities of exceeding other multiples of the standard error.

Deviation	Chance of obtaining a Positive Error exceeding x	Chance of Error lying outside the Range $-x$ to x
0·6745 σ	0·25	0·50
σ	0·1587	0·3173
1·2816 σ	0·1	0·2
1·6449 σ	0·05	0·1
1·9600 σ	0·025	0·05
2·3263 σ	0·01	0·02
2·5758 σ	0·005	0·01
3·0902 σ	0·001	0·002

Thus there is only one chance in a 100 of obtaining an error greater in absolute magnitude than about two and a half times the standard error and about one chance in 500 of exceeding three times the standard error in absolute magnitude. For practical purposes, therefore, we can say that errors are not likely to occur outside a range of two or three times the standard error.

Another criterion of accuracy which is often used is the *Average Error* or *Mean Deviation* (a.e.) ; this is the mean of all the errors, ignoring signs. For the normal curve it can be shown that

$$\text{a.e.} = \sqrt{\frac{2}{\pi}}\,\sigma = 0{\cdot}7979\,\sigma, \qquad . \qquad . \qquad (7)$$

so that the average error is about 0·8 time the standard error. The probability that an error occurs outside the range ±a.e. is equal to 0·425. In Fig. 19.1 the probable error, the average error, and the standard error have been indicated. These three measures can be readily converted into each other by means of equations (6) and (7) : the standard error is the most convenient from a theoretical point of

view, but the probable error is most commonly referred to in surveying. The p.e. $=0{\cdot}845$ a.e.

So far only the theoretical relations between these quantities and their connection with the probability curve have been considered. In practice, however, we are concerned with the estimation of the measures of dispersion from a limited set of data. Two cases arise here, according as we do or do not know the true value of the quantity measured. Triangulation is an example of the first case, in which the three angles of each triangle should sum exactly to 180° (+spherical excess): if not, the error in each triangle, *i.e.* the " triangulation error ", can be computed. If an observer obtains triangulation errors $x_1, x_2, \ldots x_n$, then an estimate of his standard error is

$$s=\sqrt{\Sigma x^2/n}, \quad . \quad . \quad . \quad (8)$$

and an estimate of his p.e. is $0{\cdot}6745s$. Thus if a number of observers are engaged upon triangulation, a comparison of the reliability and precision of each can be made. Similarly the degree of accuracy that may be expected with a certain surveyor, using certain instruments under certain conditions, may be estimated. It should be remembered, however, that this quantity s or the derived p.e. are only *estimates*, based on a particular sample, of his accuracy, and obviously more reliance can be placed on a large sample than a small sample. It is beyond the scope of this note to deal with this point in detail, and reference should be made to works on mathematical statistics.

In many, if not most, surveying measurements, the true value of the measured quantity is unknown. If $\bar{x}$ be the assumed true value and $x_1, x_2, \ldots$ be the deviations from it, then it can usually be shown that the theory of least squares is applicable, *i.e.* the most likely value of $\bar{x}$ is that for which the sum of the squares of the deviations is a minimum. This condition is satisfied when $\bar{x}$ is the arithmetic mean of the observations and the deviations $x_1, x_2, \ldots$ are then known as *apparent* or *residual* errors. Under this condition the best estimate of the standard error σ is

$$s=\sqrt{\Sigma x^2/(n-1)}\,; \quad . \quad . \quad . \quad (9)$$

the best estimate of the p.e. is then $0{\cdot}6745s$ and of the a.e., $0{\cdot}7979s$.

It will be noted that, whereas in the case where the true error is known we divide the sum of squares of the true error by n (eqn. (8)), in the case where it is not known we divide it by $n-1$. The reason for this is a technical one rather beyond the scope of this note, but it will be observed that for samples of a reasonable size the effect of using n instead of $n-1$ is not large. For rather a similar reason it will be noted that we could calculate an average error $\Sigma|x|n$ from the sample ($|x|$ denoting the absolute value); this, however, tends on the average to be slightly less than the true a.e., and it needs to be multiplied by a constant depending on n in order to make it unbiased. For this and other reasons it is preferable to use the formula (9).

Reference has been made to the fact that if a sample of measurements of an unknown quantity is available, the best estimate of

that quantity is the arithmetic mean $\bar{x}$. It can then be shown that the standard error of this mean is equal to $\sigma/\sqrt{n}$, so that the best estimate of standard error is

$$\frac{s}{\sqrt{n}}=\sqrt{\frac{\Sigma x^2}{n(n-1)}}, \qquad (10)$$

where the x's denote the residual errors as before. We can then give the p.e. of the mean as $\frac{0\cdot6745s}{\sqrt{n}}$; this implies that there is a 50 per cent chance that the observed mean differs from the true mean by less than this quantity, but it should be remembered that this only applies to large samples. Similar chances with the same proviso would apply to other multiples of the estimated standard error.

Other results following from the theory of errors, which may have applications in surveying calculations, are :

(1) If A is the product of variables B, C, D, then

$$e_A=\sqrt{(BCe_D)^2+(CDe_B)^2+(BDe_C)^2}$$

where e_A, e_B, e_C, and e_D are the errors (s.e., p.e., a.e.) in A, B, C and D respectively.

(2) If $A=B\pm C\pm D\pm \ldots$ then

$$e_A=\sqrt{e_B{}^2+e_C{}^2+e_D{}^2+\ldots}$$

As an instance of the application of this rule may be mentioned the case of triangular error. Thus, if the p.e. of a single triangle of a system is $\pm 2''$, the p.e. of a single angle is $\pm 2''/\sqrt{3}$. The formulae for the errors of the arithmetic mean also follow directly from this rule. (It will be noted that it is in many cases customary to insert the sign $\pm$ in front of any number which measures an error. Strictly, of course, the sign is unnecessary since the standard error or any other error is essentially a positive quantity.)

Example 1.—If with a 4-in. theodolite the p.e. of a single observation is $\pm 1'$, then by adopting a mean of four repetitions (p. 118) the p.e. is reduced to $\pm\frac{1'}{\sqrt{4}}=30''$.

Example 2.—A number of measurements of a line gave the following results:

Links	x_1.	x_1^2.
7017.	0	0
7014	−3	9
7018	+1	1
7020	+3	9
7015	−2	4
7018	+1	1
Mean = 7017	$\Sigma x_1=10$	$\Sigma x_1^2=24$

Applying equation (9) the p.e. of a single measurement

$$=\pm 0\cdot6745\sqrt{\frac{24}{5}}=\pm 1\cdot48 \text{ links nearly.}$$

Similarly, by equation (10), the p.e. of the arithmetic mean

$$= \pm \cdot 6745\sqrt{\frac{24}{6 \,.\, 5}} = \pm \cdot 60 \text{ link nearly.}$$

The length of the line would therefore be stated as

70·17 chains ± ·60 link.

Calculation from the a.e. would give the p.e. of the arithmetic mean

$$= \pm \cdot 845 \,.\, \frac{10}{6 \,.\, \sqrt{5}} = \cdot 63 \text{ link.}$$

The discrepancy between the results is due to the small number of readings. Strictly speaking, the theory of errors is not applicable in such a case, as the assumption that n is a very large number is not true. In practice, however, it is usual to apply the first-mentioned formula (10).

It must be noted here that the above result does not state that the length of the line is *correct* to within ±0·60 link.

In the first instance, it infers that the length is as likely to be between the limits 70·164 and 70·176 chains as to be outside them; it does not infer that ·60 link is the *maximum* error.

Secondly, no account is taken of cumulative or constant errors, such as errors in the length of the tape, etc.

The magnitude of the p.e., therefore, is no real criterion as to the absolute accuracy of the result, but it is a useful guide as to the skill and care with which the work has been executed.

Weighting.—When certain of the observations are made under more favourable conditions than others, or when they are themselves the means of groups of normal observations, they may be "weighted". Thus, the first observation in Example 2 above may be given a weight of $\omega = 4$, and the third observation a weight of $\omega = 2$, if the precision of these observations is considered to be $\sqrt{4}$ and $\sqrt{2}$ times, respectively, that of the remaining observations, for each of which $\omega = 1$.

The modified data of Example 2 are then analysed as follows:

		x_1.	ωx_1^2.
7017 × 4 =	28068	−0·1	4 × ·01 = 0·04
7014 × 1 =	7014	−3·1	= 9·61
7018 × 2 =	14036	+0·9	2 × ·81 = 1·62
7020 × 1 =	7020	+2·9	= 8·41
7015 × 1 =	7015	−2·1	= 4·41
7018 × 1 =	7018	+0·9	= 0·81
$\Sigma\omega = 10$	10)70171		$\Sigma\omega x_1^2 = 24{\cdot}90$

Weighted mean = 7017·1

The p.e. of the weighted mean is taken as

$$\pm \cdot 6745\sqrt{\frac{\Sigma\omega x_1^2}{(n-1) \,.\, \Sigma\omega}} = \pm \cdot 6745\sqrt{\frac{24{\cdot}90}{5 \times 10}} = \pm \cdot 476 \text{ link.}$$

The p.e. of a single observation of unit weight

$$= \pm \cdot 6745\sqrt{\frac{\Sigma\omega x_1^2}{(n-1)}} = \pm \cdot 6745\sqrt{\frac{24{\cdot}90}{5}} = \pm 1{\cdot}505 \text{ links.}$$

The p.e. of the first observation equals the p.e. of the mean of four observations of unit weight,

$$= \pm \frac{1 \cdot 505}{\sqrt{4}} = \pm \cdot 753 \text{ link.}$$

The p.e. of the third observation equals the p.e. of the mean of two observations of unit weight,

$$= \pm \frac{1 \cdot 505}{\sqrt{2}} = \pm 1 \cdot 064 \text{ links.}$$

The term " Permissible Error " is often used in surveying, and denotes the maximum allowable limit that a measurement may vary from the true value, or from a value previously adopted as correct. It includes all classes of error—cumulative, compensating, and mistakes.

For example, if triangulation stations have been located by accurate trigonometrical operations, and the detail between such points is to be filled in by means of chain surveys, the Surveyors may be told that their permissible error for the chain work is $\frac{1}{1000}$, *i.e.* that if the distance between the two stations upon which they are working differs by more than 0·1 per cent from the trigonometrical result, the work must be repeated, or the source of error discovered. If the discrepancy between the chain and trigonometrical results is smaller than the permissible error, it would be distributed evenly throughout the work so as to be as little appreciable as possible upon the finished plan.

Examples of the permissible error for various classes of work have been mentioned throughout the volume.

The value of the permissible error in any given case depends upon the scale and purpose of the plan, the instruments available, etc.

A good general rule is to so fix the permissible error that after the necessary adjustments have been made no point on the plan is likely to be displaced an appreciable amount from its true position, *e.g.* not more than $\frac{1}{100}$ in. say.

This, again, depends upon the class of work, *e.g.* for route surveys it would obviously be much too severe.

Adjustment of Observations.—It is beyond the scope of this book to describe fully the various methods of adjusting observations made in the different branches of Surveying. Some of the simpler methods have been described in the text, and notes on the theory of errors are given earlier in this chapter. It may be useful, however, to give a few illustrative examples.

Example 1.—The heights of three points A, B, C, above a datum O were determined by direct levelling, and as a check the differences in level of A and B and of B and C were also observed. The " weights " of the observations were estimated from such factors as the lengths of the lines, number (N) of instrument stations, instrument used, skill of observer. The " weights " should vary inversely as the square of the probable errors.

As an approximation, other factors being equal, the p.e.'s might be considered as proportional to $\sqrt{N}$ (see Chapter VI), and the weights inversely proportional to N, or inversely proportional to the lengths of the lines. There is room for considerable difference of opinion, usually, in judging " weights ".

The observation equations can be written :

A - O =	84·624	with $w=4$
B - O =	51·762	,, $w=4$
C - O =	62·463	,, $w=4$
A - B =	32·840	,, $w=6$
B - C =	- 10·714	,, $w=3$

We can consider two closed circuits OABO and BOCB with a common link BO.

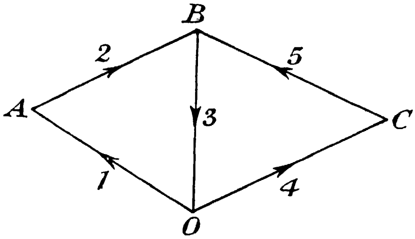


FIG. 19.2

The equations can then be rearranged :

$$\left.\begin{array}{lll} a_1 = \text{OA} = & 84{\cdot}624 & \text{weight } w_1 = 4 \\ a_2 = \text{AB} = & -32{\cdot}840 & \quad ,, \quad w_2 = 6 \\ a_3 = \text{BO} = & -51{\cdot}762 & \quad ,, \quad w_3 = 4 \\ a_4 = \text{OC} = & 62{\cdot}463 & \quad ,, \quad w_4 = 4 \\ a_5 = \text{CB} = & -10{\cdot}714 & \quad ,, \quad w_5 = 3 \end{array}\right\} \quad . \quad . \quad . \quad (1a)$$

If θ_1, θ_2, . . . are the true differences in level, then $\theta_1 + \theta_2 + \theta_3 = 0$ and $\theta_3 + \theta_4 + \theta_5 = 0$.

The observed values
and

$$\left.\begin{array}{l} a_1 + a_2 + a_3 = d \\ a_3 + a_4 + a_5 = d_1 \end{array}\right\} \quad . \quad . \quad . \quad (2)$$

Then the unknown errors are

$$e_1 = (a_1 - \theta_1),\ e_2 = (a_2 - \theta_2), \text{ etc.}$$

Therefore we can write these as

$$\left.\begin{array}{l} e_1 + e_2 + e_3 = d \\ e_3 + e_4 + e_5 = d_1 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The next condition we have is that derived from the Theory of Least Squares: the weighted sum of the squared errors should be a minimum, *i.e.*

$$w_1e_1^2 + w_2e_2^2 + w_3e_3^2 + w_4e_4^2 + w_5e_5^2 = \text{minimum}.$$

Differentiating and equating to 0, and cancelling the numerical factor of 2 from all terms, we get the " normal " equation

$$\Sigma we\delta e = w_1e_1\delta e_1 + w_2e_2\delta e_2 + w_3e_3\delta e_3 + w_4e_4\delta e_4 + w_5e_5\delta e_5 = 0 \quad . \quad (4)$$

Further, if e_1 is varied an amount δe_1, e_2 by an amount δe_2, etc., we must satisfy the conditions

$$\left.\begin{array}{l} \delta e_1 + \delta e_2 + \delta e_3 = 0 \\ \delta e_3 + \delta e_4 + \delta e_5 = 0 \end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

these equations being obtained by differentiation of equations (3). These equations (5) still hold true if we multiply throughout by constant coefficients known as correlates or correlatives. Thus, multiplying by λ_1 and λ_2 respectively, we have

$$\left.\begin{array}{l} \lambda_1\delta e_1 + \lambda_1\delta e_2 + \lambda_1\delta e_3 = 0 \\ \lambda_2\delta e_3 + \lambda_2\delta e_4 + \lambda_2\delta e_5 = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Combining equations (4) and (6), either by addition as here, or by subtraction, we get

$$(w_1e_1+\lambda_1)\delta e_1+(w_2e_2+\lambda_1)\delta e_2+(w_3e_3+\lambda_1+\lambda_2)\delta e_3 \\ +(w_4e_4+\lambda_2)\delta e_4+(w_5e_5+\lambda_2)\delta e_5=0 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The quantities δe_1, δe_2, etc. are not definitely related, so that to satisfy the equation, the coefficients of each must vanish separately, *i.e.*

$$\left.\begin{matrix} e_1=-\lambda_1/w_1 & e_2=-\lambda_1/w_2 & e_3=-(\lambda_1+\lambda_2)/w_3 \\ & e_4=-\lambda_2/w_4 & e_5=-\lambda_2/w_5 \end{matrix}\right\} \quad . \quad (8)$$

Substituting in equation (3), and changing the sign throughout,

$$\left.\begin{matrix} \lambda_1/w_1+\lambda_1/w_2+(\lambda_1+\lambda_2)/w_3=-d \\ (\lambda_1+\lambda_2)/w_3+\lambda_2/w_4+\lambda_2/w_5=-d_1 \end{matrix}\right\} . \quad . \quad . \quad . \quad (9)$$

From these two equations the coefficients λ_1 and λ_2 can be evaluated.

Then from equation (8) e_1, e_2, . . . can be calculated, and since $(a_1-\theta_1)=e_1$, etc., and knowing the values of a, we obtain the corrected values θ, *i.e.* $\theta_1=a_1-e_1$, etc., or, in other words, the correction to be applied is the same value as the error e but of opposite sign.

In the numerical example equations (2) become

$$\left.\begin{matrix} 84{\cdot}624-32{\cdot}840-51{\cdot}762= & 0{\cdot}022=d \\ -51{\cdot}762+62{\cdot}463-10{\cdot}714= & -0{\cdot}013=d_1 \end{matrix}\right\} .$$

By substitution in equation (9)

$$\lambda_1=-0{\cdot}0438 \qquad \lambda_2=+0{\cdot}0287$$

From equations (8)

$$\begin{matrix} e_1=0{\cdot}011 & e_2=0{\cdot}007 & e_3=0{\cdot}004 \\ e_4=-0{\cdot}007 & e_5=-0{\cdot}010 & \end{matrix}$$

Then as $a_1-\theta_1=e_1$, etc., $\theta_1=a_1-e_1$, *i.e.* the level differences are :

$$\begin{matrix} (1) & 84{\cdot}624-0{\cdot}011= & 84{\cdot}613 \\ (2) & -32{\cdot}840-0{\cdot}007= & -32{\cdot}847 \\ (3) & -51{\cdot}762-0{\cdot}004= & -51{\cdot}766 \\ (4) & 62{\cdot}463+0{\cdot}007= & 62{\cdot}470 \\ (5) & -10{\cdot}714+0{\cdot}010= & -10{\cdot}704 \end{matrix}$$

Hence, if the reduced level of 0 is 100·000, that of A = 184·613 ; B = 151·766 ; C = 162·470.

Had all the observations been of equal weight, we could write $w_1=w_2=w_3=w_4=w_5=1$, so that the weight symbols could have been omitted throughout.

In this case $e_1=0{\cdot}010$, $e_2=0{\cdot}010$, $e_3=0{\cdot}002$, $e_4=-0{\cdot}008$, $e_5=-0{\cdot}008$, and the reduced levels are A = 184·614 ; B = 151·764 and C = 162·471.

Example 2.—If in the previous example the levelling had consisted only of the circuit OABCO, *i.e.* if the line OB had been omitted, we would have one equation of condition, *i.e.*

$$\theta_1+\theta_2+\theta_5+\theta_4=0 \quad . \quad . \quad (1)$$

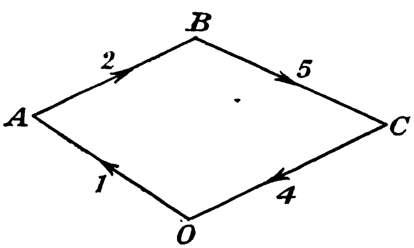


FIG. 19.3

(Or if the circuit had been between two points O and O_1 of known difference in altitude = H, then H would be substituted for 0 in equation (1).)

If the observed differences of level

$$a_1+a_2+a_5+a_4=d,$$

we have, writing $e_1=(a_1-\theta_1)$, etc., the errors

$$e_1+e_2+e_5+e_4=d. \quad . \quad . \quad (2)$$

The east square condition is that Σwe^2 shall be a minimum, *i.e.* $\Sigma we\delta e=0$ *i.e.*

$$w_1e_1\delta e_1+w_2e_2\delta e_2+w_5e_5\delta e_5+w_4e_4\delta e_4=0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Differentiating (2) and multiplying each term by the correlate λ, we have

$$\lambda\delta e_1+\lambda\delta e_2+\lambda\delta e_5+\lambda\delta e_4=0. \qquad (4)$$

Subtract (4) from (3),

$$(w_1e_1-\lambda)\delta e_1+(w_2e_2-\lambda)\delta e_2\ldots=0,$$

and since the coefficients of δe_1, δe_2, . . . must each equal 0,

$$e_1=\lambda/w_1 \quad e_2=\lambda/w_2 \quad e_5=\lambda/w_5 \quad \text{and} \quad e_4=\lambda/w_4,$$

substituting in equation (2)

$$\Sigma e=\lambda(1/w_1+1/w_2+1/w_5+1/w_4)=d. \qquad (5)$$

By substitution for w_1, w_2, w_5, and w_4, we get $\lambda=0{\cdot}035$, and hence

$$e_1=0{\cdot}009 \quad e_2=0{\cdot}006 \quad e_5=0{\cdot}012 \quad e_4=0{\cdot}009,$$

and

$$\begin{aligned}\text{OA}&=+84{\cdot}624-0{\cdot}009=+84{\cdot}615,\\ \text{AB}&=-32{\cdot}840-0{\cdot}006=-32{\cdot}846,\\ \text{BC}&=+10{\cdot}714-0{\cdot}012=+10{\cdot}702,\\ \text{CO}&=-62{\cdot}463-0{\cdot}009=-62{\cdot}472.\end{aligned}$$

In this case we should have deduced the answer more quickly by dividing the total error inversely as the reciprocals of the weights, *i.e.*

$$e_1=d/w_1\div(1/w_1+1/w_2+1/w_5+1/w_4), \text{ etc.},$$

or if the errors are proportional to $\sqrt{N}$ or $\sqrt{L}$ where L = length of line, then as $w \propto 1/L$, the total error is divided as $e_1=L_1d/\Sigma L$, $e_2=L_2d/\Sigma L$, etc. (cf. Adjustment of Traverses, pp. 140 *et seq.*).

The problem could have been solved by the method of least squares without the use of correlates, as follows: Let θ and α represent the true and observed heights of each station above the preceding one. The errors are then

$$\left.\begin{aligned}e_1&=\alpha_1-\theta_1\\ e_2&=\alpha_2-\theta_2\\ e_5&=\alpha_5-\theta_5\\ e_4&=\alpha_4-\theta_4\\ &=\alpha_4+\theta_1+\theta_2+\theta_5\end{aligned}\right\} \qquad (6)$$

But by the theory of least squares Σwe^2 is a minimum, *i.e.*

$$\Sigma we\delta e=0. \qquad (7)$$

$$\begin{aligned}&-\delta\theta_1[w_1(\alpha_1-\theta_1)-w_4(\alpha_4+\theta_1+\theta_2+\theta_5)]\\ &-\delta\theta_2[w_2(\alpha_2-\theta_2)-w_4(\alpha_4+\theta_1+\theta_2+\theta_5)]\\ &-\delta\theta_5[w_5(\alpha_5-\theta_5)-w_4(\alpha_4+\theta_1+\theta_2+\theta_5)]=0\end{aligned} \qquad (8)$$

But since $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_5$ are not entirely dependent upon one another, the coefficients of each must = 0, *i.e.*

$$\left.\begin{aligned}w_1(\alpha_1-\theta_1)-w_4(\alpha_4+\theta_1+\theta_2+\theta_5)=0\\ w_2(\alpha_2-\theta_2)-w_4(\alpha_4+\theta_1+\theta_2+\theta_5)=0\\ w_5(\alpha_5-\theta_5)-w_4(\alpha_4+\theta_1+\theta_2+\theta_5)=0\end{aligned}\right\} \qquad (9)$$

Substituting for w_1, w_2, w_4, w_5, α_1, α_2, α_4, α_5, we have

$$\begin{aligned}8\theta_1+4\theta_2+4\theta_5-588{\cdot}348&=0,\\ 4\theta_1+10\theta_2+4\theta_5-52{\cdot}812&=0,\\ 4\theta_1+4\theta_2+7\theta_5-281{\cdot}994&=0,\end{aligned}$$

from which $\theta_1=84{\cdot}615$, $\theta_2=-32{\cdot}846$, $\theta_5=10{\cdot}702$, $\theta_4=-62{\cdot}472$, agreeing with the previous result. The student is advised to apply this method to Example 1 and verify the results there obtained by the use of correlates.

Example 3.—A round of angles θ_1, θ_2, θ_3, θ_4 was observed at a station O, and the sum exceeded 360° by an amount E. The angles were measured n_1, n_2, n_3, n_4 times and averages taken, so that the weights are in the same ratio, *i.e.*

$$w_1=n_1 : w_2=n_2 : \text{etc.}$$

Let the true values of the angles be represented by θ, and the observed values by a. Then

$$\left.\begin{aligned} e_1 &= a_1 - \theta_1 \\ e_2 &= a_2 - \theta_2 \\ e_3 &= a_3 - \theta_3 \\ e_4 &= a_4 - \theta_4 \\ &= a_4 - 360^\circ + \theta_1 + \theta_2 + \theta_3 \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

$$\Sigma we^2 = \text{minimum} \quad \therefore \quad \Sigma w(a-\theta)^2 = \text{minimum}. \quad . \quad . \quad (2)$$

Differentiating with respect to θ_1, θ_2, θ_3 in turn, cancelling 2 throughout and equating to zero, we have

$$\left.\begin{aligned} -w_1(a_1-\theta_1) + w_4(a_4 - 360 + \theta_1 + \theta_2 + \theta_3) = 0 \\ -w_2(a_2-\theta_2) + w_4(a_4 - 360 + \theta_1 + \theta_2 + \theta_3) = 0 \\ -w_3(a_3-\theta_3) + w_4(a_4 - 360 + \theta_1 + \theta_2 + \theta_3) = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

From these three normal equations we can find θ_1, θ_2, θ_3, and hence θ_4. The fact that $\Sigma a = \Sigma\theta + \mathrm{E}$ is allowed for in equations (1). If equations (2) are differentiated twice, the results are positive and indicate that a *minimum* value is obtained for Σwe^2.

To solve, using the method of correlates, we have

$$\Sigma e = e_1 + e_2 + e_3 + e_4 = \mathrm{E} \quad . \quad . \quad . \quad . \quad (4)$$

$\Sigma we^2 = \text{minimum}$, *i.e.*

$$w_1e_1\delta e_1 + w_2e_2\delta e_2 + w_3e_3\delta e_3 + w_4e_4\delta e_4 = 0. \quad . \quad . \quad . \quad (5)$$

By differentiation of equation (4) and multiplying by the constant λ we have

$$\lambda\delta e_1 + \lambda\delta e_2 + \lambda\delta e_3 + \lambda\delta e_4 = 0, \quad . \quad . \quad . \quad . \quad (6)$$

combining equations (5) and (6) and equating to zero the coefficients of δe_1, δe_2, . . . , we have

$$e_1 = \lambda/w_1 \quad e_2 = \lambda/w_2, \quad \text{etc.},$$

and these are subtracted from the observed values a_1, a_2, . . . , to give the probable values for θ_1, θ_2, . . .

As an example, if the observed angles are $a_1 = 85^\circ\ 44'\ 30''$: $a_2 = 92^\circ\ 16'\ 44''$: $a_3 = 100^\circ\ 26'\ 22''$: $a_4 = 81^\circ\ 32'\ 22''$: $w_1 = 2$: $w_2 = w_3 = 3$: $w_4 = 4$: by substitution in equations (3) we find

$$\theta_1 = 85^\circ\text{-}44'\text{-}30''\cdot71 \qquad \theta_2 = 92^\circ\text{-}16'\text{-}41''\cdot47$$
$$\theta_3 = 100^\circ\text{-}26'\text{-}22''\cdot47 \qquad \theta_4 = 81^\circ\text{-}32'\text{-}22''\cdot35$$

Alternatively, if the method of correlates is used, we have $\Sigma a = 359^\circ\text{-}59'\text{-}58''$, so that the total error E is $-2''$.

From equations (6) and (4) $\lambda(1/w_1 + 1/w_2 \ldots) = -2''$. Hence $\lambda = -1\cdot41''$, and from (6)

$$e_1 = -0\cdot71'' \qquad e_2 = -0\cdot47'' \qquad e_3 = -0\cdot47'' \qquad e_4 = -0\cdot35''.$$

The corrections have the opposite signs to the errors and hence θ_1, θ_2, θ_3, θ_4 have the same values as those earlier derived from equation (3).

Example 4.—The angles of a quadrilateral with diagonals may be adjusted, using the method of correlates, as follows: Let the angles be denoted by the numbers 1–8 as shown in Fig. 19.4, left-hand angles by odd and right-hand angles by even numbers. Let the true values of the angles be θ_1, θ_2, . . . , and the observed values a_1, a_2, . . . Then the errors are $(a_1 - \theta_1) = e_1$, $(a_2 - \theta_2) = e_2$, etc.

The equations of condition may be written $\theta_1 + \theta_2 \ldots + \theta_8 = 360^\circ + e_s$, (1) where e_s is the spherical excess, if this is significant.

$$\theta_1 + \theta_2 = \theta_5 + \theta_6, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$\theta_3 + \theta_4 = \theta_7 + \theta_8, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$
$$\Sigma \log \sin \theta_L = \Sigma \log \sin \theta_R. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Other equations could have been written down, *e.g.*

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = \theta_5 + \theta_6 + \theta_7 + \theta_8,$$

but this is not independent of those already used—it may be obtained from (2) and (3). (For equation (4) see p. 403.) Assume, to make the solution more general, that the observations have weights w_1, w_2, w_3, etc. If the observations are of equal weight, the w symbols can be omitted as $w_1 = w_2 = \ldots = w_8 = 1$. Then by the theory of least squares, the weighted sum of the squares of the errors must be a minimum, *i.e.*

$$\Sigma\, we^2 = \text{minimum}. \qquad (5)$$

If the sum of the observed angles a_1, a_2, . . . , a_8 is equal to $360° + es_5 + E_o$ say, we get from equation (1)

$$e_1 + e_2 + e_3 \ldots + e_8 = E_o. \qquad (6)$$

Similarly, from equations (2) and (3) we get

$$e_1 + e_2 - e_5 - e_6 = E_1, \qquad (7)$$

$$e_3 + e_4 - e_7 - e_8 = E_2. \qquad (8)$$

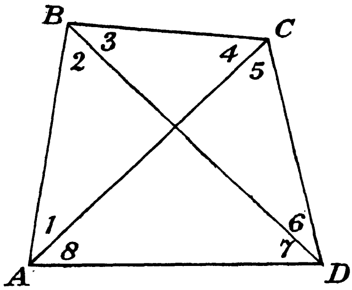


FIG. 19.4

Referring to equation (4), we find, on substitution n the observational equations that $\Sigma \log \sin a_L - \Sigma \log \sin a_R = E'_3$. But if the error in the angle (1) is $(a_1 - \theta_1) = e_1$, and e_1 is small, then the error in $\log \sin a_1 = v'_1 \times e_1$, where e_1 is expressed in seconds and v'_1 is the logarithm for 1 second difference in angle at the value a_1.

For convenience, instead of using v'_1 and E'_3 we can use $v_1 = 10^7 . v'_1$ and $E_3 = 10^7 . E_3{}^1$, *i.e.* v_1 and E_3 are the last figures in seven-figure logarithms. Hence

$$\left.\begin{array}{l} v_1e_1 + v_3e_3 + v_5e_5 + v_7e_7 \\ - v_2e_2 - v_4e_4 - v_6e_6 - v_8e_8 = E_3 \end{array}\right\} \qquad (9)$$

the factor 10^7 cancelling throughout.

From these equations (5)–(9) the normal equations can be formed by differentiation with respect to e_1, e_2, . . . etc.

From equation (5), cancelling the numerical factor 2 throughout, we get

$$w_1e_1\delta e_1 + w_2e_2\delta e_2 \ldots + w_8e_8\delta e_8 = 0. \qquad (5a)$$

From equation (6) we get

$$\delta e_1 + \delta e_2 + \delta e_3 \ldots + \delta e_8 = 0.$$

Hence

$$\lambda_1\delta e_1 + \lambda_1\delta e_2 + \lambda_1\delta e_3 \ldots + \lambda_1\delta e_8 = 0, \qquad (6a)$$

and similarly from equations (7), (8), and (9) we get

$$\lambda_2\delta e_1 + \lambda_2\delta e_2 - \lambda_2\delta e_5 - \lambda_2\delta e_6 = 0 \qquad (7a)$$

$$\lambda_3\delta e_3 + \lambda_3\delta e_4 - \lambda_3\delta e_7 - \lambda_3\delta e_8 = 0, \qquad (8a)$$

$$\lambda_4 v_1\delta e_1 - \lambda_4 v_2\delta e_2 + \lambda_4 v_3\delta e_3 \ldots - \lambda_4 v_8\delta e_8 = 0. \qquad (9a)$$

Now considering the five normal equations (5*a*)–(9*a*), on addition the coefficients of each term must vanish independently, *i.e.* the coefficients of δe_1

$$= w_1e_1 + \lambda_1 + \lambda_2 + \lambda_4 v_1 = 0$$

and similarly for the other terms

$$\left.\begin{array}{l} w_2e_2 + \lambda_1 + \lambda_2 - \lambda_4 v_2 = 0 \\ w_3e_3 + \lambda_1 + \lambda_3 + \lambda_4 v_3 = 0 \\ w_8e_8 + \lambda_1 - \lambda_3 - \lambda_4 v_8 = 0 \end{array}\right\} \qquad (10)$$

We can now form four equations, and proceed to determine the four correlates λ_1, λ_2, λ_3, and λ_4. This is done by substituting for e_1, e_2, . . . , e_8 from equations (10) in equations (6), (7), (8), and (9), and solving for λ_1, λ_2, λ_3, λ_4.

Having determined these quantities, we can find e_1, e_2, . . . , e_8 by substitution in equations (10) and the adjusted values of the angles θ_1, θ_2, . . . can finally be obtained from the equations $\theta_1 = a_1 - e_1$, $\theta_2 = a_2 - e_2$, etc. An example is given on p. 405, which it is recommended should be worked through by the

student. There, as will be seen, different equations of condition have been selected, such as $\theta_1+\theta_2+\theta_3+\theta_8=180°+$ spherical excess, the latter amounting to 03·87″. The sum of the observed angles, *i.e.* $a_1+a_2+a_3+a_8=180°\text{-}00'\text{-}03{\cdot}84''$, so that the error is $-0{\cdot}03''$ and the required correction $+0{\cdot}03''$.

The symbols x_1, x_2, etc. represent corrections, *i.e.* e_1, e_2, etc., with the signs changed. The numerical coefficients in the fourth equation represent v, the last figures in seven-place logarithms for 1″ differences in the various angles, *e.g.*

$$\begin{aligned}\log\sin 62°\text{-}32'\text{-}20'' &= 9{\cdot}9480822\\ \log\sin 62°\text{-}32'\text{-}10'' &= 9{\cdot}9480713\\ \hline & \quad 0{\cdot}0000109\end{aligned}$$

i.e. difference for $1''=0{\cdot}0000010{\cdot}9$, *i.e.* $v_1=10{\cdot}9$.

Similarly the figure 9·00 represents

$$\log\sin a_L-\log\sin a_R=\cdot 0000009,\ \textit{i.e.}\ E_3=9{\cdot}00.$$

As a check on the working $\Sigma v^2=5688$, and the correlates

$$\lambda_1=-0{\cdot}0702 \qquad \lambda_2=-0{\cdot}4203 \qquad \lambda_3=0{\cdot}1404 \qquad \lambda_4=-0{\cdot}0024.$$

Example 5.—The angles of a polygon with a central point may be adjusted as follows: The equations of condition, assuming the spherical excess can be ignored, are:

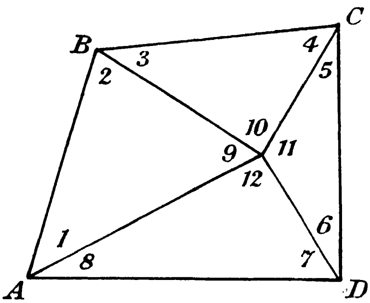


FIG. 19.5

$$\theta_1+\theta_2+\theta_9=180, \quad (1)$$
$$\theta_3+\theta_4+\theta_{10}=180, \quad (2)$$
$$\theta_5+\theta_6+\theta_{11}=180, \quad (3)$$
$$\theta_7+\theta_8+\theta_{12}=180, \quad (4)$$
$$\theta_9+\theta_{10}+\theta_{11}+\theta_{12}=360°, \quad (5)$$
$$\Sigma\log\sin\theta_L=\Sigma\log\sin\theta_R. \quad (6)$$

We could write also $\theta_1+\theta_2\ \ldots\ +\theta_8=360°$, but this is not an independent equation, since it can be derived from those already stated.

As in the previous example, but assuming all angles to have equal weights, we have

$$\Sigma e^2=\text{minimum},\ \textit{i.e.}\ \Sigma e(\delta e)=0 \quad (7)$$

Then if
$$a_1+a_2+a_9=180°+E_1$$
$$=\theta_1+\theta_2+\theta_9+E_1,$$
$$(a_1-\theta_1)+(a_2-\theta_2)+(a_9-\theta_9)=E_1,$$
$$e_1+e_2+e_9=E_1. \quad (8)$$

Similarly
$$e_3+e_4+e_{10}=E_2, \quad (9)$$
$$e_5+e_6+e_{11}=E_3, \quad (10)$$
$$e_7+e_8+e_{12}=E_4, \quad (11)$$
$$e_9+e_{10}+e_{11}+e_{12}=E_5. \quad (12)$$
$$\Sigma\log\sin a_L-\Sigma\log\sin a_R=E_6,$$

i.e.
$$\left.\begin{matrix}v_1e_1+v_3e_3+v_5e_5+v_7e_7\\ -v_2e_2-v_4e_4-v_6e_6-v_8e_8\end{matrix}\right\}=E_6. \quad (13)$$

From equations (7) we get by differentiation,

$$e_1\delta e_1+e_2\delta e_2+e_3\delta_3\ \ldots\ +e_{12}\delta e_{12}=0. \quad (7a)$$

From equations (8) we get

$$\delta e_1+\delta e_2+\delta e_9=0.$$

And thence, multiplying throughout by λ_1

$$\lambda_1\delta e_1+\lambda_1\delta e_2+\lambda_1\delta e_9=0 \quad (8a)$$

Similarly from equations 9–13

$$\lambda_2\delta e_3+\lambda_2\delta e_4+\lambda_2\delta e_{10}=0, \qquad (9a)$$

$$\lambda_3\delta e_5+\lambda_3\delta e_6+\lambda_3\delta e_{11}=0, \qquad (10a)$$

$$\lambda_4\delta e_7+\lambda_4\delta e_8+\lambda_4\delta e_{12}=0, \qquad (11a)$$

$$\lambda_5\delta e_9+\lambda_5\delta e_{10}+\lambda_5\delta e_{11}+\lambda_5\delta e_{12}=0, \qquad (12a)$$

$$\left.\begin{array}{l}\lambda_6 v_1\delta e_1-\lambda_6 v_2\delta e_2+\lambda_6 v_3\delta e_3-\lambda_6 v_4\delta e_4\\ +\lambda_6 v_5\delta e_5-\lambda_6 v_6\delta e_6+\lambda_6 v_7\delta e_7-\lambda_6 v_8\delta e_8\end{array}\right\}=0. \qquad (13a)$$

When we combine the 7 equations (7a)–(13a) the coefficients of δe_1, δe_2, . . δe_{12} must each equal 0, *i.e.*

$$\begin{array}{ll}\text{for } \delta e_1, & e_1+\lambda_1+\lambda_6 v_1=0,\\ \text{for } \delta e_2, & e_2+\lambda_1-\lambda_6 v_2=0,\\ & \text{etc.;}\end{array}$$

hence

$$\left.\begin{array}{l}e_1=-\lambda_1-\lambda_6 v_1\\ e_2=-\lambda_1+\lambda_6 v_2, \text{ etc.}\end{array}\right\}. \qquad (14)$$

We can now substitute these values for e_1, e_2, etc. in equations (8)–(13), and so obtain six simultaneous equations from which the six unknown correlates $\lambda_1, \lambda_2, \ldots, \lambda_6$ can be evaluated. Equation (8), for instance, becomes

$$-\lambda_1-\lambda_6 v_1-\lambda_1+\lambda_6 v_2-\lambda_1-\lambda_5=\mathrm{E}_1$$

or

$$3\lambda_1+\lambda_5+\lambda_6(v_1-v_2)=-\mathrm{E}_1 . \qquad (15)$$

Then knowing $\lambda_1, \lambda_2, \ldots, \lambda_6$, from equations (14) $e_1, e_2, \ldots, e_{12}$ can be readily obtained.

These are errors, and the corrections will have the opposite signs. The corrected angles are $\theta_1=\alpha_1-e_1 : \theta_2=\alpha_2-e_2$, etc.

Many surveyors replace errors by corrections throughout, as in the examples referred to on p. 405.

Here $x=-e$ and equation (8) would become

$$x_1+x_2+x_9=-\mathrm{E}_1 \text{ etc.} \qquad (8b)$$

The student is recommended to work through the Example 2 on p. 405, using the data there given, but adopting the method described above. For this purpose it may be assumed that the observed value of angle 12 (*i.e.* N.) was 135°-09′-33″·73; that the spherical excess in the triangles SNE, ENK, KNL, LNS was 0″·42, 0″·17, 0″·26, 0″·70 respectively. Then the fourth equation of condition will be replaced by two equations

$$x_4+x_5+x_{12}-0{\cdot}54=0,$$
$$x_9+x_{10}+x_{11}+x_{12}-0{\cdot}43=0.$$

As a check on the work

$$\begin{array}{lll}\lambda_1=-0{\cdot}0396 & \lambda_2=+0{\cdot}2041 & \lambda_3=-0{\cdot}4495\\ \lambda_4=+0{\cdot}0468 & \lambda_5=-0{\cdot}0479 & \lambda_6=+0{\cdot}01152\end{array}$$

Example 6.—It is frequently desired to calculate the most probable values of quantities such as $a, b, c, \ldots$ where various combinations of these quantities have been observed. Thus a, b, c, may be parts of a base line of total length $a+b+c$; or a, b, c, may be adjoining angles, together making up an angle $a+b+c$ at a trigonometrical station, etc.

Suppose the observed values are

$$\begin{array}{rll}a=&20^\circ-10'-14'', & \text{weight } 2\\ b=&30^\circ-15'-20'', & \quad,, \quad 2\\ c=&42^\circ-02'-16'', & \quad,, \quad 2\\ a+b=&50^\circ-25'-37'', & \quad,, \quad 3\\ b+c=&72^\circ-17'-34'', & \quad,, \quad 3\\ a+b+c=&92^\circ-27'-52'', & \quad,, \quad 1\end{array}$$

Multiply each line by the weight of the observation, and arrange as follows:

$$\begin{aligned}
2a \qquad\qquad &= 40°\text{-}20'\text{-}28'' \\
2b \qquad &= 60°\text{-}30'\text{-}40'' \\
2c &= 84°\text{-}04'\text{-}32'' \\
3a + 3b \qquad &= 151°\text{-}16'\text{-}51'' \\
3b + 3c &= 216°\text{-}52'\text{-}42'' \\
a + b + c &= 92°\text{-}27'\text{-}52''
\end{aligned}$$

Sum all the items containing a (*i.e.* lines 1, 4 and 6); then those containing b (*i.e.* lines 2, 4, 5 and 6); then those containing c (*i.e.* lines 3, 5 and 6).

$$\begin{aligned}
6a + 4b + c &= 284°\text{-}05'\text{-}11'' \\
4a + 9b + 4c &= 521°\text{-}08'\text{-}05'' \\
a + 4b + 6c &= 393°\text{-}25'\text{-}06''
\end{aligned}$$

These are the normal equations in a, b and c respectively, and the symmetry of the coefficients on the left may be noted.

Solving these three simultaneous equations, the most probable values of a, b and c are found to be:

$$\begin{aligned}
a &= 20°\text{-}10'\text{-}15{\cdot}9'' \\
b &= 30°\text{-}15'\text{-}20{\cdot}2'' \\
c &= 42°\text{-}02'\text{-}14{\cdot}9''
\end{aligned}$$

APPENDIX I

TRIGONOMETRICAL FORMULAE

FORMULAE IN PLANE TRIGONOMETRY

$$s=\frac{a+b+c}{2}.$$

$$\mathrm{S}=\text{area}.$$

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c} \qquad (1)$$

$$a^2=b^2+c^2-2bc\cos A \qquad (2)$$

$$\sin^2 A+\cos^2 A=1 \qquad (3)$$

$$\sin(A\pm B)=\sin A\cos B\pm\sin B\cos A \qquad (4)$$

$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B \qquad (5)$$

$$\sin 2A=2\sin A\cos B \qquad (6)$$

$$\cos 2A=\cos^2 A-\sin^2 A \qquad (7)$$

$$=1-2\sin^2 A \qquad (8)$$

$$=2\cos^2 A-1 \qquad (9)$$

$$\sin A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \qquad (10)$$

$$\sin A-\sin B=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \qquad (11)$$

$$\cos A+\cos B=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \qquad (12)$$

$$\cos A-\cos B=-2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \qquad (13)$$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B} \qquad (14)$$

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{a-b}{a+b} \qquad (15)$$

$$\tan\tfrac{1}{2}A=\frac{\sin A}{1+\cos A}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \qquad (16)$$

$$\cot\tfrac{1}{2}A=\frac{\sin A}{1-\cos A} \qquad (17)$$

$$\sin\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}} \qquad (18)$$

$$\cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}} \qquad (19)$$

$$\mathrm{S}=\sqrt{s(s-a)(s-b)(s-c)} \qquad (20)$$

$$=\tfrac{1}{2}b.c.\sin A \qquad (21)$$

$$S = \frac{a^2 \sin B \sin C}{2.\sin A} \quad . \quad . \quad . \quad . \quad (22)$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \quad . \quad . \quad (23)$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \quad . \quad . \quad (24)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad . \quad . \quad . \quad (25)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad . \quad . \quad (26)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad . \quad . \quad (27)$$

$$\sin 1'' = 0{\cdot}000004848 \quad . \quad . \quad . \quad (28)$$

$$\log \sin 1'' = 6{\cdot}6855748 \quad . \quad . \quad . \quad (29)$$

Formulae in Spherical Trigonometry

$$s = \frac{a+b+c}{2}.$$

$$S = \text{area}.$$

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \quad . \quad . \quad . \quad (1)$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad . \quad . \quad (2)$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \quad . \quad (3)$$

$$\cot A \sin B = \cot a \sin c - \cos B \cos c, \quad . \quad . \quad (4)$$

$$\text{or } \cos B = \cot a \tan c \text{ when } A = 90^\circ \quad . \quad . \quad (4a)$$

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c \quad . \quad . \quad (5)$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c \quad . \quad . \quad (6)$$

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}C \quad . \quad . \quad (7)$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2}C \quad . \quad . \quad (8)$$

$$\tan \tfrac{1}{2}A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s . \sin (s-a)}} \quad . \quad . \quad (9)$$

$$\sin \tfrac{1}{2}A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b . \sin c}} \quad . \quad . \quad (10)$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b . \sin c}} \quad . \quad . \quad (11)$$

Ten formulae for right-angled triangles may be obtained from two rules, known as Napier's Rules of Circular Parts. Thus, if the angle C in the spherical triangle ABC is 90°, and if five other parts taken in order round the triangle are arranged as in the diagram, then

The sine of any one part, known as the middle part, is equal to

(1) the product of the tangents of two adjacent parts, or

(2) the product of the cosines of the two opposite parts, *e.g.*

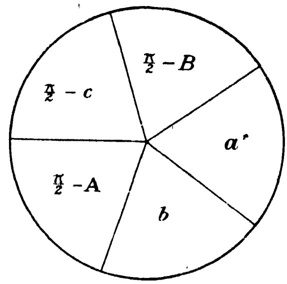


FIG. A.1

$$\sin a = \tan\left(\frac{\pi}{2} - B\right) \tan b = \cot B . \tan b,$$

and $$\sin a = \cos\left(\frac{\pi}{2} - A\right) \cos\left(\frac{\pi}{2} - c\right) = \sin A \sin c,$$

etc.

APPENDIX II

THE STAR ALMANAC FOR LAND SURVEYORS

FORMERLY the *Nautical Almanac* contained all the astronomical data which the Surveyor needed to enable him to calculate azimuth, latitude, and time from his field observations.

Now, however, various modifications have been made, and continue to be made, in the arrangement of the *Nautical Almanac* itself ; and several new publications have been introduced.

Many of the data on the fixed stars, for instance, are now omitted from the *Nautical Almanac* and are given in *Apparent Places of Fundamental Stars*. This volume gives the mean and apparent places of 1535 stars, and is intended for astronomers requiring maximum accuracy. The *Nautical Almanac* gives the mean places of a selection of 1078 of these stars, for the beginning of the year only, and to 0·1 s. and 1″. This is adequate for many purposes, but for geodetic and similar work reference must be made to *Apparent Places of Fundamental Stars*.

Much constant, or nearly constant, data is likewise being omitted from the *Nautical Almanac*, and is to be published as a separate *Supplement*, though it is not yet available (1958).

The *Abridged Nautical Almanac*, referred to in Chapter XVII, and intended mainly for the use of seamen, has been redesigned, and the tables rearranged.

The *Star Almanac for Land Surveyors* is a new publication, first issued in 1951. To quote from the Introduction, this " has been expressly designed to meet the requirements of the surveyor, with the object of providing him, in a compact volume, with astronomical data to the precision he really requires, and in the form which he prefers ".

Both in the *Nautical Almanac* and in the *Star Almanac*, the expression Universal Time (U.T.) has now been adopted as an alternative to, or in place of, Greenwich Mean Time (G.M.T.) (see p. 457).

In the *Nautical Almanac*, Apparent Declination is given to 0″·1, and the Equation of Time to 0·01 s., as is Right Ascension. As explained on p. 458, the Equation of Time, there denoted by E, is the quantity to be added to mean solar time to give apparent solar time, *i.e.*

$$\text{G.A.T.} = \text{G.M.T.} + \text{E}.$$

In the *Star Almanac*, the quantities tabulated for the Sun, for every 6 h. of U.T., are R, Declination, and E. The values for R and E are given to 0·1 s. and those for Declination to 0′·1. Here the italic E is used to denote the Equation of Time + 12 h. Hence, because the hour angle of the Sun is measured from upper transit, and solar time from lower transit, G.H.A. Sun = G.A.T. + 12 h. (deducting 24 h. if necessary), and

$$\text{G.H.A. Sun} = \text{U.T.} + E.$$

Similarly,

$$\text{G.H.A. Aries} = \text{U.T.} + R$$

where G.H.A. Aries signifies the hour angle of the First Point of Aries from the Greenwich meridian, *i.e.* G.S.T.

From this equation, various other equations may be readily derived, as on p. 458, *e.g.*

$$\text{L.M.T.} = \text{L.S.T.} - R.$$

The symbol R was first used, in 1929, in the *Abridged Almanac.*

The semi-diameter of the sun is given to 0′·1 and is an average value for the half month concerned. (In the *Nautical Almanac*, it is given to 0″·01—*cf.* example on p. 480.)

An innovation is the use of " critical tables " for certain interpolations, *e.g.* for the interpolation of R.

As an example, if we require to find R at, say, 9 h. 40 m. 28·2 s. U.T. on 3rd June 1954, we find from the main table that at 6 h. on this date, R = 16 h. 44 m. 38·6 s., and at 12 h., R = 16 h. 45 m. 37·7 s. We need, therefore to interpolate between these values for an interval of 3 h. 40 m. 28·2 s. after 6 h. U.T.

This can be done in the usual manner, to give a correction amounting to 36·2 s. to be added to 16 h. 44 m. 38·6 s., giving the required value of R = 16 h. 45 m. 14·8 s.

The interpolation can be done much more quickly by reference to the special table. This shows that the correction, for all intervals between the " critical " values of 3 h. 40 m. 03·5 s. and 3 h. 40 m. 40·0 s., is 36·2 s., and hence it applies to our interval of 3 h. 40 m. 28·2 s.

In the same way, the correction would have been 36·1 s. for all intervals between the critical values of 3 h. 39 m. 26·9 s. and 3 h. 40 m. 03·5 s.; and it would have been 36·3 s. for intervals between 3 h. 40 m. 40·0 s. and 3 h. 41 m. 16·5 s.

Tables are given in the *Star Almanac*, also, for 650 stars: right ascensions to 0·1 s. and declinations to 1″. The tabulated values are those for the beginning of each month, and interpolation tables enable the quantities for any specific date to be rapidly determined.

In addition, the apparent positions of three northern and two southern circumpolar stars are given in another table—to the nearest 1 s. in R.A. and the nearest 1″ in declination.

Pole Star Tables are given in the *Star Almanac* and in the *Nautical Almanac*, and, in a different form, in the *Abridged Nautical Almanac.*

On p. 491, special formulae are derived for the determination of latitude from extra-meridian observations on *Polaris*. The form of these equations has been changed slightly, so that using the same notation, the formula for latitude becomes:

$$l = a - p \cos \mathrm{P} + \tfrac{1}{2}p \sin p \sin^2 \mathrm{P} \tan l$$

(the symbol h is used in the *Star Almanac* instead of P).

This equation is transformed into a more convenient form for tabulation

$$\text{latitude} = \text{corrected observed altitude} + a_0 + a_1 + a_2$$

and the values of the three quantities a_0, a_1, and a_2 are tabulated.

In the same tables, quantities b_0, b_1, and b_2 are also tabulated, so that

azimuth (Z) may be determined from the equation

$$\text{azimuth} = (b_0 + b_1 + b_2) \sec (\text{latitude}).$$

The six quantities are given to a precision of 0·1′.

Refraction coefficients are given in the form

$$r = f \times r_0,$$

where r_0 is the mean refraction for the apparent altitude, and f is the correcting factor to allow for temperature and pressure differences from the normal. Both quantities r_0 and f are tabulated, the former to the nearest 1″. Parallax corrections are incorporated.

The methods described in Chapters XVI and XVII are, of course, unchanged, but if the degree of accuracy there indicated is required, the data for the stars (but not for the sun) must be obtained from the *Apparent Places of the Fundamental Stars* instead of from the *Nautical Almanac.*

If a lesser degree of accuracy is sufficient, and for most surveying purposes it is, then the data can be obtained from the *Star Almanac.* Much of it is no longer given in the *Nautical Almanac*, though data for the sun and special Pole Star Tables are included in both the *Nautical Almanac* and the *Star Almanac* to the accuracies mentioned above.

BAROMETRIC SURVEYING

Owing to the increased importance of aneroid determinations of height, particularly in connection with aviation, a Government Interdepartmental Committee was appointed to study the problems. This Committee recommended the formulae given below, and published tables to facilitate the computations. (*Aneroid Tables based on a Standard Atmosphere and a Standard Lapse Rate.* H.M.S.O., 1935.)

It is assumed that the air temperature decreases at a constant rate of 3·566° F. per 1000 feet increase in height, *i.e.* that the " lapse rate " has this constant value.

To derive the recommended formula, changing the notation of p. 235, let p, ρ, and v, respectively, represent the pressure per unit area, the density per unit volume, and the volume per unit weight of dry air at a height of h feet, while p_e, ρ_e, and v_e represent the corresponding quantities at sea-level.

T, the absolute temperature at a height h is equal to $T_e - k \,.\, h$ where T_e is the absolute temperature at sea-level and k is the lapse rate for dry air.

Then we know $\qquad pv/T = \text{constant} = c$, say,

$$\text{also } v = 1/\rho$$

$$\therefore \frac{p}{\rho(T_e - kh)} = \frac{p_e}{\rho_e T_e} = c$$

$$\therefore p = c\rho\,(T_e - kh)$$

We also know that $dp = -g\rho dh$, the sign being negative since p decreases as h increases,

$$\therefore p = \frac{-c\,dp}{g\,dh}\,(T_e - kh)$$

$$\text{or } \frac{dp}{p} = \frac{-g}{c} \frac{dh}{(T_e - kh)}$$

$$\therefore \log p = \frac{g}{ck} \log (T_e - kh) + \log A$$

$$i.e.\ p = A\ (T_e - kh)^{g/ck}$$

$$\text{But } p = p_e \text{ when } h = 0$$

$$i.e.\ p_e = A\ T_e^{g/ck}$$

$$\therefore p = p_e \left(\frac{T_e - kh}{T_e}\right)^{g/ck}$$

For standard conditions, this reduces to

$$p = p_e[(288 - 0{\cdot}00198\ h)/288]^{5{\cdot}256}$$

where p and p_e are in inches of mercury and h is in feet.

It is recommended that aneroid barometers should be graduated to agree with this formula, and tables are provided for determining h and for applying temperature and humidity corrections for non-standard conditions.

The temperature correction is

$$h_c - h = 0{\cdot}001929\ h\ (t - t_u)$$

where h_c is the required corrected value of the height in feet and h that deduced for the standard temperature (59° F.), t is the observed temperature in °F., and t_u the tabulated temperature for the height h, *i.e.* 59° − 0·003566 h.

ORDNANCE SURVEY

In Chapter XIII, a short account was given of the original Ordnance Survey, and the new maps are referred to in Chapter I. A few further particulars are given here, and the reader is referred to the Ordnance Survey publication, *Constants, Formulae, and Methods used in Transverse Mercator Projection* (H.M.S.O., 1s. net), and Major M. Hotine's articles on "The Re-triangulation of Great Britain" in the *Empire Survey Review*, vol. iv, nos. 25, 26, and 27.

In preparing maps on a plane surface, for small areas rectangular co-ordinates suffice, but for large areas some compromise has to be made. There are many different methods of "projection", each with its own advantages and disadvantages. The original maps used Cassini co-ordinates, the country being divided into counties, or groups of counties, each with its own meridian. Unfortunately, at the boundaries of the county divisions, there were apt to be discrepancies because of the distortions inherent in the projection employed.

In the new maps, it was decided to employ a different projection, a modified Transverse Mercator System, and to employ one set of co-ordinates for the whole country. It is beyond the scope of this book to describe this projection in detail, but a simple description of it is given in the small book *A Key to Maps*, by H. S. L. Winterbotham, or fuller particulars in larger treatises (see p. 406).

However, its great advantage is that, though the scale of the map varies slightly from point to point, at any one point it is the same in all

directions : the shapes of small areas are the same on the map as on the ground, and directions are the same on both also.

In attempting to recompute the old secondary triangulation observations for the new system of co-ordinates, difficulties arose because many records had been destroyed by enemy action, many of the field stations were no longer to be found, and lines formerly clear between station points were no longer so.

It was decided, therefore, to undertake a new secondary triangulation survey, and this made it worth while to do a new primary triangulation also, since many of the original triangulation points were no longer available. Primary work was commenced in 1936, and secondary work in 1939, but both were interrupted during the war.

The sides of the new primary triangles are 40-60 km. in length, except in the flat eastern counties where they are nearer 20 km. Geodetic Tavistock theodolites were used, and, as a rule, primary rays were observed by double-face directions on each of sixteen different zeros, the observations being done at night sighting to electric beacon lights. The average range of the sixteen directions to a ray was 3″, and the average triangular misclosure 1·2″.

The same instruments were used for the Secondary work, eight zeros being taken, and on Tertiary work, four. The observations were made by day, to lamps, but sometimes in tertiary work opaque signals were used. The secondary points were about 12 km. apart, tertiary points 7 km. Some fourth-order points were also observed as required, 1 to 2 km. apart, using a $3\frac{1}{2}''$ Tavistock theodolite.

The old primary triangulation is referred to on p. 392. It formed a very strong network, which provided 920 condition equations for the fixation of 218 points. The average triangulation error was 2·8″, and the agreements between the Salisbury Plain, Lough Foyle, and Lossiemouth bases are quoted on p. 406.

To take advantage of this work, a number of stations are common to both the new and the old primary triangulation networks. The new network is quite self-contained, but the scale and the orientation were chosen so that the best fit was obtained for eleven common points.

The distance between two old points, and the direction of the line joining them, were first of all assumed correct. On these assumptions, the co-ordinates of nine other common points were calculated. Then, from the differences between the old and new co-ordinates, by means of least square calculations, scale and azimuth adjustments to the assumed values were computed to give the best fit for the eleven points. These adjustments amounted to only 0·00001537 in scale and 0·5084″ in azimuth.

MISCELLANEOUS EXAMPLES

1. (U. of L.) Referring to the diagram Fig. M.E.1., A, B, C, D are four points in a town (fixed on the ground and all invisible from one another) through which the centre line of a storm-water sewer (to be laid by tunnelling) is to pass in plan.

The sewer is to run straight from A to B and from D to C, and these straight portions are to be connected by a curve EF as shown.

It is desired to sink a shaft at one end of the curve, and it is found, by means of a rough plan, that a radius of 600 ft. will bring the end E on to some vacant

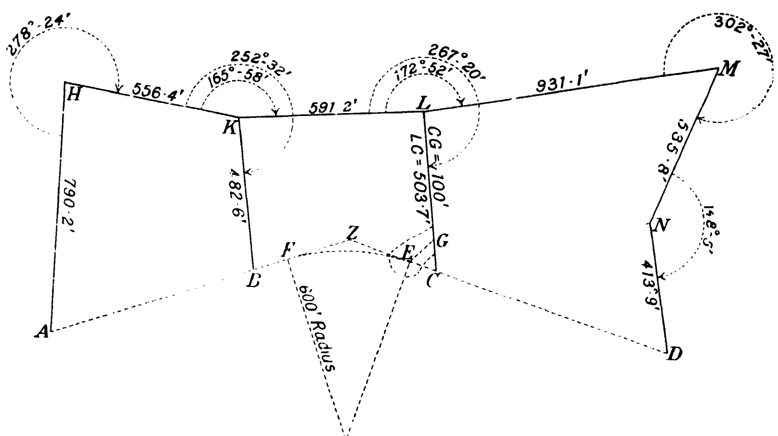


FIG. M.E.1

ground, and that E will be invisible from C but visible from a point G 100 ft. from C along CL.

A traverse is run along the roads, and the angles and distances shown are obtained.

Taking A as the origin of co-ordinates, and the bearing of AH as zero (and working to one decimal of a foot and to the nearest minute only), you are required to calculate :

(*a*) The co-ordinates of all points except G, E, F, and Z.

(*b*) The bearings of AB and DC.

(*c*) The angles HAB, NDC, and CZB (required for setting out the direction of AB and DC), and the distances AB, DC.

(*d*) The bearing and length of BC.

(*e*) The angles ZBC and ZCB, and the distances BZ and CZ.

(*f*) The tangent distance ZF or ZE, and the distances BF and CE.

(*g*) The co-ordinates of G and E and the bearing of GE.

(*h*) The angle CGE and the distance GE (required for fixing the position of E).

(*i*) The table of tangential angles, and the total length of the curve required for setting out the curve in 50 ft. chords from E. The angles are to be set off from EZ as zero.

(*j*) The depth of the shaft at E down to invert level, given that the reduced

level of the ground at E is 57·4 ft., the invert levels at D and A are to be 32·1 and 27·6 respectively, and the gradient is to be uniform along the centre line.

(*N.B.*—If you prefer to work otherwise than by co-ordinates, you may omit Nos. (*a*), (*b*), (*d*), and (*g*).

In (*j*), if you have been unable to find the whole length of sewer, you may assume a length.)

2. (I.C.E.) In order to ascertain the direction of the underground flow of water, borings were put down at three points A, B, and C, B being true north of A, and C lying westward of A and B.

The distance from A to B was 3000 ft., B to C 4000 ft., and A to C 5000 ft. The observed underground water-levels above datum were at A 130 ft., at B 170 ft., at C 10 ft.

What is the direction of underground flow ?

3. (U. of L.) A line AB bears N. 25° E., and a line AC bears N. 115° W. The surface of a plane stratum of rock falls from A to B 1 vertical to 15 horizontal, and rises from A to C 1 vertical to 25 horizontal. Find the reduced level of a stratum of rock at a point P 3000 ft. north of A, the reduced level of A being 120 ft. A graphical method may be used, provided great accuracy is observed.

4. (U. of L.) When setting out the centre line for a tunnel between the two ends A and B, an observatory station C is chosen on the top of the hill from which both A and B are visible, but it is not on the centre line of the tunnel Let D be a point on a vertical through C.

The horizontal projection of the angle ACB is 45°-58′, the vertical angle ACD is 49°-45′, and the vertical angle BCD is 57°-42′. The horizontal projection of CA is 750 yds., and of CB 800 yds.

Find the horizontal distance between A and B, and the difference in level.

5. (U. of L.) It is required to determine the exact distance between two inaccessible stations C and D located on the north side of a hill by observations taken from a base-line AB on the south side of the hill ; from a station A on this base-line the station C can be seen, but not the station D ; and from a station B on this base-line the station D can be seen, but not C. There is no station on the base-line from which both C and D can be seen. State precisely how you would carry out this work in the field, what observations you would make, and what calculations would be required.

6. (U. of L.) The slope of a certain piece of ground (which may be regarded as a plane surface) is 1 in 4. On the surface of this ground a line AB, 375 ft. long, is laid out at a gradient of 1 in 9.

Find the slope of the ground in a direction at right angles to AB as seen in plan.

If AB be the centre line of the formation for a path 10 ft. wide, horizontal transversely, and with side slopes of 2 horizontal to 1 vertical, calculate the volume of the earth to be moved in making the path. There is to be neither cutting nor filling along the centre line.

7. (U. of L.) A line AB 500 ft. along is set out. The theodolite is set up at A and at B, observations of points C and D being made and the following angles recorded :

Bearing of AB . . .	25 degrees	E. of N.
,, AC . . .	69 ,,	W. of N.
,, AD . . .	20 ,,	,,
,, BC . . .	113 ,,	,,
,, BD . . .	80 ,,	,
Altitude of C from A . .	20 ,,	
Altitude of D from A . .	18 ,,	

The point A is 156 ft. above datum. Determine the heights of C and D above datum and the horizontal distance of C from D.

8. A vertical wall, 10 ft. high, and built upon level ground, in latitude 52° N., faces due East. Calculate approximately how many hours the face of the wall will be exposed to the rays of the sun on a clear day when the sun's declination is (*a*) 20° N. and (*b*) 20° S.

What will be the approximate width of the shadow, normal to the wall, at 10 A.M. on each occasion ?

9. It is desired to make an observation for Azimuth in the evening of a certain date, in latitude 52° N. and longitude 4° W. The altitude of a star near the Prime Vertical is to be determined. What are the known or assumed data in the spherical triangle, and what data are calculated to help in the selection of a suitable star from the *N.A.* ? An observation for Azimuth was made on β Pegasi on Jan. 31, 1932.

The observed bearing of β from A	$=04°\text{-}15'\text{-}30''$
Corrected altitude	$=36°\text{-}10'\text{-}30''$
Right Ascension	=23 h. 00 m. 27 s.
Declination	$=27°\text{-}42'\text{-}50''$
Latitude	$=52°\text{-}00'\text{-}00''$

Find the bearing of A from the true meridian.

If Greenwich sidereal time at 0 h. Feb. 1 = 08 h. 40 m. 00 s., what was the G.M.T. of observation, the longitude being 4° W. ?

10. (U. of B.) Describe the International System of Time Signals adopted in 1912–13, and mention the July 1925 amendment to this system. How do these signals enable the longitude of a place to be accurately determined ?

From the following data of a Rhythmic Time Signal from Paris Eiffel Tower, starting at 20 h. 01 m. 00 s. G.M.T., find the error of your chronometer.

Chronometer Time of Coincidences. (At the whole Second)	Number of Dash	Number of Intervals between Dash and Coincidence
20 h. 04 m. 10 s.	1st	55
04 m. 55 s.	2nd	40
05 m. 40 s.	3rd	25
06 m. 25 s.	4th	10
07 m. 10 s.	4th	56
07 m. 55 s.	5th	41

The commencement of successive signals, whether dot or dash, are equally spaced at intervals of 60/61 parts of 1 second of Mean Time.

The number 995 was transmitted from Paris 24 hours after the above signal was received.

(NOTE.—995 signifies that the signals were $0''\cdot05$ early, 005 that they were $0''\cdot05$ late.)

11. (U. of W.) Draw a quadrilateral XYZP where XZ and YP are diagonals. Levelling was carried out between the four stations and the results were as follows :

Between	Y	and	Y	2·714 ft.	rise
,,	Y	,,	Z	2·821 ft.	rise
,,	Z	,,	P	6·500 ft.	fall
,,	P	,,	X	0·874 ft.	rise
	Y	,,	P	3·642 ft.	fall

The levelling between Y and Z and between X and P was carried out twice, the results given being the mean of the two observations. Using the method of Least Squares, find the most probable differences in level between the various points.

ANSWERS

Chapter I (p. 46)

1. Yes.

2. Simpson's Rule, $1123\frac{1}{3}$ sq. yds.; Trapezoidal Rule, 1131 sq. yds.; Average Ordinate Rule, 1099 sq. yds.

3. (1) $441{\cdot}7 + 2000 + 566{\cdot}7 = 3008{\cdot}4$ sq. links $= 0{\cdot}0301$ acre.
   (2) $462{\cdot}5 + 1975 + 550 \quad = 2987{\cdot}5$,, $= 0{\cdot}0299$,,
   (3) $483{\cdot}3 + 1900 + 533{\cdot}3 = 2916{\cdot}7$,, $= 0{\cdot}0292$,,

4. 1444 sq. ft.

5. This error is "cumulative," and the true area $= 30{\cdot}18$ acres nearly.

6. Write equation as $2 \log S = \log s + \log (s-a) + \log (s-b) + \log (s-c)$ and differentiate

$$\frac{\delta S_1}{S} = \pm \frac{a \,.\, \delta a}{4\,S^2}\left\{2s\,(s-a) - b\,.\,c\right\}, \frac{\delta S_2}{S} \text{ and } \frac{\delta S_3}{S} \text{ are similar terms,}$$

$$\text{then } \frac{\delta S}{S} = \pm \sqrt{\left\{\left(\frac{\delta S_1}{S}\right)^2 + \left(\frac{\delta S_2}{S}\right)^2 + \left(\frac{\delta S_3}{S}\right)^2\right\}}.$$

Answer $= \pm 0{\cdot}056$ sq. chain.

7. Maximum error of 1 in $11{\cdot}5$; $15{\cdot}9$, say 16 ft.

8. (*a*) $2{\cdot}186$ acres.
   (*b*) $35°$-$11'$ and $81°$-$0'$ nearly. $48°$-$41'$ and $37°$-$15'$ nearly.
   (*c*) $7{\cdot}541$ chains.
   (*d*) $\pm 0{\cdot}0104$ sq. chain.
   (*e*) $\pm 4\frac{1}{2}'$ and $\pm 3\frac{1}{2}'$ nearly.
   (*f*) $\pm 0{\cdot}4$ link nearly.

9. $1{\cdot}431$ acres.

Chapter III (p. 82)

1. (i) 59 divisions on primary $= 60$ on vernier.
   (ii) 44 divisions on primary $= 45$ on vernier.
   (iii) (*a*) 120. (*b*) 60.

2. $75°$-$30'$. $\cdot 51$ ft.

3. (*a*) $32'$-$17''{\cdot}5$. (*b*) $3'$-$7''{\cdot}5$ to be subtracted.

4. $70°$-$34'$ to nearest minute.

Chapter IV (p. 122)

1. $1'$-$36''{\cdot}6$. Yes. No effect.

2. $\frac{3}{196}\theta$. $0{\cdot}43$ inch.

Chapter V (p. 157)

1. $2{\cdot}148$ acres nearly.

2. At station 3 the N. point of needle is probably deflected about 3° towards W.

3. 710 ft. $162°$-$45'$.

4. Southings exceed Northings by 198·4 ft.; Eastings exceed Westings by 18·4 feet. The error is therefore too large to be caused by the *small* angular error that data suggest.

Mistake is probably in DE. An assumed mistake in DE of 200 ft. (which would be possible, due to a miscount of the chain lengths) will leave a discrepancy of 0·8 in latitude and 0 in departure to be balanced.

5. 86°-42′. Length of CD = 511·6 ft.

6. (1) 527·35 and 1318·71. (2) 1420·24. (3) 291° 48′ nearly.

7. (*a*) 6·317 acres. (*b*) and (*c*) 6·318 acres nearly.

8. (*a*) 1507, 128. (*b*) 1295·7, 78°-36′ E. of N. (*c*) 40°-12′.

9. AB, N. 30° E. BC, S. 60°-14′ E. CD, S. 0°-16′ E. DA, S. 2°-27′ W. AD = 475·9 ft.

Chapter VI (p. 222)

1. (1) 2·31 ft. (2) No. (3) Adjust to 6·78 on staff at A.

2. 7′; at B = 5·68, at C = 7·28.

The axis of bubble is not at right angles to the vertical axis; nor is the line of collimation parallel to the axis of bubble. The line of collimation may or may not be parallel to the axis of the telescope, and this may or may not be at right angles to the vertical axis of the instrument.

3. Last point, chainage = 500 ft. Reduced level = 35·96 ft.

4. Depth of cut at 1, 2, 3, 4 = 10·18, 7·27, 3·10, 1·36 ft. respectively. Height of bank at 5, 6 = 3·03, 2·70 ft. respectively.

6. (*a*) = 860 ft. radius. Movement on staff = 0·015 ft.
 (*b*) = 172 ft. ,, ,, ,, = 0·073 ft.
 (*c*) = 14·3 ft. ,, ,, ,, = 0·88 ft.
 (*d*) = 430 ft. ,, ,, ,, = 0·029 ft.

7. 4·30 ft. and 4·52 ft. 7·98 ft. at B.

8. 5 miles + 1·25 ft.

Chapter VII (p. 240)

1. 110·06 ft. 174·10 ft.

2. ±0·206 ft. ±0·186 ft.

3. 136·2 ft. above datum.

4. About 566·5 ft. above instrument axis.

5. $\pm\sqrt{(0{\cdot}0684)^2 + (0{\cdot}0434)^2 + (2{\cdot}146)^2} = \pm 2{\cdot}15$ ft. say in all.

6. D = 1059 ft. about. R.L. of B = 656·9 ft. about.

7. p.e. in $h_2 = \pm 0{\cdot}12$ ft.; in D = ±1·2 ft. nearly; in $h_1 = \pm 1{\cdot}0$ ft. nearly.

8. 1248·36 ft. about. $r = 10''{\cdot}35$, K = 0·0688.

9. 1748·5 ft. $r = 16''{\cdot}15$, K = 0·0797.

10. ±0·35 ft. ±0·39 ft.

11. 2082·8 ft.

12. 161 ft. below datum.

13. PQ = 684·2 ft. Altitudes, P = 342·2, Q = 411·7.

14. (*a*) 0·134 ft. (*b*) 0·115 ft.

15. (*a*) = 79·46 ft. (*b*) = 79·11 ft. Axis of instrument at B = 83·51, C = 81·27. Approximate distance *ab* = 62 chains.

CHAPTER VIII (p. 283)

1. 0·12 inch. 464·5 ft.

2. 100. 1·2 ft.

3. (1) 60·3. (4) 118·7. (9) 150·5 ft.
Levels B = 80·58. C = 112·68. D = 106·59.
(1) 86·01. (4) 94·80. (9) = 157·76 ft. above datum.

4. +0·40 or −0·37 ft. $\pm$0·07 ft. nearly.

6. $\pm$2·52 ft. in distance. $\pm$0·16 ft. in altitude.

7. 644·4 ft. 641·9 ft.

8. 531·3 ft. −3·51 ft.

10. $\pm 16 \times 4{\cdot}62 = \pm 73{\cdot}9$; 70·8 ft. above, 77·1 ft. below ; 2·6 by estimation ; 462 − 12 = 450 ft.

CHAPTER IX (p. 304)

1. Mean R.L. = 1374 ft. above datum.

2. 0·0011 in. (*i.e.* negligible) ; 0·024 in. (appreciable) ; 0·1 in. (large).

3. 300 ft. approx.

CHAPTER X (p. 333)

1. 15, 14·69, 13·78, 12·26, 10·13, 7·37, 4·00 ft.

2. Deflection angles : Right, 54′ ; 2°-54′ ; 4°-54′ ; 6°-45′. Radius, 14·327 chains. Tangent distance, 1·696 chains. Length, 3·375 chains. Curve begins, 130·55 chains ; ends, 133·925 chains.

3. (i) 0·71, 5·06, 6·98, 6·21 ft.
(ii) 0·35, 0·79, 0·87, 0·87, 0·87, 0·87, 0·74 ft. for pegs at 50 ft. intervals.

4. 17·492 and 56·587 chains. $\delta_1 = 21'{\cdot}83$, δ_2 to $\delta_{39} = 42'{\cdot}97$, $\delta_{40} = 25'{\cdot}22$; $\Delta_1 = 21'{\cdot}8$, $\Delta_2 = 1°\text{-}04'{\cdot}8$, $\Delta_3 = 1°\text{-}47'{\cdot}8$ . . . $\Delta_{40} = 28°$; $o_1 = 0{\cdot}32$ link, $o_2 = 1{\cdot}88$, o_3 to $o_{39} = 2{\cdot}5$, $o_{40} = 1{\cdot}16$ links.

5. AT = 13·267, AT_1 = 14·969, AM = 6·272, AN = 9·61, MP = 6·995, NP = 5·359, MN = 12·354 chains.

6. Co-ordinate of P from A = 2·856 and 1·754 chains ; AT = 12·324 chains ; R = 26·429 chains.
Chainages at T = 108·236 ; at P 117·915 ; at T_1 = 131·30 chains.

7. R = 380·4 ft. Tangent distances from B and C = 219·6 and 380·4 ft. respectively.

8. 81·213, 82·713, 87·496, 88·99 chains.
Deflection angles for transition curve from T, 2′·6, 19′·7, 52′·7, 1°-11′·6.
The tangent ME at E makes an angle of 2°-23′·2 with ET.
Deflection angles for circular arc from E, 41′·1, 1°-52′·7, 3°-04′·3, 4°-15′·9 . . . 11°-25′.

9. $h_1 = 0{\cdot}5$, $h_2 = 0{\cdot}4$ ft. ; L_1 = 150 ft. say $2\frac{1}{2}$ chains, L_2 = 120 ft. say 2 chains ; $s_1 = 1{\cdot}3$ links, $s_2 = 0{\cdot}66$ link ; MP = 9·332 chains, NP = 6·700 chains, AM = 8·154 chains ; AN = 12·493 chains ; chainage at T = 125·803 chains ; deflection angles for first transition curve 0′·4, 16′, 55′, 1°-11′ ; angle TEF = 2°-23′. Deflection angles for 20-chain curve from E, 17′, 1°-43′, etc., to $t_1 = 22°\text{-}30'$, to $p_1 = 23°\text{-}13'$. Chainage at t_1 = 141·506 chains, at p_1, p_2, or p_3 = 142·006 chains, if L = 1 chain ; deflection angles from $p_3 = 1°\text{-}8'$, 2°-17′, etc., to E′ = 13°-51′, to $T_{11}' = 15°$; $p_1p_2 = 0{\cdot}66$ link, $p_1p_3 = 0{\cdot}33$ link, $T_{11}'f' = \frac{1}{2}S_2 = 0{\cdot}33$ link ; chainage at E′ = 154·096 chains, at f' = 155·096 chains, at T′ = 156·096 chains.

10. (*a*) 157·80, 158·40, 158·80, 159·00, 159·00, 158·80, 158·40, 157·80, 157·00.
(*b*) 159·90, 160·65, 161·50.
(*c*) 161·30, 160·55, 159·70.

11. L = 600 ft. also. 1250 (130·00): 1300 (131·57); 1350 (132·91); . . . 1550 136·25); . . . 1650 (136·67); . . . 1800 (135·73); 1850 (135·00).

Chapter XI (p. 351)

1. 9740 sq. yds. 31,600 cub. yds.

2. Half breadths = 64·8, 47·8, 57·7, 42·6, 55·9, 41·3 ft. respectively. Volume = 12,050 cub. yds. nearly.

3. 64 cub. yds. about.

4. 96·6, 62, 83·2, 53·5, 65·25, 41·9 ft. 11,610 cub. yds. nearly.

5. +13·6, +5·6, −6·0, −15·3, −9·2, −3·9, −1·9, −1·7, +1·0 ft. about.

6. 1 in 60. Areas 19, 242, 629, 742, 559, 398, 254, 227, 19 sq. ft.; 34,660 cub. yds.

Chapter XII (p. 380)

1. 2·8 ft. per sec.; 4,350,000 cub. ft. per diem roughly.

2. $v = 0{\cdot}017 + \frac{11{\cdot}10}{t}$, where v = vel. in metres per sec., and t = number of secs. between successive contacts.

3. 81·32 cub. ft. per sec.

4. (*a*) 822,800 galls. per hr.
(*b*) 941,200 galls. per hr.

5. 3·073 cub. ft. per sec.

6. 1670, 1331, 1438 ft.

8. O, A, B, C are concyclic—failure of fix.

Chapter XIII (p. 408)

1. 63°-51′.

2. A = +4″·09. B = −3″·26. C = −5″·74.

3. +12″·52.

Chapter XIV (p. 431)

1. Sag, −6·985 ft.; pull, +2 914 ft.; temp., +1·030 ft.: Total, −3·041 ft.

2. ∓0·0828 ft.

3. 100 + 0·0104 + 0·0065 − 0·0318 = 99·9851 ft.

4. 4514·99 ft. nearly.

5. (i) +·0118 ft.; (ii) −·0119 ft.; (iii) +·0119 ft.; (iv) (*a*) +·0038 ft., (*b*) −·0099 ft.; (v) −·00027 ft. nearly; (vi) −·00019 ft.; (vii) (*a*) −·0076, (*b*) +·0057 −·0057 = 0.

Chapter XV (p. 442)

1. 1°-33′-20″; 1°-11′-20″; 88°-48′-40″.

2. 89°-24′-14″·6. 0·156 mile; 0·313 mile.

3. Spiral to pole. Great circle.

Chapter XVI (p. 462)

1. 5′-19″, or with refraction, 4′-47″.
2. 85°-00′-40″ W.
3. 5 h. 4 m. P.M. nearly.
4. 6°-57′-27″ E.
5. 10 h. 8 m. 33·7 s.

Chapter XVII (p. 502)

1. 112°-55′ from north.
2. 24°-2′-8″ from south, if sun is setting.
3. (i) 4″·3 : 0 ; 5″·8 : 0.
   (ii) 4″·5 : 0 ; 6″·0 : 0.
   (*a*) 3′-53″. (*b*) 28″. (*c*) 37″.
4. 89°-39′-36″. 71°-19′-36″ nearly.
5. 1°-18′-9″ S. (see Chambers' tables for refraction corrections).
6. N. 40°-25′-22″·05, if sun is south of Zenith.
   N. 3°-36′-59″·25, ,, north ,,
7. N. 41°-07′-54″·8, ,, south ,,
   N. 1°-23′-23″·6, ,, north ,,
8. 10°-46′-22″·2 W.
9. (*a*) ±10·8 s. (*b*) ±0·63 s.
10. Polaris and Benetnasch . Z = 20″·5 W. of N. ; t = 49·7 seconds
    Polaris and Mizar . Z = 10′-03″·2 W. of N. ; t = 24 m. 25·35 s.
    δ Cassiopeiae and Mizar . Z = 03′-54″·6 W. of N. ; t = 29·1 seconds
11. 42 m. 14·4 s.

Miscellaneous (p. 551)

2. S. 71°-34′ W. nearly.
3. 83 ft. below datum nearly.
4. 605·3 yds. nearly. B is 129·2 yds. above A.
6. (*a*) 1 in 4·47. (*b*) 107·28 cub. yds.
7. 331·3 ft. 337·2 ft. 436·5 ft.
8. (*a*) 7 h. 51 m. (*b*) 4 h. 9 m. (*a*) 6·1 ft. (*b*) 20·3 ft.
9. 265°-44′-01″·2 : 19 h. 00 m. 18·2 s.
10. 2 m. 15·24 s.
11. + 2·747, + 2·835, − 6·472, + 0·890, − 3·637.

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THE END